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The Frequency of t-Practical Numbers

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Abstract

Hausman and Shapiro gave an estimate for the number of practical numbers $n, n \leq x$ to be

$$O\left({}^{x}/_{(\log x)^{\beta}}\right),$$

for every positive $\beta < \frac{1}{2}(\frac{1}{\log 2} - 1)^2$. In this paper, we generalize Hausman and Shapiro bound by proving the number of t-practical numbers $n, (n \leq x), (t \geq 1)$ to be

$$O\left({}^{x}/_{(\log x)^{\beta}}\right),$$

for every positive $\beta < \frac{1}{2}(\frac{1}{\log 2} - 1)^2$, and $1 \leq t \leq exp.((\log x)^{\delta_1})$ for any δ_1 satisfying

$0 < \delta_1 < 1 - (1 + \sqrt{2\beta}) \log 2$. We mean by the t-practical number n, the number in which every integer $1 \leq m \leq tn$ is of the form

$$m = \sum_{d|n} c_d\, d, \quad 0 \leq c_d \leq t$$

Keywords: Bound for the t-practical numbers, Number of t-practical numbers $n \leq x$, Practical numbers.

تكرّر الاعداد العملية ذات التكرار t, $(t \geq 1)$

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الخلاصة:

في هذا البحث تم حساب تكرر الاعداد العملية $n \leq x, n$ ذات التكرار $(t \geq 1), t$ للقيمة الثابتة $0 < \beta < \frac{1}{2}(\frac{1}{\log 2} - 1)^2, \beta$ لكي يكون

$$O\left({}^{x}/_{(\log x)^{\beta}}\right)$$

لكل قيم t بحيث

$$0 < t \leq \exp.\{(\log x)^{\delta_1}\}$$

ولأي δ_1 في الفترة

$$0 < \delta_1 < 1 - (1 + \sqrt{2\beta}) \log 2$$

وهذه النتيجة تغطي كافة الاعداد العملية $n \leq x, n$ بتكرار $(t \geq 1), t$ حيث ان الحالة التي يكون فيها t=1 تمثل التكرر المعطى في نتيجة كل من Hausman, Shapiro المشار اليها في البحث.

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1. Introduction

The t- practical number is a generalization of practical numbers, n, defined in an earlier work [1], where Margenstern conjectured that the number of practical numbers $n, n \leq x$ is denoted by P(x) and

$$P(x) \sim \lambda \frac{x}{\log x}$$

with $\lambda \cong 1.341$. Further results related to P(x) were given by Weingartner [2], who showed that

$$P(x) = \frac{cx}{\log x}\{1 + O(\frac{\log\log x}{\log x})\}$$

for a positive constant c and $x \geq 3$. These results were demonstrated by Wang and Sun [3] to be

$$P(x) \sim \frac{cx}{\log x}$$

as $x \to \infty$.

In this paper, our generalization covered all t-practical numbers and the case when t=1 implies the number of practical numbers given by Hausman and Shapiro, which was established in [4].

2. Preliminary Results and Definitions

Definition (2.1) [1]: Let $n \geq 1$. Then n is called t-practical number if every integer v $m \ni 1 \leq m < n$ is having the form

$$m = \sum_{d|n} c_d d, \quad c_d = 0, 1.$$

Definition (2.2): The integer $n, n \geq 1$ is called t-practical number if every integer $m, 1 \leq m \leq tn, (t \geq 1)$ is written as

$$m = \sum_{d|n} c_d d, \quad (1 \leq c_d \leq t).$$

Definition (2.3) [5]: The function ω (n) is defined to be the number of distinct prime factors of distinct prime factors of n, where $\omega(n) = r$, when $n = p^{a_1}{}_1 \dots p^{a_r}{}_r$

The following Lemmas will be required.

Lemma (2.1) [4]: For ω(n) and any $\varepsilon > 0$ (possibly a function of x), the number of $n \leq x$ such that $\omega(n) > (1+\varepsilon)\log\log x$ is

$$O\left({x}\Big/{(\log x)^{\varepsilon^2/2}}\right)$$

The function O is uniform in ε.

Lemma (2.2): Suppose that $n = n^*.p_1 \dots p_i, (p_i, n^*) = 1$ and p_i are distinct primes, then if for $i = 1, 2, \dots, l, p_i \leq t\sigma(n^*.p_1, \dots p_{i-1}) + 1$ then

$$p_i \leq [t\sigma(n^*) + i]^{2^{i-1}} \qquad \dots (1)$$

($\sigma(n)$ is the sum of divisors of n).

