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Solving Fuzzy Differential Equations by Using Power Series

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Abstract

In this paper, the series solution is applied to solve third order fuzzy differential equations with a fuzzy initial value. The proposed method applies Taylor expansion in solving the system and the approximate solution of the problem which is calculated in the form of a rapid convergent series; some definitions and theorems are reviewed as a basis in solving fuzzy differential equations. An example is applied to illustrate the proposed technical accuracy. Also, a comparison between the obtained results is made, in addition to the application of the crisp solution, when the α-level equals one.

Keywords: Taylor expansion; third order Fuzzy differential equations; Residual power series method.

حل المعادلات التفاضلية الضبابية باستخدام متسلسلة القوى

رشا حسين ابراهيم

قسم الرياضيات، كلية التربية الاساسية، الجامعه المستنصرية، بغداد، العراق

الخلاصه:

في هذا البحث ، يتم تطبيق حل المتسلسلة لحل المعادلات التفاضلية الغامضة من الدرجة الثالثة مع القيمة اولية غامضة. الطريقة المقترحة تحصل على توسع تايلور في حل النظام والحل التقريبي للمشكلة يتم حسابها في شكل سلسلة متقاربة سريعة ؛ تتم مراجعة بعض التعاريف والنظريات كأساس في حل المعادلات التفاضلية الغامضة. تم تطبيق مثال لتوضيح الدقة للتقنية المقترحة. أيضا ، يتم إجراء مقارنة بين النتائج التي تم الحصول عليها ، وكذلك مع الحل الواضح ، عندما يكون مستوى المستوى القطع يعادل واحد.

1-Introduction

Fuzzy differential equations (FDEs) have started to grow rapidly. The concept of FDEs was first introduced by Chang and Zadeh [1]. Later, Dubois and Prade [2] expanded the principle approach in solving FDEs. Kaleva [3] and Seikkala [4] managed to solve FDEs with the fuzzy initial value problems (FIVPs), which appear when the modeling of these problems is imperfect and its nature is under uncertainty. Hence, studying and finding solution of FIVPs are extremely necessary for different applications, particularly when they involve uncertain parameters or uncertain initial conditions. The basic concepts in fuzzy set theory will play a major role in solving fuzzy differential equations [5].
In many cases, it is difficult or impossible to find the exact solutions of differential equations. Alternatively, we can find the approximate solutions using some techniques such as finite difference methods or finite element methods, see for instance [6].

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In this paper, the RPS method was employed to solve third-order fuzzy differential equations. The approximate solution is represented in the form of power series. Moreover, the approximate solution and all its derivatives (if they exist) converge to the exact solution and all its derivatives, respectively. The suggested algorithm created a swiftly convergent series with an easily computable components using a symbolical calculation program. Series expansions are used in numerical calculations, especially for quick estimates that are made manually. Often, we express FDEs in terms of serial expansions. However, the RPS theory is an analytical method for solving different types of ordinary and partial differential equations [7]. The classical higher order, i.e. Taylor series method, is computationally expensive for large orders and proper for the linear problems. On the other hand, the suggested method is an alternate procedure for earning analytic Taylor series solutions of systems of FIVPs.

The purpose of this paper is to develop the implementation of the residual power series method for earning an analytical solution for the first-order fuzzy DE of the following form [3]:

$$\begin{aligned} y'(x) &= f\big(x, y(x)\big) \\ y(x_0) &= y^0 \end{aligned} \tag{1.1}$$

and the third-order differential equations in the following form:

$$y'''(x) + g\big(x, y(x), y'(x), y''(x)\big) = h\big(x, y(x), y'(x), y''(x)\big) \tag{1.2}$$
$$y(x_0) = y^0 \quad , y'(x_0) = y^1 \ , \ , y''(x_0) = y^2$$

where $-\infty < x_0 \le x \le x_0 + a < \infty$, $f: [x_0, x_0 + a] \times R_F \to R_F$ and g, $h : [x_0, x_0 + a] \times R_F \to R_F$ are fuzzy-number-valued functions, $y(x)$ is an unknown function of variable (x) to be specified, y^0, y^1, y^2 are fuzzy numbers, and(x_0) , (a) are real constants with a>0.

The structure of the paper includes the following: In section 2, we provide some important definitions and basic results to be used in this paper. In section 3, we present the theory of fuzzy DEs. In section 4, the main idea of the Residual power series method is introduced. In section 5, we illustrate the proposed method by
a numerical example.

2. Preliminaries

In this section, we present basic concepts for fuzzy calculus and concept of fuzzy derivative; we will adopt strongly generalized differentiability.

Definition 2.1. [8]. A fuzzy number y is a fuzzy set: R → [0, 1] which satisfies the following requirements:

(i) y is upper semicontinuous function,
(ii) $y(x) = 0$ outside some interval [c, d],
(iii) There are real numbers a, b such that $c \le a \le b \le d$ for which
(a) $y(x)$ is monotonic increasing on [c, a],
(b) $y(x)$ is monotonic decreasing on [b, d],
(c) $y(x) = 1$ on [a, b].

We will let $R_F$denotes the set of fuzzy numbers on R. Obviously, $R \subset R_F$, where R is understood as $R \cong R\chi = \{\chi_{\{x\}} : x \in R\} \subset R_F$. The α -level represents a fuzzy number y , denoted by $[y]^\alpha$, is defined as:

$$[y]^\alpha = \begin{cases} \{s \in R : y(s) \ge \alpha\}, & \alpha \in (0,1], \\ \{s \in R : y(s) > \alpha\}, & \alpha = 0. \end{cases}$$

It is clear that the α -Level representation of a fuzzy number y is a compact convex subset of R. Thus, if y is a fuzzy number, then $[y]^\alpha = [\underline{y}(\alpha), \overline{y}(\alpha)]$, where $\underline{y}(\alpha)$= min $\{s : s \in [y]^\alpha\}$ and $\overline{y}(\alpha)$= max$\{s : s \in [y]^\alpha\}$ $\forall\ \alpha \in [0, 1]$. Sometimes, we will write $\underline{y}_\alpha$ and $\overline{y}_\alpha$ as replacements of $\underline{y}(\alpha)$ and $\overline{y}(\alpha)$, respectively, and $\forall\ \alpha \in [0,1]$.

Theorem 2.1. [9]. Suppose that $\underline{y}$: [0, 1] → R and $\overline{y}$: [0, 1] → R, which satisfies the following conditions:

(i) $\underline{y}$ is a bounded increasing left continuous function on (0, 1],
(ii) $\overline{y}$ is a bounded decreasing left continuous function on (0, 1],
(iii) $\underline{y}$ and $\overline{y}$ are right continuous functions at $\alpha = 0$,
(iv) $\underline{y}(\alpha) \le \overline{y}(\alpha)$ on [0, 1], then y : R → [0, 1], defined by

$$y(s) = \sup\{\alpha: \underline{y}(\alpha) \le s \le \overline{y}(\alpha)\}, \quad (2.1)$$

is a fuzzy number given by $[\underline{y}(\alpha), \overline{y}(\alpha)]$. Moreover, if $y : R \to [0, 1]$ is a fuzzy number given by $[\underline{y}(\alpha), \overline{y}(\alpha)]$, then the functions $\underline{y}(\alpha)$ and $\overline{y}(\alpha)$ satisfy the conditions (i-iv).

Definition 2.2. [10] Let $y = [x_0, x_0 + a] \to R_F$ and $x^* \in [x_0, x_0 + a]$. We say that x is strongly generalized differentiable at x^* , if there exists an element $y'(x^*) \in R_F$ such that:

(i) $\forall$ h > 0 sufficiently close to 0, the H-differences $y(x^* + h) \ominus y(x^*)$, $y(x^*) \ominus y(x^* - h)$ exist and $\lim_{h\to 0+} \frac{y(x^*+h) \ominus y(x^*)}{h} = \lim_{h\to 0+} \frac{y(x^*) \ominus y(x^*-h)}{h} = y'(x^*)$

(ii) $\forall$ h > 0 sufficiently close to 0, the H-differences $y(x^*) \ominus y(x^* + h)$, $y(x^* - h) \ominus y(x^*)$ exist and $\lim_{h\to 0+} \frac{y(x^*) \ominus y(x^*+h)}{h} = \lim_{h\to 0+} \frac{y(x^*-h) \ominus y(x^*)}{h} = y'(x^*)$.

Definition 2.3. [11] Let $y = [x_0, x_0 + a] \to R_F$. If y is differentiable in the concept (i) of Definition 2.2, then we say that y is (1)-differentiable on $[x_0, x_0 + a]$ and its derivative is denoted by $D_1^{(1)} y$, and we have $D_2^{(1)} y$ for (2)-differentiability **.**

Definition 2.4. [12] Let $y = [x_0, x_0 + a] \to R_F$ and m = 1, 2. if $D_n^{(1)} y$ exists and it is (m)-differentiable. We say that y is (n, m) - differentiable on $[x_0, x_0 + a]$. The second derivatives of y are denoted by $D_{n,m}^{(2)} y$

For n, m = 1, 2.

Now, we present the definition for the third-order derivatives founded on the selection of derivative type in each differentiation step. For a given fuzzy function $y = [x_0, x_0 + a] \to R_F$, we have two possibilities to get the derivatives of y: $D_1^{(1)} y(x)$ and $D_2^{(1)} y(x)$. And , in each of these possibilities, there are four derivatives.

Definition 2.5. Let $y = [x_0, x_0 + a] \to R_F$ and m = 1, 2. If $D_n^{(1)} y$ exists and it is (m)-differentiable, then we say that y is (n, m) - differentiable on $[x_0, x_0 + a]$. The third derivatives of y are denoted by $D_{n,m}^{(3)} y$ for n, m = 1, 2.

