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Semi -*T*- Small Submodules

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Abstract
Let R be a ring with identity and T be a submodule of a left R- module$W$. A submodule N of W is called T- small in W, denoted by$N \overset{\ll}{_T} W$, in case for any submodule X of$W$, $T \subseteq X+N$ implies$T \subseteq X$. A Submodule N of W is called semi -T- small in W, denoted by $N \overset{\ll}{_{S-T}} W$, provided for submodule X of W, $T \subseteq X+N$ implies that $T \subseteq X+Rad(W)$. We studied this concept which is a generalization of the small submodules and obtained some related results

Keywords: Small Submodules, T-Small Submodules, Semi -T- Small Submodules.

المقاسات الجزئية الصغرى من النمط T

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الخلاصه
لتكن R حلقة ابدالية ذات عنصر محايد و T مقاس جزئي من المقاس W المعرف على R يدعى المقاس الجزئي N من W بانه مقاس جزئي شبه اصغر من النمط T. اذا كان $T \subseteq N+X$ يؤدي هذا الى ان $T \subseteq X+Rad(W)$.درسنا هذا المفهوم الذي هو تعميم لمقاسات الجزئية الصغرى وحصلنا على بعض النتائج ذات العلاقة.

1- INTRODUCTION

In this paper, all rings have identity elements and all modules are left unitary. Let R be a ring and W be an R- module. Recall that a submodule N of W is small, denoted by $N \ll W$, if for any submodule X of W, $X+N=W$ implies that $X=W$. More details about small submodules can be found in earlier reports [1,2,3]. Following Beyranvand and Moradi [4], let T be a submodule of a module W. A submodule N of a module W is called T- small in w, denoted by $N \overset{\ll}{_T} W$, in case for any submodule X of W, $T \subseteq X+N$ implies $T \subseteq X$.

In this work, we introduce the concept of semi-T- small submodules. Let T be a submodule of a module W. A submodule N of a module W is called semi-T- small in W, denoted by $N \overset{\ll}{_{S-T}} W$, in case for any submodule X of W, $T \subseteq X+N$ implies $T \subseteq X+Rad(W)$ (where $Rad\,(W)$ is The Jacobson radical of W). Some properties of this kind of submodules are considered.

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2- SEMI -T- SMALL SUBMODULES

Definition 2.1 Let $\boldsymbol{R}$ be a ring and $\boldsymbol{T}$ be a submodule of an $\boldsymbol{R}$-module $\boldsymbol{W}$. A submodule $\boldsymbol{N}$ of $\boldsymbol{W}$ is called semi-$\boldsymbol{T}$-small in $\boldsymbol{W}$, denoted by $\boldsymbol{N} \overset{\ll}{_{S-T}} \boldsymbol{W}$, in case that for any submodule $\boldsymbol{X} \leq \boldsymbol{W}$, $\boldsymbol{T} \subseteq \boldsymbol{X} + \boldsymbol{N}$ implies that $\boldsymbol{T} \subseteq \boldsymbol{X} + \boldsymbol{Rad}(\boldsymbol{W})$.

Remarks and Examples 2.2

1. Let W be an R-module and T be a submodule of W. Then every T-small submodule is semi $-T-$ small submodule

Proof: It is clear.

2. The converse of (1) is not true, i.e. semi-T-small submodule need not to be T-small submodule; for example: consider $\mathbb{Z}_{12}$ as $\mathbb{Z}$- module and $T =<\bar{2}>, N =<\bar{3}>, X =<\bar{4}>$. Then $T =<\bar{2}>\subseteq<\bar{4}> + <\bar{3}>= \mathbb{Z}_{12}$. Since $T \nsubseteq X$, thus N is not T-small in $\mathbb{Z}_{12}$. But $X + Rad(\mathbb{Z}_{12}) =<\bar{4}> + <\bar{6}>=<\bar{2}>$, thus $T = X + Rad(\mathbb{Z}_{12})$. Also $T =<\bar{2}>\subseteq<\bar{2}> + <\bar{3}>= \mathbb{Z}_{12}$ (where $X =<\bar{2}>$) and thus $<\bar{2}>\subseteq<\bar{2}>= X$. Therefore $T \subseteq X + Rad(\mathbb{Z}_{12}) = X$. Hence N is semi-T-small.

