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On Existence and Uniqueness of an Integrable Solution for a Fractional Volterra Integral Equation on R^+

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Abstract
In this paper, by using the Banach fixed point theorem, we prove the existence and uniqueness theorem of a fractional Volterra integral equation in the space of Lebesgue integrable $L_1(R^+)$ on unbounded interval $[0,\infty)$.

Keywords: space of Lebesgue integrable - Banach fixed point theorem - fractional Volterra integral equation - Superposition Operator.

حول وجود ووحدانية حل لمعادلة فولتيرا التكاملية الكسرية في R^+

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الخلاصه
في هذا البحث استخدمت نظرية باناخ للنقطة الثابتة وبرهنا وجود ووحدانية نظرية لحل معادلة فولتيراالتكاملية الكسرية في فضاء ليبيج التكاملي على فترة غير محدودة من الصفر الى مالانهاية.

1. Introduction

Since the last century, time-dependent problems of non-linear differential equations and integral equation have been studied by many authors; see [1-7].

The subject of nonlinear fractional integral equation considered as an important branch of mathematics because it is used for solving in many fields such as physics, engineering and economics [1-4].

In this paper, we will prove the existence and uniqueness theorem of a fractional Volterra integral equation in the space of Lebesgue integrable $L_1(R^+)$ on unbounded interval $[0,\infty)$of the type :

$$x(t) = g(t)\, f(t,x(t)) + h(t) + \int_0^t \frac{e^{-(t-s)}(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s,x(s)\big)ds, \tag{1.1}$$

where $0 < \alpha < 1, \quad t > 0.$

2. Preliminaries

Let R be the field of real number, R^+ be the interval $[0,\infty)$. If A is a Lebesgue measurable subset of R, then the symbol mas(A) stands for the Lebesgue measure of A.
Further, we denote by $L_1(A)$ the space of all real functions, defined and Lebesgue measurable on the set A. The norm of a function $x\varepsilon\, L_1(A)$ is defined in the standard way by the formula,

$$\|x\| = \|L_1(A)\| = \int_A |x(t)|\, dt$$

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Obviously, $L_1(A)$ forms a Banach space under this norm. The space $L_1(A)$ is called the Lebesgue space. In the case when $A = R^+$, we write L_1 instead of $L_1(R^+)$.

One of the most important operators studied in the nonlinear functional analysis is the so- called the superposition operator [8]. Now, let us assume that $A \subset R$ is a given interval bounded.

Definition 2.1 [8]**:** Assume that a function $f(t,x) = f: I \times R \to R$ satisfies the so-called Carathéodory condition, i. e. it is measurable in t for any $x \in R$ and continuous in x for almost all $t \in I$. Then to every function $x = x(t)$ which is measurable on I we may assign the function $(Fx)(t) = f\big(t, x(t)\big)$, $t \in I$. The operator F defined in such a way is said to be the **superposition operator** generated by the function f.

Theorem 2.1 [9]

The superposition operator F generated by a function f maps continuously the space $L^1(I)$ into itself if and only if $|f(t,x)| \le a(t) + b|x|$ for all $t \in I$ and $x \in R$, where $a(t)$ is a function from $L^1(I)$ and b is a nonnegative constant.

This theorem was proved by Krasnoselskii [9] in the case when I is a bounded interval. The generalization to the case of an unbounded interval I was given by Appell and Zabrejko [8].

Definition 2.2 [10] : A function $f : A \to R^m$, $A \subset R^n$ is said to be Lipschitz continuous if there exists a constant L, $L > 0$ (called the Lipschitz constant of f on A) such that

$|f(x) - f(y)| \le L\,|x - y|$, for all $x, y \in A$.

Definition 2.3 [11] Let (X, d) be a metric space and $T : X \to X$ is called contraction mapping, if there exist a number $\gamma < 1$, such that : $d(Tx, Ty) \le \gamma\, d(x,y)$, $\forall\, x, y \in X$.

Theorem 2.2 [12] : Let X be a closed subset of a Banach space E and $T : X \to X$ be a cont-raction, then T has a unique fixed point.