The principle of the derivative properties is known and can be found in previous articles [11,12, 13]. In this paper, we extend the theorem proved in two of those articles [11,12]

Theorem 2.2. Let y, $D_i^{(1)} y: [x_0 , x_0 + a] \to R_F$, i = 1, 2 where[$\underline{y}_\alpha(x), \overline{y}_\alpha(x)]$,$\forall\ \alpha \in [0, 1]$:

(i) if y is (1)-differentiable," then $\underline{y}_\alpha , \overline{y}_\alpha$ are differentiable functions" and $\left[D_1^{(1)} y(x)\right]^\alpha = [\ \underline{y'}_\alpha(x), \overline{y'}_\alpha(x)]$

(ii) if y is (2)-differentiable," then $\underline{y}_\alpha , \overline{y}_\alpha$ are differentiable functions" and

$$\left[D_2^{(1)} y(x)\right]^\alpha = [\ \overline{y'}_\alpha(x),\ \underline{y'}_\alpha(x)]$$

(iii) if $D_1^{(1)} y$ is (1)- differentiable, "then $\underline{y'}_\alpha , \overline{y'}_\alpha$ are differentiable functions" and

$$\left[D_{1,1}^{(2)} y(x)\right]^\alpha = [\ \underline{y''}_\alpha(x), \overline{y''}_\alpha(x)]$$

(iv) if $D_1^{(1)} y$ is (2)- differentiable, "then $\underline{y'}_\alpha , \overline{y'}_\alpha$ are differentiable functions" and

$$\left[D_{1,2}^{(2)} y(x)\right]^\alpha = [\ \overline{y''}_\alpha(x), \underline{y''}_\alpha(x)]$$

(v) if $D_1^{(1)} y$ is (1)- differentiable, "then $\underline{y'}_\alpha , \overline{y'}_\alpha$ are differentiable functions " and

$$\left[D_{1,1}^{(3)} y(x)\right]^\alpha = [\ \underline{y'''}_\alpha(x), \overline{y'''}_\alpha(x)]$$

(vi) if $D_1^{(1)} y$ is (2)- differentiable, "then $\underline{y'}_\alpha , \overline{y'}_\alpha$ are differentiable functions " and

$$\left[D_{1,2}^{(3)} y(x)\right]^\alpha = [\ \overline{y'''}_\alpha(x), \underline{y'''}_\alpha(x)]$$

(vii) if $D_2^{(1)} y$ is (1)-differentiable, "then $\underline{y'}_\alpha , \overline{y'}_\alpha$ are differentiable functions" and

$$\left[D_{2,1}^{(2)} y(x)\right]^\alpha = [\ \overline{y''}_\alpha(x) , \underline{y''}_\alpha(x)]$$

(viii) if $D_2^{(1)}y$ is (2)-differentiable, "then $\underline{y'}_\alpha, \overline{y'}_\alpha$ are differentiable functions" and

$$\left[D_{2,2}^{(2)}\, y(x)\right]^\alpha = [\ \underline{y''}_\alpha(x), \overline{y''}_\alpha(x)]$$

(ix) if $D_2^{(1)}y$ is (1)-differentiable, "then $\underline{y'}_\alpha, \overline{y'}_\alpha$ are differentiable functions" and

$$\left[D_{2,1}^{(3)}\, y(x)\right]^\alpha = [\,\overline{y'''}_\alpha(x)\underline{y'''}_\alpha(x)]$$

(x) if $D_2^{(1)}y$ is (2)-differentiable, " then $\underline{y'}_\alpha, \overline{y'}_\alpha$ are differentiable functions" and

$$\left[D_{2,2}^{(3)}\, y(x)\right]^\alpha = [\,\underline{y'''}_\alpha(x), \overline{y'''}_\alpha(x)]$$

3- Theory of third-order fuzzy differential equations

In this section, we study the theory of third- order fuzzy differential equations under strongly generalized differentiability. Furthermore, we present an algorithm to solve these types of problems, which consists of eight classical ODEs systems for fuzzy DE (1.2). The fuzzy solution of DE (1.1) depends on the choice of the derivative type in the fuzzy setting. To solve the DE (1.2), we give the definition below.

Definition 3.1 [11] Let $y = [x_0, x_0 + a] \to R_F$ and $n \in \{1, 2\}$.), if $D_n^{(1)}y$ exists and y and $D_n^{(1)}y$ satisfy fuzzy DE (1.1), we say that y is a (n)-solution of fuzzy DE (1.1**).**

The following definition of $f(x, y(x))$ is a conclusion to the extension principle of Zadeh when $y(x)$ is a fuzzy number [4]: $f\big(x, y(x)\big)(s) = \sup\{y(x)(\tau): s = f(x, \tau), s \in R\}$. Thus, according to the theory of Nguyen [14, 15], it follows that:

$$[f(x, y(x)]^\alpha = f(x, [y(x)]^\alpha) = \left[\underline{f}_\alpha\big(x, y(x)\big), \bar{f}_\alpha\big(x, y(x)\big)\right]$$

where the two expression endpoint functions $\underline{f}_\alpha$ and $\bar{f}_\alpha$ are defined, respectively, as:

$$\underline{f}_\alpha(x, y(x)) = \min\{f(x, [y(x)]^\alpha\} = f_{1,\alpha}\left(x, \underline{y}_\alpha(x), \overline{y}_\alpha(x)\right),$$

$$\overline{f}_\alpha(x, y(x)) = \max\{f(x, [y(x)]^\alpha\} = f_{2,\alpha}\left(x, \underline{y}_\alpha(x), \overline{y}_\alpha(x)\right).$$

Similarly, taking into account the type of differentiability, we can write

$$\left[g(x, y(x), y'(x), y''(x))\right]^\alpha$$

$$= \left[g_{1,\alpha}\left(\underline{y}_\alpha(x), \overline{y}_\alpha(x), \underline{y'}_\alpha(x), \overline{y'}_\alpha(x), \underline{y''}_\alpha(x), \overline{y''}_\alpha(x)\right), g_{2,\alpha}\left(\underline{y}_\alpha(x), \overline{y}_\alpha(x), \underline{y'}_\alpha(x), \overline{y'}_\alpha(x), \underline{y''}_\alpha(x), \overline{y''}_\alpha(x)\right)\right]$$

$$\left[h(x, y(x), y'(x), y''(x))\right]^\alpha$$

$$= \left[h_{1,\alpha}\left(\underline{y}_\alpha(x), \overline{y}_\alpha(x), \underline{y'}_\alpha(x), \overline{y'}_\alpha(x), \underline{y''}_\alpha(x), \overline{y''}_\alpha(x)\right), h_{2,\alpha}\left(\underline{y}_\alpha(x), \overline{y}_\alpha(x), \underline{y'}_\alpha(x), \overline{y'}_\alpha(x), \underline{y''}_\alpha(x), \overline{y''}_\alpha(x)\right)\right]$$

The objective of the next algorithm is to implement a procedure to solve the fuzzy DE (1.1) in a parametric form, in terms of its α-levels representation.

Algorithm 3.1[16]: To find the solutions of fuzzy DE (1.1), we discuss the following two cases:

Case I. if y(x) is (1)-differentiable, then $[y'(x)]^\alpha = [\ \underline{y'}_\alpha(x), \overline{y'}_\alpha(x)]$ and solving fuzzy DE (1.1) translates into the following subroutine:

(i) Solve the following system of ODEs for $\underline{y}_\alpha(x)$, $\overline{y}_\alpha(x)$:

$$\begin{aligned} &\underline{y'}_\alpha(x) = f_{1,\alpha}\left(x, \underline{y}_\alpha(x), \overline{y}_\alpha(x)\right) \\ &,\overline{y'}_\alpha(x) = f_{2,\alpha}\left(x, \underline{y}_\alpha(x), \overline{y}_\alpha(x)\right) \end{aligned} \tag{3.1}$$

subject to the initial conditions

$$\underline{y}_\alpha(x_0) = \underline{y}_\alpha^0 \quad , \ \overline{y}_\alpha(x_0) = \overline{y}_\alpha^0 \tag{3.2}$$

(ii) Ensure that the solutions $[\underline{y}_\alpha(x),\ \overline{y}_\alpha(x)]$ and $[\underline{y}'_\alpha(x), \overline{y}'_\alpha(x)$ are valid level sets for each $\alpha \in [0, 1]$. Further, use the identity (2.1) to construct a (1)-solution $y(x)$.

Case II. If $y(x)$ is (2)-differentiable, then $[y'(x)]^\alpha = [\ \overline{y}'_\alpha(x), \underline{y}'_\alpha(x)]$ and solving fuzzy DE (1.1) translates into the following subroutine:

(i) Solve the following system of ODEs for $\underline{y}_\alpha(x),\ \overline{y}_\alpha(x)$:

$$\underline{y}'_\alpha(x) = f_{2,\alpha}\left(x, \underline{y}_\alpha(x), \overline{y}_\alpha(x)\right)$$
$$\overline{y}'_\alpha(x) = f_{1,\alpha}\left(x, \underline{y}_\alpha(x), \overline{y}_\alpha(x)\right)$$

subject to the initial conditions

$$\underline{y}_\alpha(x_0) = \underline{y}^0_\alpha \quad , \ \overline{y}_\alpha(x_0) = \overline{y}^0_\alpha$$

(ii) Ensure that the solutions $[\underline{y}_\alpha(x),\ \overline{y}_\alpha(x)]$ and $\left[\overline{y}'_\alpha(x), \underline{y}'_\alpha(x)\right]$ are valid level sets for each $\alpha \in [0, 1]$. Further, use the identity (2.1) to construct a (2)-solution $y(x)$.