3. Let W be an R- module. If $Rad(W) = 0$, then the two concepts T-small submodules and semi-T-small submodules are equivalent.

4. If $T = 0$ in the last definition, then every submodule of W is semi $-T-$ small.

5. If $T \neq 0$ and $Rad(W) = 0$, then $N \overset{\ll}{_{S-T}} W$ implies that $T \nsubseteq N$, for if not then $T \subseteq N + (0)$ and hence $T \subseteq (0)$, which is a contradiction.

6. It is clear that $0 \overset{\ll}{_{S-T}} W$, for an R-module W.

Proposition 2.3: Let $\boldsymbol{W}$ be an $\boldsymbol{R}$-module, $\boldsymbol{L} \leq \boldsymbol{K} \leq \boldsymbol{W}$, and $\boldsymbol{T} \leq \boldsymbol{W}$. If $\overset{\ll}{_{S-T}} \boldsymbol{W}$ **then** $\boldsymbol{L} \overset{\ll}{_{S-T}} \boldsymbol{W}$.

Proof : Suppose that $T \subseteq L + X$, where X is a submodule of W, thus $T \subseteq L + K$. But $K \overset{\ll}{_{S-T}} W$, therefore $T \subseteq X + Rad(W)$. Hence $L \overset{\ll}{_{S-T}} W$.

Proposition 2.4: Let $\boldsymbol{W}$ be an $\boldsymbol{R}$-module with $\boldsymbol{N} \leq \boldsymbol{K} \leq \boldsymbol{W}$ and $\boldsymbol{T} \leq \boldsymbol{K}$. If $\boldsymbol{N} \overset{\ll}{_{S-T}} \boldsymbol{K}$ then $N \overset{\ll}{_{S-T}} W$.

Proof: Suppose that $N \overset{\ll}{_{S-T}} K$. To show that $N \overset{\ll}{_{S-T}} W$, let $T \subseteq N + X$, where X is a submodule of W. Thus $T \subseteq N + (X \cap K)$, but $N \overset{\ll}{_{S-T}} K$. Therefore $T \subseteq (X \cap K) + Rad(K)$. Since $Rad(K) \subseteq Rad(W)$ [5], thus $T \subseteq X + Rad(W)$ and therefore $N \overset{\ll}{_{S-T}} W$.

Before we give the converse of the last proposition, we need the following Lemma [6].

Lemma 2.5 [6, Lemma (1.33), p22]: Let M be a module, then $Rad(M) = 0$ if and only if $Rad(N) = Rad(M) \cap N$, for every submodule N if M.

Now, we have the following.

Proposition 2.6: Let T, K and N be submodules of an R- module W such that $T \leq K$, $N \leq K \leq W$ and $Rad(W) = 0$. If $N \overset{\ll}{_{S-T}} W$, then $N \overset{\ll}{_{S-T}} K$.

Proof: Suppose that $N \overset{\ll}{_{S-T}} W$ and $Rad(W) = 0$. To show that $N \overset{\ll}{_{S-T}} K$, let $T \subseteq N + X$, where X is a submodule of K. Since $N \overset{\ll}{_{S-T}} W$, thus $T \subseteq X + Rad(W)$. But $T \subseteq K$, thus $T \subseteq (X + Rad(W) \cap K) = X + Rad(W) \cap K$, and by Lemma (2.5), $T \subseteq X + Rad(K)$. Thus $N \overset{\ll}{_{S-T}} K$.