Proof: We have that

$$p_i \leq t\sigma(n^* p_1 \dots p_{i-1}) + 1 \qquad \dots (2)$$

We shall proceed by induction on *i* to show that

$$t\sigma(n^* p_1 \dots p_{i-1}) \leq [t\sigma(n^*) + i - 1]^{2^{i-1}} \qquad \dots (3)$$

for $i = 1$, then (3) is true. As an induction hypothesis, assume that (3) is true for $i - 1$. Then we write

$$t\sigma(n^* p_1 \dots p_i) = t\sigma(n^* p_1 \dots p_{i-1})(p_i + 1)$$
$$\leq [t\sigma(n^* p_1 \dots p_{i-1})][t\sigma(n^* p_1 \dots p_{i-1}) + 2]$$

and by the induction hypothesis, we have

$$t\sigma(n^* p_1 \dots p_i) \leq [t\sigma(n^*) + i - 1]^{2^{i-1}}[(t\sigma(n^*) + i - 1)^{2^{i-1}} + 2]$$
$$t\sigma(n^* p_1 \dots p_i) \leq [t\sigma(n^*) + i - 1]^{2^i} + 2[t\sigma(n^*) + i - 1]^{2^{i-1}} + 1$$
$$t\sigma(n^* p_1 \dots p_i) \leq (t\sigma(n^*) + i)^{2^i}$$

Therefore, inequality (3) is true for *i*. Using (2) and (3), we get

$$p_i \leq t\sigma(n^* p_1 \dots p_{i-1}) + 1 \leq [t\sigma(n^*) + i - 1]^{2^{i-1}} + 1$$

Then,

$$p_i \leq [t\sigma(n^*) + i]^{2^{i-1}}$$

This is the required solution.

3. The Frequency of t-Practical Numbers

The following results were proved by Hausman and Shapiro [4].

Theorem (3.1) [4]: the number of practical numbers n less than or equal to x is

$$O\left({x}/{(\log x)^{\beta}} \right)$$

for every fixed positive $\beta < \frac{1}{2}(\frac{1}{\log 2} - 1)^2$.

Lemma (3.1) [4]: For ω(n) equal to the number of distinct prime factors of n and any given $\varepsilon > 0$ (possibly function of x), the number of $n \leq x$, such that $\omega(n) > (1+\varepsilon)\log\log x$, is

$$O\left({x}/{(\log x)^{\varepsilon^2/2}} \right)$$

where the O uniform in ε.

A generalization of theorem (3.1) [4] is the substance of the following main theorem.

Theorem (3.2): Let $t \geq 1$ and $0 < \beta < \frac{1}{2}(\frac{1}{\log 2} - 1)^2$. Then the number of t-practical numbers $n \leq x$ is $O\left({x}/{(\log x)^{\beta}} \right)$

This estimate is a uniform for t in the range

$$1 \leq t \leq exp.((\log x)^{\delta_1})$$

for any δ_1 with $0 < \delta_1 < 1 - (1+\sqrt{2\beta})\log 2$.

Proof: Let $0 < \varepsilon < (\frac{1}{\log 2} - 1)$. Then form Lemma (3.1) [4], the number of t-practical numbers $n \leq x \ni \omega(n) > (1+\varepsilon)\log\log x$ is

$$O\left({x}/{(\log x)^{\varepsilon^2/2}} \right)$$

Now, we consider the t-practical numbers $n \leq x$ with $\omega(n) \leq (1+\varepsilon)\log\log x$ and n to have the form $q_1 \dots q_k . p_1 \dots p_l$,

where q_j, p_i are distinct primes $p_i \neq q_j$, for $(1 \leq i \leq l), (1 \leq j \leq k)$

and

$$q_j \leq t+1 < p_1 < \dots < p_l \quad \dots (1)$$

Since $\omega(n) = l + k$, then

$(l+k) \leq (1+\varepsilon)\log\log x$.

By setting $n^* = q_1 \dots q_k$, then from (1) we have

$$n^* \leq (t+1)^{(1+\varepsilon)\log\log x} \quad \dots (2)$$

or

$$n^* \leq exp.((2\log\log x).\log(t+1)) \quad \dots (3)$$

Therefore the t-practical numbers are of the form

$$n^* p_1 \dots p_l \quad \dots (4)$$

with $l < (1+\varepsilon)\log\log x$ and p_i larger than primes dividing n^*. From Lemma (2.2), we have

$$p_i \leq t\sigma(n^*.p_1 \dots p_{i-1}) + 1 \quad \dots (5)$$

and

$$p_i \leq [t\sigma(n^*) + i]^{2^{i-1}} \quad \dots (6)$$

From [5], we have

$\sigma(n^*) = O(n^* \log\log n)$,

Then from (3), we get

$$\sigma(n^*) = O(n^* \log\log n) \leq exp.(c_1 \log\log x).\log(t+1) \quad \dots (7)$$

where $c_1 > 0$ is a constant. Therefore (6) and (7) imply that

$$p_i \leq [t.exp.(c_1(\log\log x).\log(t+1) + i]^{2^{i-1}} \quad \dots (8)$$

Since $i \leq l(1+\varepsilon)\log\log x$, then (8) becomes

$$p_i \leq [t.exp.(c_2 \log\log x).\log(t+1)]^{2^{i-1}} \quad \dots (9)$$

and $c_2 > 0$ is a constant. From (3) and (9), the number of t-practical numbers having the form (4) is at most .