Next, we study the properties of solutions of fuzzy DE (1.2) with respect to different types of differentiability in order to solve such fuzzy problems.

Definition 3.2. Let $y = [x_0, x_0 + a] \to R_F$ and $n, m \in \{1,2\}$. We say that y is a (n, m) - solution of fuzzy Differential Equations (1.1), if $D^{(1)}_n y(x)$ and $D^{(2)}_{n,m} y(x), D^{(3)}_{n,m} y(x)$ exist and $x, D^{(1)}_n y(x)$ and $D^{(2)}_{n,m} y(x), D^{(3)}_{n,m} y(x)$ satisfy fuzzy DE (1.2).

Let y be an (n, m) -solution for fuzzy DE (1.2). To find it, we apply Theorem 2.2 and, considering the initial values, we can transform fuzzy DE (1.2) to a system of third-order ODEs. Therefore, the possible ODEs systems for this type of fuzzy problems are eight, as follows:

Algorithm 3.2:

To find the solutions of the fuzzy differential equation (1.2) in term of its α-level representation, we consider the following cases:

Case I. For (1, 1)-differentiable, consider the differentiability of y and $D^{(1)}_1 y$ in the sense (i) , (iii) and (v) of Theorem 2.2. Then we get the following (1, 1)-system of ODEs:

$$\underline{y}'''_\alpha(x) + g_{1,\alpha}\left(\underline{y}_\alpha(x), \overline{y}_\alpha(x), \underline{y}'_\alpha(x), \overline{y}'_\alpha(x), \underline{y}''_\alpha(x), \overline{y}''_\alpha(x)\right) = h_{1,\alpha}\left(\underline{y}_\alpha(x), \overline{y}_\alpha(x), \underline{y}'_\alpha(x), \overline{y}'_\alpha(x), \underline{y}''_\alpha(x), \overline{y}''_\alpha(x)\right)$$
$$\overline{y}'''_\alpha(x) + g_{2,\alpha}\left(\underline{y}_\alpha(x), \overline{y}_\alpha(x), \underline{y}'_\alpha(x), \overline{y}'_\alpha(x), \underline{y}''_\alpha(x), \overline{y}''_\alpha(x)\right) = h_{2,\alpha}\left(\underline{y}_\alpha(x), \overline{y}_\alpha(x), \underline{y}'_\alpha(x), \overline{y}'_\alpha(x), \underline{y}''_\alpha(x), \overline{y}''_\alpha(x)\right)$$

subject to the initial conditions

$$\underline{y}_\alpha(x_0) = \underline{y}^0_\alpha \quad , \ \overline{y}_\alpha(x_0) = \overline{y}^0_\alpha$$
$$\underline{y}'_\alpha(x_0) = \underline{y}^1_\alpha \ , \ \overline{y}'_\alpha(x_0) = \overline{y}^1_\alpha$$
$$, \underline{y}''_\alpha(x_0) = \underline{y}^2_\alpha , \quad \overline{y}''_\alpha(x_0) = \overline{y}^2_\alpha$$

Case II. For (1, 2)-differentiable, consider the differentiability of y and $D^{(1)}_1 y$ in the sense (i) , (iv) and (vi) of Theorem 2.2. Then we get the following (1, 2)-system of ODEs:

$$\overline{y}'''_\alpha(x) + g_{1,\alpha}\left(\underline{y}_\alpha(x), \overline{y}_\alpha(x), \underline{y}'_\alpha(x), \overline{y}'_\alpha(x), \underline{y}''_\alpha(x), \overline{y}''_\alpha(x)\right) = h_{1,\alpha}\left(\underline{y}_\alpha(x), \overline{y}_\alpha(x), \underline{y}'_\alpha(x), \overline{y}'_\alpha(x), \underline{y}''_\alpha(x), \overline{y}''_\alpha(x)\right)$$
$$\underline{y}'''_\alpha(x) + g_{2,\alpha}\left(\underline{y}_\alpha(x), \overline{y}_\alpha(x), \underline{y}'_\alpha(x), \overline{y}'_\alpha(x), \underline{y}''_\alpha(x), \overline{y}''_\alpha(x)\right) = h_{2,\alpha}\left(\underline{y}_\alpha(x), \overline{y}_\alpha(x), \underline{y}'_\alpha(x), \overline{y}'_\alpha(x), \underline{y}''_\alpha(x), \overline{y}''_\alpha(x)\right)$$

subject to the initial conditions

$$\underline{y}_\alpha(x_0) = \underline{y}^0_\alpha \quad , \ \overline{y}_\alpha(x_0) = \overline{y}^0_\alpha$$
$$\underline{y}'_\alpha(x_0) = \underline{y}^1_\alpha \ , \ \overline{y}'_\alpha(x_0) = \overline{y}^1_\alpha$$
$$, \overline{y}''_\alpha(x_0) = \underline{y}^2_\alpha , \quad \underline{y}''_\alpha(x_0) = \overline{y}^2_\alpha$$

Case III. For (1, 1)-differentiable, consider the differentiability of y and $D_1^{(1)}y$ in the sense (i), (iii) and (vi) of Theorem 2.2. Then we get the following (1, 1)-system of ODEs:

$$\overline{y}'''_{\alpha}(x) + g_{1,\alpha}\left(\underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x), \underline{y}'_{\alpha}(x), \overline{y}'_{\alpha}(x), \underline{y}''_{\alpha}(x), \overline{y}''_{\alpha}(x)\right) = h_{1,\alpha}\left(\underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x), \underline{y}'_{\alpha}(x), \overline{y}'_{\alpha}(x), \underline{y}''_{\alpha}(x), \overline{y}''_{\alpha}(x)\right)$$

$$\underline{y}'''_{\alpha}(x) + g_{2,\alpha}\left(\underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x), \underline{y}'_{\alpha}(x), \overline{y}'_{\alpha}(x), \underline{y}''_{\alpha}(x), \overline{y}''_{\alpha}(x)\right) = h_{2,\alpha}\left(\underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x), \underline{y}'_{\alpha}(x), \overline{y}'_{\alpha}(x), \underline{y}''_{\alpha}(x), \overline{y}''_{\alpha}(x)\right)$$

subject to the initial conditions

$$\underline{y}_{\alpha}(x_0) = \underline{y}^0_{\alpha} \quad , \ \overline{y}_{\alpha}(x_0) = \overline{y}^0_{\alpha}$$
$$\underline{y}'_{\alpha}(x_0) = \underline{y}^1_{\alpha} \ , \ \overline{y}'_{\alpha}(x_0) = \overline{y}^1_{\alpha}$$
$$, \underline{y}''_{\alpha}(x_0) = \underline{y}^2_{\alpha}, \quad \overline{y}''_{\alpha}(x_0) = \overline{y}^2_{\alpha}$$

Case IV. For (1, 2)-differentiable, consider the differentiability of y and $D_1^{(1)}y$ in the sense (i), (iv) and (v) of Theorem 2.2. Then we get the following (1, 2)-system of ODEs:

$$\underline{y}'''_{\alpha}(x) + g_{1,\alpha}\left(\underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x), \underline{y}'_{\alpha}(x), \overline{y}'_{\alpha}(x), \underline{y}''_{\alpha}(x), \overline{y}''_{\alpha}(x)\right) = h_{1,\alpha}\left(\underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x), \underline{y}'_{\alpha}(x), \overline{y}'_{\alpha}(x), \underline{y}''_{\alpha}(x), \overline{y}''_{\alpha}(x)\right)$$

$$\overline{y}'''_{\alpha}(x) + g_{2,\alpha}\left(\underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x), \underline{y}'_{\alpha}(x), \overline{y}'_{\alpha}(x), \underline{y}''_{\alpha}(x), \overline{y}''_{\alpha}(x)\right) = h_{2,\alpha}\left(\underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x), \underline{y}'_{\alpha}(x), \overline{y}'_{\alpha}(x), \underline{y}''_{\alpha}(x), \overline{y}''_{\alpha}(x)\right)$$

subject to the initial conditions

$$\underline{y}_{\alpha}(x_0) = \underline{y}^0_{\alpha} \quad , \ \overline{y}_{\alpha}(x_0) = \overline{y}^0_{\alpha}$$
$$\underline{y}'_{\alpha}(x_0) = \underline{y}^1_{\alpha} \ , \ \overline{y}'_{\alpha}(x_0) = \overline{y}^1_{\alpha}$$
$$\overline{y}''_{\alpha}(x_0) = \underline{y}^2_{\alpha}, \quad \underline{y}''_{\alpha}(x_0) = \overline{y}^2_{\alpha}$$