Proposition 2.7: Let $\boldsymbol{W}$ be a module with submodules $\boldsymbol{N_1}$, $\boldsymbol{N_2}$ and $\boldsymbol{T}$. Then $\boldsymbol{N_1} \overset{\ll}{_{S-T}} \boldsymbol{W}$ and $\boldsymbol{N_2} \overset{\ll}{_{S-T}} \boldsymbol{W}$ if and only if $\boldsymbol{N_1} + \boldsymbol{N_1} \overset{\ll}{_{S-T}} \boldsymbol{W}$.

Proof: ($\Rightarrow$): Suppose that $N_1 \overset{\ll}{_{S-T}} W$ and $N_2 \overset{\ll}{_{S-T}} W$. To prove that $N_1 + N_2 \overset{\ll}{_{S-T}} W$, suppose that $T \subseteq N_1 + N_2 + X$, where X is a submodule of W. Thus $T \subseteq N_1 + (N_2 + X)$. Since $N_1 \overset{\ll}{_{S-T}} W$, then $T \subseteq N_2 + X + Rad(W) = N_2 + (X + Rad(W))$. But $N_2 \overset{\ll}{_{S-T}} W$, thus $T \subseteq X + Rad(W) + Rad(W) = X + Rad(W)$. Hence $N_1 + N_2 \overset{\ll}{_{S-T}} W$.

($\Leftarrow$): Now, suppose that $N_1 + N_2 \overset{\ll}{_{S-T}} W$. To show that $N_1 \overset{\ll}{_{S-T}} W$, suppose that $T \subseteq N_1 + X$, where X is a submodule of W, thus $T \subseteq N_1 + N_2 + X$. Since $N_1 + N_2 \overset{\ll}{_{S-T}} W$, hence $T \subseteq X + Rad(W)$. Thus $N_1 \overset{\ll}{_{S-T}} W$. Similarly, we can prove that $N_2 \overset{\ll}{_{S-T}} W$.

Proposition 2.8: Let $\boldsymbol{W}$ be an $\boldsymbol{R}$- module with $\boldsymbol{K_1 \leq W_1 \leq W}$ and $\boldsymbol{K_2 \leq W_2 \leq W}$. Then $\boldsymbol{K_1 \underset{S-T}{\ll} W_1}$ and $\boldsymbol{K_2 \underset{S-T}{\ll} W_2}$ if and only if $\boldsymbol{K_1 + K_2 \underset{S-T}{\ll} W_1 + W_2}$.

Proof: First assume that $K_1 \underset{S-T}{\ll} W_1$ and $K_2 \underset{S-T}{\ll} W_2$. By Proposition (2.4), $K_1 \underset{S-T}{\ll} W_1 + W_2$. Also, by Proposition (2.7), $K_1 + K_2 \underset{S-T}{\ll} W_1 + W_2$.

Conversely, suppose that $K_1 + K_2 \underset{S-T}{\ll} W_1 + W_2$. To show that $K_1 \underset{S-T}{\ll} W_1$, suppose that $T \subseteq K_1 + X$, where X is a submodule of W_1. Thus $T \subseteq K_1 + K_2 + X$, but $K_1 + K_2 \underset{S-T}{\ll} W_1 + W_2$. Therefore $K_1 \underset{S-T}{\ll} W_1$. Similarly, $K_2 \underset{S-T}{\ll} W_2$.

Theorem 2.9: Let $\{\boldsymbol{T_i}\}_{\boldsymbol{i \in I}}$ be an indexed set of submodules of an $\boldsymbol{R}$ –module $\boldsymbol{W}$, and $\boldsymbol{K}$ be a submodule of $\boldsymbol{W}$. If for each $\boldsymbol{i \in I}$, $\boldsymbol{K \underset{S-T_i}{\ll} W}$, then $\boldsymbol{K \underset{S-\sum_{i\in I} T_i}{\ll} W}$.

Proof: suppose that $\sum T_i \subseteq K + X$, for some $X \subseteq W$, then for each $i \in I$, $T_i \subseteq K + X$, and by hypothesis, $T_i \subseteq X + Rad(W)$. Thus $\sum_{i\in I} T_i \subseteq X + Rad(W)$.