Definition 2.4 [13]: Let $[a,b]$ $(-\infty < a < b < \infty)$ be a finite interval on the real axis R, the Riemann-Liouville fractional integral $I^{\alpha}_{a^+}f$ of the order $\alpha \epsilon C(\mathcal{R}(\alpha) > 0)$ is define by :

$$I^{\alpha}_{a^+}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{f(s)ds}{(t-s)^{1-\alpha}} \qquad (\ t > a\ ;\ \mathcal{R}(\alpha) > 0\ ).$$

Definition 2.5 [14] : Let $[a,b]$ $(-\infty < a < b < \infty)$ be a finite interval on the real axis R, the Riemann-Liouville fractional integral $I^{\alpha}_{a^+}f$ of the order $\alpha \epsilon C(\mathcal{R}(\alpha) \ge 0)$ is define by :

$$D^{\alpha}_{a^+}f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t \frac{f(s)ds}{(t-s)^{1-n+\alpha}} \quad (t > a\,; n = [\mathcal{R}(\alpha)] + 1)$$

where $[\mathcal{R}(\alpha)]$ denotes the integral part of $\mathcal{R}(\alpha)$. " i.e. $[\mathcal{R}(\alpha)]$ it satisfies

$[\mathcal{R}(\alpha)] \le \mathcal{R}(\alpha) \le [\mathcal{R}(\alpha)] + 1.$ "

3. Existence Theorem

Define the operator H associated with integral equation (1.1) which takes the following form:

$$Hx = Ax + Bx. \qquad (3.1)$$

where $(Ax)(t) = g(t)\, f(t, x(t))$,

$$(Bx)(t) = h(t) + \int_0^t \frac{e^{-(t-s)}(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, f\big(s, x(s)\big)ds$$
$$= h(t) + KFx(t),$$

where, $(Kx)(t) = \int_0^t \frac{e^{-(t-s)}(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, x(s)\, ds$,

$Fx = f(t,x)$ are linear operators at superposition, respectively.

We shall treat the equation (3.1) under the following assumptions.

Assume that :

i) g : $R^+ \to R$ is a bounded function such that : $M = \sup_{t \in R^+} |g(t)|$,

and $h : R^+ \to R$, such that $h \in L_1(R^+)$.

ii) $f : R^+ \times R \to R$ satisfies Lipschitz condition with positive constant L such that

$|f(t, x(t)) - f(t, y(t))| \le L|x(t) - y(t)|$, for all $t \in R^+$.

iii) $LM + L < 1$.

Now, for the existence of a unique solution of our equation, we need to prove the following theorem .

Theorem 3.1 : If the assumptions (i)-(iii) are satisfied, then the equation (1.1) has a unique solution, where $x \in L_1(R^+)$.

Proof : Firstly, we will prove that $H : L_1(R^+) \to L_1(R^+)$.

Secondly, will prove that H is a contraction .
Consider the operator H as :

$$Hx(t) = g(t)\, f\big(t, x(t)\big) + h(t) + \int_0^t \frac{e^{-(t-s)}(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, f\big(s, x(s)\big)ds$$

Then our equation (1.1) becomes

$$x(t) = Hx(t).$$

We notice that by assumption (ii), we have

$$\begin{aligned}|f(t,x)| &= |f(t,x) - f(t,0) + f(t,0)| \\ &\le |f(t,x) - f(t,0)| + |f(t,0)| \\ &\le L\,|x - 0| + |f(t,0)| \\ &\le L\,|x| + a(t)\end{aligned}$$

Where $|f(t,0)| = a(t)$
To prove that $H : L_1(R^+) \to L_1(R^+)$.
Let $x \in L_1(R^+)$,
then we have

$$\begin{aligned}\int_0^\infty |Hx(t)|dt &= \int_0^\infty \left|g(t)f(t,x(t)) + h(t) + \int_0^t \frac{e^{-(t-s)}(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s,x(s)\big)ds\right|dt \\ &\le \int_0^\infty |g(t)|\big|f\big(t,x(t)\big)\big|dt \\ &\quad + \int_0^\infty |h(t)|dt + \int_0^\infty \left|\int_0^t \frac{e^{-(t-s)}(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s,x(s)\big)ds\right| dt \\ &\le M\int_0^\infty [a(t) + L|x(t)|]\, dt + \|h\| \\ &\quad + \int_0^\infty \int_{t=s}^\infty \frac{e^{-(t-s)}(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s,x(s)\big)dt\, ds\end{aligned}$$