Case V. For (2, 1)-differentiable, consider the differentiability of y and $D_2^{(1)}y$ in the sense (ii) , (vii) and (ix) of Theorem 2.2. Then we get the following (2,1)-system of ODEs:

$$\overline{y}'''_{\alpha}(x) + g_{1,\alpha}\left(\underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x), \underline{y}'_{\alpha}(x), \overline{y}'_{\alpha}(x), \underline{y}''_{\alpha}(x), \overline{y}''_{\alpha}(x)\right) = h_{1,\alpha}\left(\underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x), \underline{y}'_{\alpha}(x), \overline{y}'_{\alpha}(x), \underline{y}''_{\alpha}(x), \overline{y}''_{\alpha}(x)\right)$$

$$\underline{y}'''_{\alpha}(x) + g_{2,\alpha}\left(\underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x), \underline{y}'_{\alpha}(x), \overline{y}'_{\alpha}(x), \underline{y}''_{\alpha}(x), \overline{y}''_{\alpha}(x)\right) = h_{2,\alpha}\left(\underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x), \underline{y}'_{\alpha}(x), \overline{y}'_{\alpha}(x), \underline{y}''_{\alpha}(x), \overline{y}''_{\alpha}(x)\right)$$

subject to the initial conditions

$$\underline{y}_{\alpha}(x_0) = \underline{y}^0_{\alpha} \quad , \ \overline{y}_{\alpha}(x_0) = \overline{y}^0_{\alpha}$$
$$\overline{y}'_{\alpha}(x_0) = \underline{y}^1_{\alpha} \ , \ \underline{y}'_{\alpha}(x_0) = \overline{y}^1_{\alpha}$$
$$\overline{y}''_{\alpha}(x_0) = \underline{y}^2_{\alpha}, \quad \underline{y}''_{\alpha}(x_0) = \overline{y}^2_{\alpha}$$

Case VI. For (2, 2)-differentiable, consider the differentiability of y and $D_2^{(1)}y$ in the sense (ii) ,(viii) and (x) of Theorem 2.2. Then we get the following (2,2)-system of ODEs:

$$\underline{y}'''_{\alpha}(x) + g_{1,\alpha}\left(\underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x), \underline{y}'_{\alpha}(x), \overline{y}'_{\alpha}(x), \underline{y}''_{\alpha}(x), \overline{y}''_{\alpha}(x)\right) = h_{1,\alpha}\left(\underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x), \underline{y}'_{\alpha}(x), \overline{y}'_{\alpha}(x), \underline{y}''_{\alpha}(x), \overline{y}''_{\alpha}(x)\right)$$

$$\overline{y}'''_{\alpha}(x) + g_{2,\alpha}\left(\underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x), \underline{y}'_{\alpha}(x), \overline{y}'_{\alpha}(x), \underline{y}''_{\alpha}(x), \overline{y}''_{\alpha}(x)\right) = h_{2,\alpha}\left(\underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x), \underline{y}'_{\alpha}(x), \overline{y}'_{\alpha}(x), \underline{y}''_{\alpha}(x), \overline{y}''_{\alpha}(x)\right)$$

subject to the initial conditions

$$\underline{y}_{\alpha}(x_0) = \underline{y}^0_{\alpha} \quad , \ \overline{y}_{\alpha}(x_0) = \overline{y}^0_{\alpha}$$

$$\overline{y'}_{\alpha}(x_0) = \underline{y}^1_{\alpha} \ , \ \underline{y'}_{\alpha}(x_0) = \overline{y}^1_{\alpha}$$
$$\underline{y}''_{\alpha}(x_0) = \underline{y}^2_{\alpha} \ , \ \overline{y}''_{\alpha}(x_0) = \overline{y}^2_{\alpha}$$

Case VII. For (2, 1)-differentiable, consider the differentiability of y and $D_2^{(1)}y$ in the sense (ii) , (viii) and (x) of Theorem 2.2. Then we get the following (2,3)-system of ODEs:

$$\underline{y}'''_{\alpha}(x) + g_{1,\alpha}\left(\underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x), \underline{y'}_{\alpha}(x), \overline{y'}_{\alpha}(x), \underline{y''}_{\alpha}(x), \overline{y''}_{\alpha}(x)\right) = h_{1,\alpha}\left(\underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x), \underline{y'}_{\alpha}(x), \overline{y'}_{\alpha}(x), \underline{y''}_{\alpha}(x), \overline{y''}_{\alpha}(x)\right)$$
$$\overline{y}'''_{\alpha}(x) + g_{2,\alpha}\left(\underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x), \underline{y'}_{\alpha}(x), \overline{y'}_{\alpha}(x), \underline{y''}_{\alpha}(x), \overline{y''}_{\alpha}(x)\right) = h_{2,\alpha}\left(\underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x), \underline{y'}_{\alpha}(x), \overline{y'}_{\alpha}(x), \underline{y''}_{\alpha}(x), \overline{y''}_{\alpha}(x)\right)$$

subject to the initial conditions

$$\underline{y}_{\alpha}(x_0) = \underline{y}^0_{\alpha} \ , \ \overline{y}_{\alpha}(x_0) = \overline{y}^0_{\alpha}$$
$$\overline{y'}_{\alpha}(x_0) = \underline{y}^1_{\alpha} \ , \ \underline{y'}_{\alpha}(x_0) = \overline{y}^1_{\alpha}$$
$$\overline{y}''_{\alpha}(x_0) = \underline{y}^2_{\alpha}, \ \underline{y}''_{\alpha}(x_0) = \overline{y}^2_{\alpha}$$

Case VIII. For (2, 2)-differentiable, consider the differentiability of y and $D_2^{(1)}y$ in the sense (ii) , (viii) and (ix) of Theorem 2.2. Then we get the following (2, 4)-system of ODEs:

$$\overline{y}'''_{\alpha}(x) + g_{1,\alpha}\left(\underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x), \underline{y'}_{\alpha}(x), \overline{y'}_{\alpha}(x), \underline{y''}_{\alpha}(x), \overline{y''}_{\alpha}(x)\right) = h_{1,\alpha}\left(\underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x), \underline{y'}_{\alpha}(x), \overline{y'}_{\alpha}(x), \underline{y''}_{\alpha}(x), \overline{y''}_{\alpha}(x)\right)$$
$$\underline{y}'''_{\alpha}(x) + g_{2,\alpha}\left(\underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x), \underline{y'}_{\alpha}(x), \overline{y'}_{\alpha}(x), \underline{y''}_{\alpha}(x), \overline{y''}_{\alpha}(x)\right) = h_{2,\alpha}\left(\underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x), \underline{y'}_{\alpha}(x), \overline{y'}_{\alpha}(x), \underline{y''}_{\alpha}(x), \overline{y''}_{\alpha}(x)\right)$$

Subject to the initial conditions

$$\underline{y}_{\alpha}(x_0) = \underline{y}^0_{\alpha} \ , \ \overline{y}_{\alpha}(x_0) = \overline{y}^0_{\alpha}$$
$$\overline{y'}_{\alpha}(x_0) = \underline{y}^1_{\alpha} \ , \ \underline{y'}_{\alpha}(x_0) = \overline{y}^1_{\alpha}$$
$$, \underline{y}''_{\alpha}(x_0) = \underline{y}^2_{\alpha}, \ \overline{y}''_{\alpha}(x_0) = \overline{y}^2_{\alpha}$$

Theorem 3.1. [15] Let $n, m \in \{1,2\}$ and $[y(x)]^{\alpha} = \left[\underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x)\right]$ be an (n, m)-solution for fuzzy DE (1.1) on $[x_0, x_0 + a]$. Then, $\underline{y}_{\alpha}(x)$ and $\overline{y}_{\alpha}(x)$ are the solutions of the associated (n, m)- systems.

Theorem 3.2. [15] Let, $m \in \{1,2\}$ and $\tilde{\underline{y}}_{\alpha}(x), \tilde{\overline{y}}_{\alpha}(x)$ solve the (n, m) - system on $[x_0, x_0 + a]$. $\forall$ $\alpha \in [0, 1]$. Let $[y(x)]^{\alpha} = \left[\tilde{\underline{y}}_{\alpha}(x), \tilde{\overline{y}}_{\alpha}(x)\right]$. If $\tilde{y}(x)$ has level sets on $[x_0, x_0 + a]$, and $D^{(2)}_{n,m}\tilde{y}(x)$ exists, then $\tilde{y}(x)$ is a (n, m) - solution for fuzzy DE (1.1).

4-The Residual Power Series Method for Solving Fuzzy Deferential Equations

In this section, to get series solutions for systems of Initial Value Problems with initial conditions, we use our theorem of the RPS. At first, we interpreted and analyzed the RPS theorem for solving IVPs for fuzzy DE(1.1) with respect to (1)-differentiability only. The RPS theorem expresses the solutions of IVPs (3.1) and (3.2) as a power series expansion about the initial point $x = x_0$, we assume that these solutions take the form

$\underline{y}_{\alpha}(x) = \sum_{n=0}^{\infty} \underline{y}_{\alpha,n}(x)$ and $\overline{y}_{\alpha}(x) = \sum_{n=0}^{\infty} \overline{y}_{\alpha,n}(x)$ where $\underline{y}_{\alpha,n}$ and $\overline{y}_{\alpha,n}$ are two terms of approximations and they are given as:

$$\underline{y}_{\alpha,n}(x) = c_n(\alpha)(x - x_0)^n,$$
$$\overline{y}_{\alpha,n}(x) = d_n(\alpha)(x - x_0)^n.$$

Obviously, when n = 0, since $\underline{y}_{\alpha,0}(x)$ and $\overline{y}_{\alpha,n}(x)$ satisfy the initial conditions (3.2), where $\underline{y}_{\alpha,0}(x)$ and $\overline{y}_{\alpha,n}(x)$ are the initial approximations of $\underline{y}_{\alpha}(x)$ and $\overline{y}_{\alpha}(x)$, respectively, we have $c_0(\alpha) = \underline{y}_{\alpha,0}(x_0) = \underline{y}_{\alpha}(x_0)$ and $d_0(\alpha) = \overline{y}_{\alpha,0}(x_0) = \overline{y}_{\alpha}(x_0)$.