Corollary 2.10: Let $\boldsymbol{K_1}$ and $\boldsymbol{K_2}$ be submodules of an $\boldsymbol{R}$-module $\boldsymbol{W}$ such that $\boldsymbol{K_1 \underset{S-K_2}{\ll} W}$ and $\boldsymbol{K_2 \underset{S-K_1}{\ll} W}$. Then $\boldsymbol{K_1 \cap K_2 \underset{S-K_1+k_2}{\ll} W}$.

Proof: Since $K_1 \underset{S-K_2}{\ll} W$, then by Proposition (2.3), $K_1 \cap K_2 \underset{S-K_2}{\ll} W$ and $K_1 \cap K_2 \underset{S-K_1}{\ll} W$. Also, by theorem (2.9),$K_1 \cap K_2 \underset{S-K_1+K_2}{\ll} W$.

We introduce the following concept.

Definition 2.11: Let $\boldsymbol{W}$, $\boldsymbol{H}$ be two right $\boldsymbol{R}$-modules and $\boldsymbol{0 \neq T \leq W}$. An $\boldsymbol{R}$-epimorphism $\boldsymbol{F: W \to H}$ is called semi- $\boldsymbol{T}$-small in case that $\boldsymbol{Kerf \underset{S-T}{\ll} W}$.

Proposition 2.12: Let $\boldsymbol{K}$ and $\boldsymbol{0 \neq T}$ be two submodules of Left $\boldsymbol{R}$ – module $\boldsymbol{W}$. The following statements are equivalent:

1. $K \underset{S-T}{\ll} W$.
2. The natural epimorphism $P_K: W \to W/K$ is semi-T-small.
3. For every R –module F and R –homomorphism $h: F \to W$, $T \subseteq K + Imh$ implies that $T \subseteq Imh + Rad(W)$

Proof: (1) $\Rightarrow$(2): Let $P_K: W \to W/K$ be the natural epimorphism and suppose that $T \subseteq KerP_K + X$, where X is a submodule of W. But $KerP_K = K$, thus $T \subseteq K + X$ and since $K \underset{S-T}{\ll} W$, therefore $T \subseteq X + Rad(W)$. Hence $KerP_K \underset{S-T}{\ll} W$, i.e. P_K is semi-T-small.

(2) $\Longrightarrow$(3): It is clear.

(3)$\Longrightarrow$(1): Suppose that $T \subseteq K + X$, for some $X \leq W$. Let $i: X \longrightarrow W$ be the inclusion homomorphism. Then $T \subseteq K + Imi = K + X$ and by (3) $T \subseteq X + Rad(W) = Im\, i + Rad(W)$.

Lemma 2.13: Let $\boldsymbol{W}$ and $\boldsymbol{F}$ be $\boldsymbol{R}$- modules and $\boldsymbol{f: W \to F}$ be $\boldsymbol{R}$- homomorphism. If $\boldsymbol{K}$ and $\boldsymbol{T}$ are submodules of $\boldsymbol{W}$ such that $\boldsymbol{K \underset{S-T}{\ll} W}$, then $\boldsymbol{f(K) \underset{S-f(T)}{\ll} F}$. In particular, if $\boldsymbol{K \underset{S-T}{\ll} W \ll F}$, then $\boldsymbol{K \underset{S-T}{\ll} F}$.

Proof: We may assume that $f(T) \neq O$. Let $f(T) \subseteq f(K) + X$, for some $X \leq F$. We claim that $T \subseteq K + f^{-1}(X)$. Let $t \in T$, then $f(t) = f(k) + x$, for some $k \in K$ and $x \in X$. Then $f(t-k) \in X$ and so $t - k \in f^{-1}(X)$. This implies that $t \in K + f^{-1}(X)$, but $K \ll W$, therefore $T \subseteq f^{-1}(X) + Rad(W)$. Thus $f(T) \subseteq X + f\big(Rad(W)\big) \subseteq X + Rad(f)$, i.e. $f(k) \underset{s-f(T)}{\ll} F$.