$$[exp.\{(2\log\log x).\log(t+1\}].[t.exp.\{c_2\log\log x).\log(t+1)\}]^{1+2+\cdots+2^{l-1}}$$
$$[exp.\{(2\log\log x).\log(t+1\}].[t.exp.\{c_2\log\log x).\log(t+1)\}]^{2^{l-1}}$$
$$\leq [exp.(c_3\log\log x).\log(t+1)^{2^{(1+\varepsilon)\log\log x}+1}.t^{2^{(1+\varepsilon)\log\log x}} \quad \ldots (10)$$

where $c_3 > 0$ is a constant. Since,

$$2^{(1+\varepsilon)\log\log x} = e^{\log(2^{\log\log x})^{1+\varepsilon}} = (e^{\log 2^{\log\log x}})^{1+\varepsilon}$$
$$= e^{(\log\log x.\log 2)1+\varepsilon} = (e^{\log\log x})^{(1+\varepsilon)\log 2}$$

therefore

$$2^{(1+\varepsilon)\log\log x} = (\log x)^{(1+\varepsilon)\log 2} \quad \ldots (11)$$

and the number of t-practical numbers given in (10) is at most

$$[exp.\{(c_3\log\log x).\log(t+1)\}]^{1+\log x^{(1+\varepsilon)\log 2}}.t^{(\log x)^{(1+\varepsilon)\log 2}-1} \quad \ldots (12)$$

where

$$t^{(\log x)^{(1+\varepsilon)\log 2}-1} = e^{\{(\log x)^{(1+\varepsilon)\log x}-1\}.\log t}$$

i.e.

$$t^{(\log x)^{(1+\varepsilon)\log 2}-1} = exp.\{(\log t).(\log x)^{(1+\varepsilon)\log 2} - \log t\} \quad \ldots (13)$$

where for any $\varepsilon > 0$ with $(1+\varepsilon)\log 2 < 1$, we have for sufficiently large x

$$1 + (\log x)^{(1+\varepsilon)\log 2} \leq 2(\log x)^{(1+\varepsilon)\log 2} \quad \ldots (14)$$

Hence, from (13) and (14), the number of t- practical numbers $n \leq x$ given in (12) becomes at most

$$exp.\{(2c_3\log\log x).(t+1)^{(\log x)^{(1+\varepsilon)\log 2}}\}.\exp\{(\log t)(\log x)^{(1+\varepsilon)\log 2} - \log t\}$$

or

$$\exp\{(c_4\log\log x).\log(t+1)\,(\log x)^{(1+\varepsilon)\log 2}\}.\exp\{(\log t).(\log x)^{(1+\varepsilon)\log 2} - \log t\} \quad \ldots (15)$$

where $c_4 > 0$ is a constant and $c_4\log\log x > 1$. Therefore

$$(c_4\log\log x).\log(t+1) > \log t$$

Thus (15) becomes at most

$$exp.\{(c_5\log\log x).(\log t)\,(\log x)^{(1+\varepsilon)\log 2} - \log t\}$$

and

$$exp.[(\log t)\left\{(c_5\log\log x).(\log x)^{(1+\varepsilon)\log 2} - 1\right\}]$$
$$\leq exp.\{(c_5\log\log x).(\log t).(\log x)^{(1+\varepsilon)\log 2}\} \quad \ldots (16)$$

where $c_5 > 0$ is a constant. Since $(1+\varepsilon)\log 2 < 1$, then, for large x,

$$(\log\log x).(\log x)^{(1+\varepsilon)\log 2} \leq (\log x)^{1-\delta}$$

with $\delta > 0$ is a constant chosen such that $0 < \delta < 1 - (1+\varepsilon)\log 2$.

Thus, the number of t-practical numbers $n \leq x$ given in (16) is at most

$$O[exp.\left\{\frac{c_5(\log t).\log x}{(\log x)^{\delta}}\right\}] \quad \ldots (17)$$

for t in the range of $1 \leq t \leq exp.\{(\log x)^{\delta_1}\}$, where $0 < \delta_1 < \delta$ or (17) is written as

$$O[exp.\left\{\frac{(\log x)}{(\log x)^{\eta}}\right\}]$$

and $\eta = \delta - \delta_1$ Since $\eta > 0$, then

$$O\left(x\Big/(\log x)^{\varepsilon^2/2}\right)$$

which is the bound for the t-practical numbers $n \leq x$ with $0 < \varepsilon < (\frac{1}{\log 2} - 1)$. By putting $\beta = \frac{\varepsilon^2}{2}$, then the number of t- practical numbers $n \leq x$ is

$$O\left(x/(\log x)^{\beta}\right)$$

Provided that $0 < \beta < \frac{1}{2}(\frac{1}{\log 2} - 1)^2$.

This ends the proof and Theorem (3.2) will cover a wide range of t-practical numbers n under the same bound given by Hausman and Shapiro [4], where Theorem (3.2) implies Theorem (3.1) [4] when $t = 1$.

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