On the other hand, if we choose $\underline{y}_{\alpha,0}(x_0) = \underline{y}_{\alpha}(x_0)$ and $\overline{y}_{\alpha,0}(x_0) = \overline{y}_{\alpha}(x_0)$ approximations as initials of $\underline{y}_{\alpha}(x)$ and $\overline{y}_{\alpha}(x)$, respectively, then we can calculate $\underline{y}_{\alpha,n}(x)$ and $\overline{y}_{\alpha,n}(x)$, for n = 1, 2, 3, . . . and approximate the solutions $\underline{y}_{\alpha}(x)$ and $\overline{y}_{\alpha}(x)$ for system of ODEs (3.1) and (3.2) by the k-th-truncated series

$$\begin{aligned} \psi_{\underline{y}_{\alpha,k}(x)} &= \sum_{n=0}^{k} c_n(\alpha)(x - x_0)^n \\ \psi_{\overline{y}_{\alpha,k}(x)} &= \sum_{n=0}^{k} d_n(\alpha)(x - x_0)^n \end{aligned} \tag{4.1}$$

before applying the RPS method for solving the system of ODEs (3.1) and (3.2), we define the residual functions for this system as follows:

$$\begin{aligned} R_{1,\alpha}(x) &= \underline{y}'_{\alpha}(x) + f_{1,\alpha}\left(x, \underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x)\right) \\ R_{2,\alpha}(x) &= \overline{y}'_{\alpha}(x) + f_{2,\alpha}\left(x, \underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x)\right) \end{aligned} \tag{4.2}$$

It is clear that $R_{1,\alpha}(x) = R_{2,\alpha}(x) = 0 \ \forall\, x \in [x_0, x_0 + a]$ and $\alpha \in [0, 1]$. In order to approximate the solution, substitute the expansion of $\underline{y}_{\alpha}(x)$ and $\overline{y}_{\alpha}(x)$ in Eq. (4.2) to get

$$\begin{aligned} R_{1,\alpha}(x) &= \sum_{n=1}^{\infty} n c_n(\alpha)(x - x_0)^{n-1} + f_{1,\alpha}\left(x, \sum_{n=0}^{k} c_n(\alpha)(x - x_0)^n, \sum_{n=0}^{k} d_n(\alpha)(x - x_0)^n\right) \\ R_{2,\alpha}(x) &= \sum_{n=1}^{\infty} n d_n(\alpha)(x - x_0)^{n-1} + f_{2,\alpha}\left(x, \sum_{n=0}^{k} c_n(\alpha)(x - x_0)^n, \sum_{n=0}^{k} d_n(\alpha)(x - x_0)^n\right). \end{aligned} \tag{4.3}$$

To obtain the first approximate solution put $x = x_0$ in Eq. (4.3) and using $R_{1,\alpha}(x_0) = R_{2,\alpha}(x_0) = 0$, we have

$$\begin{aligned} c_1(\alpha) &= f_{1,\alpha}\left(x_0, c_0(\alpha), d_0(\alpha)\right) \\ d_1(\alpha) &= f_{2,\alpha}\left(x_0, c_0(\alpha), d_0(\alpha)\right) \end{aligned} \tag{4.4}$$

Using the 1th-truncated series, the first approximation of ODEs (3.1) and (3.2) can be written as follows:

$$\begin{aligned} \psi_{\underline{y}_{\alpha,1}}(x) &= \underline{y}_{\alpha}(x_0) + f_{1,\alpha}\left(x_0, \underline{y}_{\alpha}(x_0), \overline{y}_{\alpha}(x_0)\right)(x - x_0) \\ \psi_{\overline{y}_{\alpha,1}}(x) &= \overline{y}_{\alpha}(x_0) + f_{2,\alpha}\left(x_0, \underline{y}_{\alpha}(x_0), \overline{y}_{\alpha}(x_0)\right)(x - x_0) \end{aligned}$$

And to find the second approximation, we differentiate Eq. (4.3) respect to(x), and using $\square'_{1,\square}(x_0) = \square'_{2,\square}(x_0) = 0$ to obtain the following results

$$\begin{aligned} c_2(\alpha) = \frac{1}{2}\Bigg[&\frac{\partial}{\partial x} f_{1,\alpha}\left(x_0, c_0(\alpha), d_0(\alpha)\right) + c_1(\alpha)\frac{\partial}{\partial \underline{y}_{\alpha}} f_{1,\alpha}\left(x_0, c_0(\alpha), d_0(\alpha)\right) \\ &+ d_1(\alpha)\frac{\partial}{\partial \overline{y}_{\alpha}} f_{1,\alpha}\left(x_0, c_0(\alpha), d_0(\alpha)\right)\Bigg] \\ d_2(\alpha) = \frac{1}{2}\Bigg[&\frac{\partial}{\partial x} f_{2,\alpha}\left(x_0, c_0(\alpha), d_0(\alpha)\right) + c_1(\alpha)\frac{\partial}{\partial \overline{y}_{\alpha}} f_{2,\alpha}\left(x_0, c_0(\alpha), d_0(\alpha)\right) \\ &+ d_1(\alpha)\frac{\partial}{\partial \overline{y}_{\alpha}} f_{2,\alpha}\left(x_0, c_0(\alpha), d_0(\alpha)\right)\Bigg] \end{aligned} \tag{4.5}$$

The second approximation of ODEs (3.1) and (3.2) using 2th-truncated series, can be written as:

$$\psi_{\underline{y}_{\alpha,2}}(x) = \underline{y}_\alpha(x_0) + f_{1,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right)(x-x_0) + \frac{1}{2}\left[\frac{\partial}{\partial x} f_{1,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right)\right.$$
$$+ f_{1,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right)\frac{\partial}{\partial \underline{y}_\alpha} f_{1,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right)$$
$$\left. + f_{2,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right)\frac{\partial}{\partial \overline{y}_\alpha} f_{1,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right)\right](x-x_0)^2$$

$$\psi_{\overline{y}_{\alpha,2}}(x) = \overline{y}_\alpha(x_0) + f_{2,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right)(x-x_0) + \frac{1}{2}\left[\frac{\partial}{\partial x} f_{2,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right)\right.$$
$$+ f_{1,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right)\frac{\partial}{\partial \underline{y}_\alpha} f_{2,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right)$$
$$\left. + f_{2,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right)\frac{\partial}{\partial \overline{y}_\alpha} f_{2,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right)\right](x-x_0)^2$$

Similarly, to find the third approximation, we differentiate of Eq. (4.3) with respect to(x), and using the fact $R'_{1,\alpha}(x_0) = R'_{2,\alpha}(x_0) = 0$ to obtain the following results

$$c_3(\alpha) = \frac{1}{6}\left[\frac{\partial}{\partial x} f_{1,\alpha}\left(x_0, c_0(\alpha), d_0(\alpha)\right) + c_2(\alpha)\frac{\partial}{\partial \underline{y}_\alpha} f_{1,\alpha}\left(x_0, c_0(\alpha), d_0(\alpha)\right)\right.$$
$$\left. + d_2(\alpha)\frac{\partial}{\partial \overline{y}_\alpha} f_{1,\alpha}\left(x_0, c_0(\alpha), d_0(\alpha)\right)\right] \qquad (4.6)$$
$$d_3(\alpha) = \frac{1}{6}\left[\frac{\partial}{\partial x} f_{2,\alpha}\left(x_0, c_0(\alpha), d_0(\alpha)\right) + c_2(\alpha)\frac{\partial}{\partial \overline{y}_\alpha} f_{2,\alpha}\left(x_0, c_0(\alpha), d_0(\alpha)\right)\right.$$
$$\left. + d_2(\alpha)\frac{\partial}{\partial \overline{y}_\alpha} f_{2,\alpha}\left(x_0, c_0(\alpha), d_0(\alpha)\right)\right]$$

The third approximation for the system of ODEs (3.1) and (3.2) using 3th-truncated series is as follows :

$$\psi_{\underline{y}_{\alpha,3}}(x) = \underline{y}_\alpha(x_0) + f_{1,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right)(x-x_0) + \frac{1}{2}\left[\frac{\partial}{\partial x} f_{1,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right)\right.$$
$$+ f_{1,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right)\frac{\partial}{\partial \underline{y}_\alpha} f_{1,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right)$$
$$\left. + f_{2,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right)\frac{\partial}{\partial \overline{y}_\alpha} f_{1,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right)\right](x-x_0)^2$$
$$+ \frac{1}{6}\left[\frac{\partial}{\partial x} f_{1,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right) c_2(\alpha)\frac{\partial}{\partial \underline{y}_\alpha} f_{1,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right)\right.$$
$$\left. + d_2(\alpha)\frac{\partial}{\partial \overline{y}_\alpha} f_{2,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right)\right](x-x_0)^3$$

$$\psi_{\overline{y}_{\alpha,3}}(x) = \overline{y}_\alpha(x_0) + f_{2,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right)(x-x_0) + \frac{1}{2}\left[\frac{\partial}{\partial x} f_{2,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right)\right.$$
$$+ f_{1,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right)\frac{\partial}{\partial \underline{y}_\alpha} f_{2,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right)$$
$$\left. + f_{2,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right)\frac{\partial}{\partial \overline{y}_\alpha} f_{2,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right)\right](x-x_0)^2$$
$$+ \frac{1}{6}\left[\frac{\partial}{\partial x} f_{2,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right) + c_2(\alpha)\frac{\partial}{\partial \overline{y}_\alpha} f_{2,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right)\right.$$
$$\left. + d_2(\alpha)\frac{\partial}{\partial \overline{y}_\alpha} f_{2,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right)\right](x-x_0)^3$$

In order to find the k-th approximate solution for system of ODEs (3.1) and (3.2), it is enough to substitute the

k-th -truncated series $\psi_{\underline{y}_{\alpha,k}}(x)$ and $\psi_{\overline{y}_{\alpha,k}}(x)$ instead of the expansion of $\underline{y}_\alpha(x)$ and $\overline{y}_\alpha(x)$, respectively, into the residual Eq. (4.3), and then apply the same procedure since it is easy to show that $\underline{y}_\alpha^{(s)}(x_0) = \psi_{\underline{y}_{\alpha,k}}^{(s)}(x_0)$ and $\overline{y}_\alpha^{(s)}(x_0) = \psi_{\overline{y}_{\alpha,k}}^{(s)}(x_0)$ for each $s \le k$.