A submodule V of an R-module is called a supplement of a submodule U of W, if V is a minimal element in the set of submodules $\leq W$, with U+F = W. Equivalently, V is a supplement of U if W = U+V and U $\cap$ V<<V [7].

Proposition 2.14: Let $\boldsymbol{N}$ and $\boldsymbol{T}$ be submodules of an $\boldsymbol{R}$- module $\boldsymbol{W}$ and $\boldsymbol{N'}$ be a supplement of $\boldsymbol{N}$ in $\boldsymbol{W}$. If $\boldsymbol{N \underset{S-T}{\ll} W}$, then $\boldsymbol{T \subseteq N' + Rad(W)}$.

Proof: It is clear.

Theorem 2.15: Let $\boldsymbol{K}$ be a submodule of an $\boldsymbol{R}$-module $\boldsymbol{W}$ and $\boldsymbol{K'}$ is a supplement of $\boldsymbol{K}$ in $\boldsymbol{W}$. The following are equivalent:

1. $K' \underset{S-K}{\ll} W$;
2. For each submodule N of W, the relation $K + N = W$ implies that $K' \subseteq N + Rad(W)$.

Proof: (1)$\Longrightarrow$(2): It is clear.

(2)⟹(1): Suppose that $K' \subseteq K + X$ for some $X \leq W$. Since $W = K + K' \subseteq K + X$, we have $W = K + X$, and by assumption, $K' \subseteq X + Rad(W)$.

3- ADDITIONAL RESULTS ABOUT SEMI-T-SMALL SUBMODULES

In this section, we provide new results about semi – T – small submodules and start by the following proposition.

Proposition 3.1: Let $\boldsymbol{K}, \boldsymbol{N}$ and $\boldsymbol{L}$ be submodules of a module $\boldsymbol{W}$ such that $\boldsymbol{K} \subseteq \boldsymbol{N} \subseteq \boldsymbol{L} \subseteq \boldsymbol{W}$. If $\frac{\boldsymbol{L}}{\boldsymbol{K}} \underset{S-\frac{T+K}{K}}{\ll} \frac{\boldsymbol{W}}{\boldsymbol{K}}$, then $\frac{\boldsymbol{L}}{\boldsymbol{N}} \underset{S-\frac{T+N}{N}}{\ll} \frac{\boldsymbol{W}}{\boldsymbol{N}}$ and $\frac{\boldsymbol{N}}{\boldsymbol{K}} \underset{S-\frac{T+N}{N}}{\ll} \frac{\boldsymbol{W}}{\boldsymbol{K}}$.

Proof: Let $\frac{L}{K} \underset{S-\frac{T+N}{N}}{\ll} \frac{W}{K}$. To show that $\frac{L}{N} \underset{S-\frac{T+N}{N}}{\ll} \frac{W}{N}$, let $\frac{T+N}{N} \subseteq \frac{L}{N} + \frac{X}{N}$, for some submodule $\frac{X}{N}$ of $\frac{W}{N}$, then $\frac{T+N}{N} \subseteq \frac{L+X}{N}$. Hence $T \subseteq T + N \subseteq L + X$, so $\frac{T+K}{K} \subseteq \frac{L+X}{K}$, therefore $\frac{T+K}{K} \subseteq \frac{L}{K} + \frac{X}{K}$. But $\frac{L}{K} \underset{S-\frac{T+N}{N}}{\ll} \frac{W}{K}$, therefore $\frac{T+K}{K} \subseteq \frac{X}{K} + Rad\left(\frac{W}{K}\right) \subseteq \frac{X}{K} + \frac{Rad(W)}{K}$.

So $T \subseteq T + K \subseteq X + Rad(W)$, hence $\frac{T+N}{N} \subseteq \frac{X}{N} + \frac{Rad(W)+N}{N} \subseteq \frac{X}{N} + \frac{Rad(W)}{N} \subseteq \frac{X}{N} + Rad\left(\frac{W}{N}\right)$. Thus $\frac{L}{N} \underset{S-\frac{T+N}{N}}{\ll} \frac{W}{N}$.