Let $J = \int_{t=s}^\infty \frac{e^{-(t-s)}(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, dt$
let $X = t - s \to dX = dt$
where $t = s \to X = 0$, $t = \infty \to X = \infty$
then ;

$$J = \int_{t=s}^\infty \frac{e^{-(t-s)}(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, dt = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-X}X^{\alpha-1}dX = \frac{\Gamma(\alpha)}{\Gamma(\alpha)} = 1$$

Then, we get

$$\begin{aligned}\int_0^\infty |Hx(t)|dt &\le M\int_0^\infty [a(t) + L|x(t)|]\, dt + \|h\| + \int_0^\infty \big|f\big(s,x(s)\big)\big|ds \\ &\le M\,\|a\| + LM\int_0^\infty |x(t)|\, dt + \|h\| + \int_0^\infty [a(s) + L|x(s)|]ds \\ &\le M\,\|a\| + LM\|x\| + \|h\| + \|a\| + L\|x\| \\ &\le M\,\|a\| + \|h\| + \|a\| + [\,LM + L\,]\int_0^\infty |x(t)|dt\end{aligned}$$

Then

$$H : L_1(R^+) \to L_1(R^+).$$

Secondly, we prove that H is a contraction.
Let $x, y \in L_1[0,\infty)$, then

$$\begin{aligned}\int_0^\infty |Hx(t) - Hy(t)|dt &= \int_0^\infty \Big|g(t)f(t,x(t)) + h(t) + \int_0^t \frac{e^{-(t-s)}(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s,x(s)\big)ds \\ &\qquad - g(t)f\big(t,y(t)\big) - h(t) - \int_0^t \frac{e^{-(t-s)}(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s,y(s)\big)ds\Big|dt \\ &\le \int_0^\infty |g(t)|\big|f\big(t,x(t)\big) - f\big(t,y(t)\big)\big|dt \\ &\quad + \int_0^\infty \Big| \int_0^t \frac{e^{-(t-s)}(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s,x(s)\big)ds \\ &\quad - \int_0^t \frac{e^{-(t-s)}(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s,y(s)\big)ds\Big|dt \\ &\le M\int_0^\infty L|x(t) - y(t)|dt \\ &\quad + \int_0^\infty \int_0^t \left|\frac{e^{-(t-s)}(t-s)^{\alpha-1}}{\Gamma(\alpha)}\right| \big|f\big(s,x(s)\big) - f\big(s,y(s)\big)\big|\, ds\, dt \\ &\le LM\|x - y\| + \int_0^\infty \big|f\big(s,x(s)\big) - f\big(s,y(s)\big)\big|\, ds \\ &\le LM\|x - y\| + \int_0^\infty L|x(s) - y(s)|\, ds\end{aligned}$$

$$\leq LM\|x-y\| + L\int_0^{\infty}|x(s)-y(s)|\,ds$$
$$\leq LM\|x-y\| + L\|x-y\| \leq [\,LM+L\,]\|x-y\|$$

Hence, by using Banach fixed point theorem,
H has a unique point, which is the solution of the equation (1.1), where $x \epsilon L_1\,[0,\infty)$. ■

Conclusion

In this paper, by using Banach fixed point theorem we proved the existence and uniqueness theorem of a fractional nonlinear Volterra integral equation in the space of Lebesgue integrable $L_1(R^+)$ on unbounded interval $[0,\infty)$ of the type :

$$x(t) = g(t)\,f(t,x(t)) + h(t) + \int_0^t \frac{e^{-(t-s)}(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, f\big(s,x(s)\big)ds,$$

where $0 < \alpha < 1,\ t > 0$. under the following assumptions :

i) $g : R^+ \to R$ is a bounded function such that : $M = \sup_{t\in R^+}|g(t)|$,
and $h : R^+ \to R$, such that $h \in L_1\,(R^+)$.

ii) $f : R^+ \times R \to R$ satisfies Lipschitz condition with positive constant L such that
$|f(t,x(t)) - f(t,y(t))| \leq L|x(t)-y(t)|$, for all $t \in R^+$.

iii) $LM + L < 1$.

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