The following theorem is an extension of theorem shown in a previous work [14], which shows the convergence of the RPS method in the sense of (1) – differentiability.

Theorem 4.1. Suppose that $\underline{y}_\alpha(x)$ and $\overline{y}_\alpha(x)$ are the exact solutions of ODEs (3.1) and (3.2) in the sense of (1)-differentiability. Then, the approximate solution obtained by the Residual Power Series method is just the Taylor expansion of that $\underline{y}_\alpha(x)$ and $\overline{y}_\alpha(x)$.

Proof:- Suppose that the approximate solution of ODEs (3.1) and (3.2) be as follows:

$$\begin{aligned} \tilde{\underline{y}}_\alpha(x) &= c_0(\alpha) + c_1(\alpha)(x-x_0) + c_2(\alpha)(x-x_0)^2 + c_3(\alpha)(x-x_0)^3 + \cdots \\ \tilde{\overline{y}}_\alpha(x) &= d_0(\alpha) + d_1(\alpha)(x-x_0) + d_2(\alpha)(x-x_0)^2 + d_3(\alpha)(x-x_0)^3 + \cdots \end{aligned} \tag{4.7}$$

To prove the theory, we show that the coefficients c_n and d_n in Eq. (4.7) be as follows:

$$c_n(\alpha) = \frac{1}{n!}\underline{y}_\alpha^{(n)}(x_0) \quad , \quad d_n(\alpha) = \frac{1}{n!}\overline{y}_\alpha^{(n)}(x_0) \tag{4.8}$$

$\forall\, n = 0,1,2,3,\ldots$ and $\alpha \in [0,1]$, where $\underline{y}_\alpha(x)$ and $\overline{y}_\alpha(x)$ represent the exact solutions of ODEs (3.1) and (3.2).

It is clear that ,for $n = 0$, the initial conditions (3.2) give $c_0(\alpha) = \underline{y}_\alpha(x_0)$ and $d_0(\alpha) = \overline{y}_\alpha(x_0)$

, and for n=1 , substitute $x = x_0$ into Eq.(3.1) to obtain $f_{1,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right) = \underline{y}'_\alpha(x_0)$

, and $f_{2,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right) = \overline{y}'_\alpha(x_0)$. On the other hand, from Eq. (3.1), we can write

$$\begin{aligned} \tilde{\underline{y}}_\alpha(x) &= \underline{y}_\alpha(x_0) + c_1(\alpha)(x-x_0) + c_2(\alpha)(x-x_0)^2 + c_3(\alpha)(x-x_0)^3 + \cdots \\ \tilde{\overline{y}}_\alpha(x) &= \overline{y}_\alpha(x_0) + d_1(\alpha)(x-x_0) + d_2(\alpha)(x-x_0)^2 + d_3(\alpha)(x-x_0)^3 + \cdots \end{aligned} \tag{4.9}$$

by substituting Eq. (4.9) in Eq. (3.1) and then putting $x = x_0$, we get

$$\begin{aligned} c_1(\alpha) &= f_{1,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right) = \underline{y}'_\alpha(x_0) \\ d_1(\alpha) &= f_{2,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right) = \overline{y}'_\alpha(x_0) \end{aligned} \tag{4.10}$$

Further, for $n = 2$, by differentiating Eq. (3.1) with respect to x, we can get

$$\begin{aligned} \underline{y}''_\alpha(x) &= \frac{\partial}{\partial x} f_{1,\alpha}\left(x, \underline{y}_\alpha(x), \overline{y}_\alpha(x)\right) + \underline{y}'_\alpha(x)\frac{\partial}{\partial \underline{y}_\alpha} f_{1,\alpha}\left(x, \underline{y}_\alpha(x), \overline{y}_\alpha(x)\right) \\ &\quad + \overline{y}'_\alpha(x)\frac{\partial}{\partial \overline{y}_\alpha} f_{1,\alpha}\left(x, \underline{y}_\alpha(x), \overline{y}_\alpha(x)\right) \\ \overline{y}''_\alpha(x) &= \frac{\partial}{\partial x} f_{2,\alpha}\left(x, \underline{y}_\alpha(x), \overline{y}_\alpha(x)\right) + \underline{y}'_\alpha(x)\frac{\partial}{\partial \underline{y}_\alpha} f_{2,\alpha}\left(x, \underline{y}_\alpha(x), \overline{y}_\alpha(x)\right) \\ &\quad + \overline{y}'_\alpha(x)\frac{\partial}{\partial \overline{y}_\alpha} f_{2,\alpha}\left(x, \underline{y}_\alpha(x), \overline{y}_\alpha(x)\right) \end{aligned} \tag{4.11}$$

by substituting $x = x_0$ in Eq. (4.11), we can obtain

$$\underline{y}''_{\alpha}(x_0) = \frac{\partial}{\partial x} f_{1,\alpha}\left(x_0, \underline{y}_{\alpha}(x_0), \overline{y}_{\alpha}(x_0)\right) + \underline{y}'_{\alpha}(x_0) \frac{\partial}{\partial \underline{y}_{\alpha}} f_{1,\alpha}\left(x_0, \underline{y}_{\alpha}(x_0), \overline{y}_{\alpha}(x_0)\right) + \overline{y}'_{\alpha}(x_0) \frac{\partial}{\partial \overline{y}_{\alpha}} f_{1,\alpha}\left(x_0, \underline{y}_{\alpha}(x_0), \overline{y}_{\alpha}(x_0)\right) \quad (4.12)$$

$$\overline{y}''_{\alpha}(x_0) = \frac{\partial}{\partial x} f_{2,\alpha}\left(x_0, \underline{y}_{\alpha}(x_0), \overline{y}_{\alpha}(x_0)\right) + \underline{y}'_{\alpha}(x_0) \frac{\partial}{\partial \underline{y}_{\alpha}} f_{2,\alpha}\left(x_0, \underline{y}_{\alpha}(x_0), \overline{y}_{\alpha}(x_0)\right) + \overline{y}'_{\alpha}(x_0) \frac{\partial}{\partial \overline{y}_{\alpha}} f_{2,\alpha}\left(x_0, \underline{y}_{\alpha}(x_0), \overline{y}_{\alpha}(x_0)\right)$$

According to Eqs. (4.9) and (4.10), we can create the approximation system of ODEs (3.1) and (3.2) as follows:

$$\begin{aligned} \tilde{\underline{y}}_{\alpha}(x) &= \underline{y}_{\alpha}(x_0) + \underline{y}'_{\alpha}(x_0)(x - x_0) + c_2(\alpha)(x - x_0)^2 + \cdots \\ \tilde{\overline{y}}_{\alpha}(x) &= \overline{y}_{\alpha}(x_0) + \overline{y}'_{\alpha}(x_0)(x - x_0) + d_2(\alpha)(x - x_0)^2 + \cdots \end{aligned} \quad (4.13)$$

By substituting Eq. (4.13) in Eq. (4.11) and putting $x = x_0$, we can obtain:

$$2\, c_2(\alpha) = \frac{\partial}{\partial x} f_{1,\alpha}\left(x_0, \underline{y}_{\alpha}(x_0), \overline{y}_{\alpha}(x_0)\right) + \underline{y}'_{\alpha}(x_0) \frac{\partial}{\partial \underline{y}_{\alpha}} f_{1,\alpha}\left(x_0, \underline{y}_{\alpha}(x_0), \overline{y}_{\alpha}(x_0)\right) + \overline{y}'_{\alpha}(x_0) \frac{\partial}{\partial \overline{y}_{\alpha}} f_{1,\alpha}\left(x_0, \underline{y}_{\alpha}(x_0), \overline{y}_{\alpha}(x_0)\right) \quad (4.14)$$

$$2\, d_2(\alpha) = \frac{\partial}{\partial x} f_{2,\alpha}\left(x_0, \underline{y}_{\alpha}(x_0), \overline{y}_{\alpha}(x_0)\right) + \underline{y}'_{\alpha}(x_0) \frac{\partial}{\partial \underline{y}_{\alpha}} f_{2,\alpha}\left(x_0, \underline{y}_{\alpha}(x_0), \overline{y}_{\alpha}(x_0)\right) + \overline{y}'_{\alpha}(x_0) \frac{\partial}{\partial \overline{y}_{\alpha}} f_{2,\alpha}\left(x_0, \underline{y}_{\alpha}(x_0), \overline{y}_{\alpha}(x_0)\right)$$

by comparing Eqs. (4.12) and (4.14), we can get

$$c_2(\alpha) = \frac{1}{2}\underline{y}''_{\alpha}(x_0) \quad \text{and} \quad d_2(\alpha) = \frac{1}{2}\overline{y}''_{\alpha}(x_0) \quad (4.15)$$