To show that $\frac{N}{k} \underset{S-\frac{T+N}{N}}{\ll} \frac{W}{K}$, let $\frac{T+K}{K} \subseteq \frac{N}{K} + \frac{X}{K}$, for some submodule $\frac{X}{K}$ of $\frac{W}{K}$. Then $\frac{T+K}{K} \subseteq \frac{N+X}{K}$ and hence $T \subseteq T + K \subseteq N + X \subseteq L + X$. So $\frac{T+K}{K} \subseteq \frac{L+X}{K}$, therefore $\frac{T+K}{K} \subseteq \frac{X}{K} + Rad\left(\frac{W}{K}\right)$. Thus $\frac{N}{K} \underset{S-\frac{T+N}{N}}{\ll} \frac{W}{K}$.

Proposition 3.2: Let $\boldsymbol{W}$ be an $\boldsymbol{R}$-module, $\boldsymbol{L} \leq \boldsymbol{K} \leq \boldsymbol{W}$ and $\boldsymbol{T} \leq \boldsymbol{W}$. If $\boldsymbol{K} \underset{S-T}{\ll} \boldsymbol{W}$, then $\frac{\boldsymbol{K}}{\boldsymbol{L}} \underset{S-T}{\ll} \frac{\boldsymbol{W}}{\boldsymbol{L}}$.

Proof: Suppose that $K \underset{S-T}{\ll} W$. To show that $\frac{K}{L} \underset{S-T}{\ll} \frac{W}{L}$, suppose that $\frac{T}{L} \subseteq \frac{K}{L} + \frac{X}{L}$, where $X \leq W$ such that $L \leq X$. Then $\frac{T}{L} \subseteq \frac{K+X}{L}$, therefore $T \subseteq K + X$. But $K \underset{S-T}{\ll} W$, thus $T \subseteq X + Rad(W)$. Therefore $\frac{T}{L} \subseteq \frac{X+Rad(W)}{L} = \frac{X}{L} + \frac{Rad(W)+L}{L} \subseteq \frac{X}{L} + Rad\left(\frac{W}{L}\right)$. Hence $\frac{K}{L} \underset{S-T}{\ll} \frac{W}{L}$.

Proposition 3.3: Let $\boldsymbol{W} = \boldsymbol{W_1} \oplus \boldsymbol{W_2}$ be a module such that $\boldsymbol{R} = \boldsymbol{annW_1} + \boldsymbol{annW_2}$. If $\boldsymbol{H_1} \underset{S-T_1}{\ll} \boldsymbol{W_1}$ and $\boldsymbol{H_2} \underset{S-T_2}{\ll} \boldsymbol{W_2}$, then $\boldsymbol{H_1} \oplus \boldsymbol{H_2} \underset{S-T_1 \oplus T_2}{\ll} \boldsymbol{W}$.

Proof: Let $T_1 \oplus T_2 \subseteq H_1 \oplus H_2 + K$, for some submodule K of W. Since $R = annW_1 + annW_2$, then by [7, prop. 4.2, ch1] $K = K_1 \oplus K_2$, for some submodules K_1 of W_1 and submodule K_2 of W_2. Hence,

$T_1 \oplus T_2 \subseteq H_1 \oplus H_2 + K_1 \oplus K_2 = (H_1 + K_1) \oplus (H_2 + K_2)$.q

One can easily shows that $T_1 \subseteq H_1 + K_1$ and $T_2 \subseteq H_2 + K_2$. Since $H_1 \underset{S-T_1}{\ll} W_1$ and $H_2 \underset{S-T_2}{\ll} W_2$, then $T_1 \subseteq K_1 + Rad(W_1)$ and $T_2 \subseteq K_2 + Rad(W_2)$. Thus $T_1 \oplus T_2 \subseteq K_1 \oplus K_2 + Rad(W_1) \oplus Rad(W_2)$. Hence $T_1 \oplus T_2 \subseteq K_1 + K_2 + Rad(W)$ [5]. Therefore $H_1 \oplus H_2 \underset{S-T_1 \oplus T_2}{\ll} W$.