Similarly, for □ = 3 , by differentiating Eq. (3.1) with respect to x, we get

$$\underline{y}'''_{\alpha}(x) = \frac{\partial}{\partial x} f_{1,\alpha}\left(x, \underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x)\right) + \underline{y}'_{\alpha}(x) \frac{\partial}{\partial \underline{y}_{\alpha}} f_{1,\alpha}\left(x, \underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x)\right) + \overline{y}'_{\alpha}(x) \frac{\partial}{\partial \overline{y}_{\alpha}} f_{1,\alpha}\left(x, \underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x)\right) + \underline{y}''_{\alpha}(x) \frac{\partial}{\partial \underline{y}_{\alpha}} f_{1,\alpha}\left(x, \underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x)\right) + \overline{y}''_{\alpha}(x) \frac{\partial}{\partial \overline{y}_{\alpha}} f_{1,\alpha}\left(x, \underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x)\right) \quad (4.16)$$

$$\overline{y}'''_{\alpha}(x) = \frac{\partial}{\partial x} f_{2,\alpha}\left(x, \underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x)\right) + \underline{y}'_{\alpha}(x) \frac{\partial}{\partial \underline{y}_{\alpha}} f_{2,\alpha}\left(x, \underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x)\right) + \overline{y}'_{\alpha}(x) \frac{\partial}{\partial \overline{y}_{\alpha}} f_{2,\alpha}\left(x, \underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x)\right) + \underline{y}''_{\alpha}(x) \frac{\partial}{\partial \underline{y}_{\alpha}} f_{2,\alpha}\left(x, \underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x)\right) + \overline{y}''_{\alpha}(x) \frac{\partial}{\partial \overline{y}_{\alpha}} f_{2,\alpha}\left(x, \underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x)\right)$$

According to Eqs. (4.9), (4.10) and (4.15), we can create the approximation system of ODEs (3.1) and (3.2) as follows:

$$\begin{aligned} \tilde{\underline{y}}_{\alpha}(x) &= \underline{y}_{\alpha}(x_0) + \underline{y}'_{\alpha}(x_0)(x - x_0) + \frac{1}{2}\underline{y}''_{\alpha}(x_0)(x - x_0)^2 + c_3(\alpha)(x - x_0)^3 + \cdots \\ \tilde{\overline{y}}_{\alpha}(x) &= \overline{y}_{\alpha}(x_0) + \overline{y}'_{\alpha}(x_0)(x - x_0) + \frac{1}{2}\overline{y}''_{\alpha}(x_0)(x - x_0)^2 + d_3(\alpha)(x - x_0)^3 + \cdots \end{aligned} \quad (4.17)$$

By substituting Eq. (4.17) in Eq. (4.16) and putting $x = x_0$,we can get

$$6\,c_3(\alpha) = \frac{\partial}{\partial x} f_{1,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right) + \underline{y}'_\alpha(x_0)\frac{\partial}{\partial \underline{y}_\alpha} f_{1,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right)$$
$$+\overline{y}'_\alpha(x_0)\frac{\partial}{\partial \overline{y}_\alpha} f_{1,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right) + \underline{y}''_\alpha(x_0)\frac{\partial}{\partial \underline{y}_\alpha} f_{1,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right)$$
$$+\overline{y}''_\alpha(x_0)\frac{\partial}{\partial \overline{y}_\alpha} f_{1,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right) \qquad (4.18)$$
$$6\,d_3(\alpha) = \frac{\partial}{\partial x} f_{2,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right) + \underline{y}'_\alpha(x_0)\frac{\partial}{\partial \underline{y}_\alpha} f_{2,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right)$$
$$+\overline{y}'_\alpha(x_0)\frac{\partial}{\partial \overline{y}_\alpha} f_{2,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right) + \underline{y}''_\alpha(x_0)\frac{\partial}{\partial \underline{y}_\alpha} f_{2,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right)$$
$$+\overline{y}''_\alpha(x_0)\frac{\partial}{\partial \overline{y}_\alpha} f_{2,\alpha}\left(x_0, \underline{y}_\alpha(x_0), \overline{y}_\alpha(x_0)\right)$$

by comparing Eqs. (4.18) and (4.16) and putting $x = x_0$, we can conclude that $c_3(\alpha) = \frac{1}{6}\underline{y}'''_\alpha(x_0)$ and $d_3(\alpha) = \frac{1}{6}\overline{y}'''_\alpha(x_0)$.

Corollary 4.1. [16]. If either $\underline{y}_\alpha(x)$ or $\overline{y}_\alpha(x)$ is a polynomial, then the Residual Power Series method will access the exact solution.

5-Ilustrative Example

Consider the third order fuzzy initial value problem

$$\mathbf{y}'''(\mathbf{x}) = \mathbf{2}\,\mathbf{y}''(\mathbf{x}) + \mathbf{3}\,\mathbf{y}'(\mathbf{x}) \qquad \mathbf{0} \le \mathbf{x} \le \mathbf{1} \qquad (\mathbf{5.1})$$

Subject to the fuzzy initial conditions

$$\left.\begin{aligned} \mathbf{y}(0) &= (3+\boldsymbol{\alpha}, \mathbf{5}-\boldsymbol{\alpha}) \\ \mathbf{y}'(\mathbf{0}) &= (-\mathbf{3}+\boldsymbol{\alpha}, -\mathbf{1}-\boldsymbol{\alpha}) \\ \mathbf{y}''(\mathbf{0}) &= (\mathbf{8}+\boldsymbol{\alpha}, \mathbf{10}-\boldsymbol{\alpha}) \end{aligned}\right\} \qquad (5.2)$$

The eigenvalue-eigenvector solution can be found as follows:

$$y(\mathbf{x}, \alpha) = \left(-\frac{1}{3} + \frac{7}{12}\,e^{3x} + \left(\frac{11}{4} + \alpha\right)e^{-x},\ -\frac{1}{3} + \frac{7}{12}\,e^{3x} + \left(\frac{19}{4} - \alpha\right)e^{-x}\right)$$

The approximate solution $\mathbf{y}$ with α-levels is as follows: $y_{\boldsymbol{\alpha}}(x) = [\underline{y}, \overline{y}]$, $\alpha \in (0, 1]$. Hence, to create solutions in the (lower case of solution) $\underline{y}$, suppose the problem:

$$\underline{y}'''(\mathbf{x}) = \mathbf{2}\,\underline{y}''(\mathbf{x}) + \mathbf{3}\,\underline{y}'(\mathbf{x}) \qquad (\mathbf{5.3})$$

with initial conditions:

$$\underline{y}(0) = 3 + \boldsymbol{\alpha}\ ,\ \underline{y}'(\mathbf{0}) = -\mathbf{3} + \boldsymbol{\alpha}\ ,\ \underline{y}''(\mathbf{0}) = \mathbf{8} + \boldsymbol{\alpha} \qquad (\mathbf{5.4})$$

We shall look for a power series solution of about $x_0 = 0$

$$\text{let: } \underline{y}(x) = \sum_{n=0}^{\infty} c_n (x-0)^n = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots. \qquad (5.5)$$

Now, the first three coefficients are already calculated by the initial conditions:

$$3 + \alpha = \underline{y}(0) = c_0 + c_1(0) + c_2(0)^2 + c_3(0)^3 + \cdots = c_0$$

And $\underline{y}'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + \cdots$

$$-3 + \alpha = \underline{y}'(0) = c_1$$

And so $\underline{y}''(x) = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = 2c_2 + 3(2)c_3 x + \cdots$

$$8 + \alpha = \underline{y}''(0) = 2c_2$$

$$c_2 = \frac{8+\alpha}{2}$$

Similarly, by differentiating again we find

$$\underline{y}'''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2)c_n x^{n-3} = 6c_3 + 4(3)(2)c_4 x + \cdots$$

$$\underline{y}'''(0) = 2\,\underline{y}''(0) + 3\underline{y}'(0)$$

$$6c_3 = 2(8+\alpha) + 3(-3+\alpha) = 16 + 2\alpha - 9 + 3\alpha = 7 + 5\alpha$$

$$c_3 = \frac{7+5\alpha}{6}$$

Thus, if (5.5) is used to solve (5.3), we need

$$\sum_{n=3}^{\infty} n(n-1)(n-2)c_n x^{n-3} - 2\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 3\sum_{n=1}^{\infty} nc_n x^{n-1} = \tilde{0}$$

To solve the equation above, we first put the first power series in a standard form:

$$\sum_{n=0}^{\infty} (n+3)(n+2)(n+1)c_{n+3} x^n - 2\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2} x^n - 3\sum_{n=0}^{\infty} (n+1)c_{n+1} x^n = 0$$

$$\sum_{n=0}^{\infty} ((n+3)(n+2)(n+1)c_{n+3} - 2(n+2)(n+1)c_{n+2} - 3(n+1)c_{n+1})x^n = 0$$

This must vanish for all x, so the total coefficients of each power x^n must vanish separately:

$$(n+3)(n+2)(n+1)c_{n+3} - 2(n+2)(n+1)c_{n+2} - 3(n+1)c_{n+1}) = 0 \qquad \forall n$$

We thus conclude that

$$c_{n+3} = \frac{2(n+2)(n+1)c_{n+2} + \mathbf{3}(n+1)c_{n+1}}{(n+3)(n+2)(n+1)} \qquad \forall n$$

Let us suppose that c_0, c_1, c_2, c_3 are given. Then

$$, \quad c_4 = \frac{\mathbf{2}(3)(2)c_3 + \mathbf{3}(2)c_2}{(4)(3)(2)} = \frac{\mathbf{2(7+5\alpha) + 3(8+\alpha)}}{(4)(3)(2)}$$

$$c_5 = \frac{\mathbf{2}(4)(3)c_4 + \mathbf{3}(3)c_3}{(5)(4)(3)} = \frac{\mathbf{2(38+13\alpha) + 3(7+5\alpha)}}{(5)(4)(3)(2)}$$