Let W be an R – module. W is called a fully stable module if for each submodule N of W and for each R- homomorphism f from N to W, $f(N) \subseteq N$, see [8].

Proposition 3.4: Let $\boldsymbol{W} = \oplus \boldsymbol{W_i}$ be a fully stable module. If $\boldsymbol{H_i} \underset{S-T_i}{\ll} \boldsymbol{W_i}$, for each $\boldsymbol{i} \in \boldsymbol{I}$ then $\oplus \boldsymbol{H_i} \underset{S-\oplus_{i\in I} T_i}{\ll} \oplus \boldsymbol{W_i}$.

Proof: Let $W = \bigoplus_{i\in I} W_i$ be a fully stable module and $H_i \underset{S-T_i}{\ll} W_i$, for each $i \in I$. To show that $\bigoplus_{i\in I} H_i \underset{S-\oplus_{i\in I} T_i}{\ll} \bigoplus_{i\in I} W_i$, let $\bigoplus_{i\in I} T_i \subseteq \bigoplus_{i\in I} H_i + K$, for some submodule K of W.We claim that $K = \bigoplus_{i\in I} (K \cap W_i)$. To show that, for each $i \in I$, let $\pi_i: W \to W_i$ be the projection map and let $x \in K$, then $x \in \bigoplus_{i\in I} W_i$ and hence $x = \sum_{i\in I} x_i$ where $x_i \in W_i$, for all $i \in I$ and $x_i \neq 0$ for at most a finite of $i \in I$. Since W is fully stable, then $\pi_i(x) \in K$, $\forall i \in I$. Now, $\pi_i(x) = \pi_i(\sum_{i\in I} x_i) = x_i \in K \cap W_i$ and hence $x = \sum_{i\in I} x_i \in \oplus (K \cap W_i)$. Thus $K \subseteq \oplus (K \cap W_i)$. Clearly, $\bigoplus_{i\in I} (K \cap W_i) \subseteq K$. Thus $K = \bigoplus_{i\in I} (K \cap W_i)$. Now, $\bigoplus_{i\in I} T_i \subseteq (\bigoplus_{i\in I} H_i) + \left(\bigoplus_{i\in I} (K \cap W_i)\right) = \bigoplus_{i\in I} \left(H_i + (K \cap W_i)\right)$.

Therefore $T_i \subseteq H_i + (K \cap W_i)$, for each $i \in I$. Since $H_i \underset{S-T_i}{\ll} W_i$, $\forall i \in I$, then $T_i \subseteq K \cap W_i + Rad(W_i)$, $\forall i \in I$ and hence $\oplus T_i \subseteq \oplus (K \cap W_i) + Rad(W)$ [5].

Let W be an R- module. Recall that W is a multiplication module if, for each submodule N of W, there exists an ideal I of R such that $N = IW$, see [9].

Proposition 3.5

Let W be a finitely generated, faithful, and multiplication module and let I, J be ideals in R. Then $I \underset{S-T}{\ll} R$ if and only if $IW \underset{S-JW}{\ll} W$.

Proof:

Assume that $I \underset{S-J}{\ll} R$. To show that $IW \underset{S-JW}{\ll} W$. $JW \subseteq IW + X$, for some submodule X of W. Since W is a multiplication module, then $X = KW$, for some ideal K of R and hence $JW \subseteq IW + KW = (I + K)W$. But W is finitely generated faithful and a multiplication module, therefore W is a cancelation module by [9]. So $J \subseteq I + K$. Since $I \underset{S-J}{\ll} R$, then $J \subseteq K + Rad(R)$. Hence $JW \subseteq KW + Rad(R)W \subseteq X + Rad(W)$ [6]. Thus $IW \underset{S-JW}{\ll} W$.

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