Similarly, for the upper solution $\bar{y}$, suppose the problem:

$$\bar{y}'''(\mathbf{x}) = \mathbf{2}\,\bar{y}''(\mathbf{x}) + \mathbf{3}\,\bar{y}'(\mathbf{x}) \qquad (\mathbf{5.6})$$

with initial conditions:

$$\bar{y}(0) = 5 - \boldsymbol{\alpha} \quad , \ \bar{y}'(\mathbf{0}) = -\mathbf{1} - \boldsymbol{\alpha} \ , \ \bar{y}''(\mathbf{0}) = \mathbf{10} - \boldsymbol{\alpha} \qquad (\mathbf{5.7})$$

We shall look for a power series solution of about x =0

$$\text{let: } \bar{y}(x) = \sum_{n=0}^{\infty} d_n (x-0)^n = \sum_{n=0}^{\infty} d_n x^n = d_0 + d_1 x + d_2 x^2 + d_3 x^3 + \cdots \qquad \mathbf{(5.8)}$$

Now, the first three coefficients are already calculated by the initial conditions:

$$5 - \alpha = \bar{y}(0) = d_0 + d_1(0) + d_2(0)^2 + d_3(0)^3 + \cdots = d_0$$

And $\bar{y}'(x) = \sum_{n=1}^{\infty} n d_n x^{n-1} = d_1 + 2d_2 x + 3d_3 x^2 + \cdots$

$$-1 - \alpha = \bar{y}'(0) = d_1$$

And so $\bar{y}''(x) = \sum_{n=2}^{\infty} n(n-1)d_n x^{n-2} = 2d_2 + 3(2)d_3 x + \cdots.$

$$10 - \alpha = \bar{y}''(0) = 2d_2$$

$$d_2 = \frac{10-\alpha}{2}$$

Similarly, by differentiating again, we find

$$\bar{y}'''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2)d_n x^{n-3} = 6d_3 + 4(3)(2)d_4 x + \cdots$$

$$\bar{y}'''(0) = 2\bar{y}''(0) + 3\bar{y}'(0)$$

$$6d_3 = 2(10-\alpha) + 3(-1-\alpha) = 20 - 2\alpha - 3 - 3\alpha = 17 - 5\alpha$$

$$d_3 = \frac{17-5\alpha}{6}$$

Thus, if (5.8) is used to solve (5.6), we need

$$\sum_{n=3}^{\infty} n(n-1)(n-2)d_n x^{n-3} - 2\sum_{n=2}^{\infty} n(n-1)d_n x^{n-2} - 3\sum_{n=1}^{\infty} nd_n x^{n-1} = \tilde{0}$$

To solve the equation above, we first put the first power series in a standard form:

$$\sum_{n=0}^{\infty} (n+3)(n+2)(n+1)d_{n+3} x^n - 2\sum_{n=0}^{\infty}(n+2)(n+1)d_{n+2} x^n - 3\sum_{n=0}^{\infty} (n+1)d_{n+1} x^n = 0$$

$$\sum_{n=0}^{\infty} ((n+3)(n+2)(n+1)d_{n+3} - 2(n+2)(n+1)d_{n+2} - 3(n+1)d_{n+1})x^n = 0$$

This must vanish for all x, so the total coefficients of each power x^n must vanish separately:

$$(n+3)(n+2)(n+1)d_{n+3} - 2(n+2)(n+1)d_{n+2} - 3(n+1)d_{n+1}) = 0 \qquad \forall n$$

Thus, we conclude that:

$$d_{n+3} = \frac{2(n+2)(n+1)d_{n+2} + \mathbf{3}(n+1)d_{n+1}}{(n+3)(n+2)(n+1)} \qquad \forall n$$

Let us suppose *that* d_0, d_1, d_2, d_3 are given, then

$$, \quad d_4 = \frac{\mathbf{2}(3)(2)d_3 + \mathbf{3}(2)d_2}{(4)(3)(2)} = \frac{\mathbf{2(17-5\alpha) + 3(10-\alpha)}}{(4)(3)(2)}$$

$$d_5 = \frac{\mathbf{2}(4)(3)d_4 + \mathbf{3}(3)d_3}{(5)(4)(3)} = \frac{\mathbf{2(4-13\alpha) + 3(17-5\alpha)}}{(5)(4)(3)(2)}$$

For finding the fuzzy (n, m)-solutions of fuzzy DE (5.1) and (5.2), we have eight cases, as follows:
Case I: Let y (t) is (1, 1)-differentiable and consider Case I in Algorithm 2.2. On the other hand, if we determine the initial approximations as: $\underline{y}_{\alpha,2} = (3+\alpha) + (-3+\alpha)x + \frac{8+\alpha}{2}x^2, \overline{y}_{\alpha,2} = (5-\alpha) + (-1-\alpha)x + \frac{10-\alpha}{2}x^2$ then the power series expansions of solutions take the form

$$\underline{y}_{\alpha} = (3+\alpha) + (-3+\alpha)x + \frac{8+\alpha}{2}x^2 + c_3(\alpha)x^3 + \cdots$$
$$\overline{y}_{\alpha} = (5-\alpha) + (-1-\alpha)x + \frac{10-\alpha}{2}x^2 + d_3(\alpha)x^3 + \cdots$$

Consequently, the 3rd-order power series approximation of the RPS solution for (1, 1)-system relative to these initial approximations is as follows:

$$\psi_{\underline{y}_{\alpha,3}}(x) = (3+\alpha) + (-3+\alpha)x + \frac{8+\alpha}{2}x^2 + \frac{7+5\alpha}{3!}x^3$$
$$\psi_{\overline{y}_{\alpha,3}}(x) = (5-\alpha) + (-1-\alpha)x + \frac{10-\alpha}{2}x^2 + \frac{17-5\alpha}{3!}x^3$$

Case II: For (1, 2)-differentiable, consider Case II in Algorithm 2.2, we find the following solutions for (1, 2)-system:

$$\underline{y}_{\alpha} = (3+\alpha) + (-3+\alpha)x + \frac{10-\alpha}{2}x^2 + \frac{17-5\alpha}{3!}x^3$$
$$\overline{y}_{\alpha} = (5-\alpha) + (-1-\alpha)x + \frac{8+\alpha}{2}x^2 + \frac{7+5\alpha}{3!}x^3$$

Case III: : For (1, 1)-differentiable, consider Case III in Algorithm 2.2, we find the following solutions for (1, 1)-system:

$$\underline{y}_{\alpha} = (3+\alpha) + (-3+\alpha)x + \frac{8+\alpha}{2}x^2 + \frac{17-5\alpha}{3!}x^3$$
$$\overline{y}_{\alpha} = (5-\alpha) + (-1-\alpha)x + \frac{10-\alpha}{2}x^2 + \frac{7+5\alpha}{3!}x^3$$

Case IV: For (1, 2)-differentiable, consider Case IV in Algorithm 2.2, we find the following solutions for (1, 2)-system:

$$\underline{y}_{\alpha} = (3+\alpha) + (-3+\alpha)x + \frac{10-\alpha}{2}x^2 + \frac{7+5\alpha}{3!}x^3$$
$$\overline{y}_{\alpha} = (5-\alpha) + (-1-\alpha)x + \frac{8+\alpha}{2}x^2 + \frac{17-5\alpha}{3!}x^3$$

Case V: Let y (t) be (2, 1)-differentiable and consider Case V in Algorithm 3.2. Then, the (2, 1)-system yields the following solutions:

$$\underline{y}_{\alpha} = (3+\alpha) + (-1-\alpha)x + \frac{10-\alpha}{2}x^2 + \frac{17-5\alpha}{3!}x^3$$
$$\overline{y}_{\alpha} = (5-\alpha) + (-3+\alpha)x + \frac{8+\alpha}{2}x^2 + \frac{7+5\alpha}{3!}x^3$$

Case VI: For (2, 2)-differentiable, consider Case VI in Algorithm 2.2, we find the following solutions for (2, 2)-system:

$$\underline{y}_{\alpha} = (3+\alpha) + (-1-\alpha)x + \frac{8+\alpha}{2}x^2 + \frac{7+5\alpha}{3!}x^3$$
$$\overline{y}_{\alpha} = (5-\alpha) + (-3+\alpha)x + \frac{10-\alpha}{2}x^2 + \frac{17-5\alpha}{3!}x^3$$

Case VII: For (2, 1)-differentiable, consider Case VII in Algorithm 2.2, we find the following solutions for (2, 1)-system:

$$\underline{y}_{\alpha} = (3+\alpha) + (-1-\alpha)x + \frac{10-\alpha}{2}x^2 + \frac{7+5\alpha}{3!}x^3$$
$$\overline{y}_{\alpha} = (5-\alpha) + (-3+\alpha)x + \frac{8+\alpha}{2}x^2 + \frac{17-5\alpha}{3!}x^3$$

Case VIII: For (2, 2)-differentiable, consider Case VIII in Algorithm 2.2, we find the following solutions for (2, 2)-system:

$$\underline{y}_{\alpha} = (3+\alpha) + (-1-\alpha)x + \frac{8+\alpha}{2}x^2 + \frac{17-5\alpha}{3!}x^3$$
$$\overline{y}_{\alpha} = (5-\alpha) + (-3+\alpha)x + \frac{10-\alpha}{2}x^2 + \frac{7+5\alpha}{3!}x^3$$

Combining $\underline{y}$ and $\overline{y}$ represents the fuzzy solution of the fuzzy initial value problem (5.1) as $y_{\alpha}(x) = [\underline{y}(x), \overline{y}(x)]$, $\forall\ \alpha \in (0, 1]$, $x \in [0, 1]$. It is clear that for $\alpha = 1$, we find $\underline{y}(x) = \overline{y}(x)$, which is the

same exact solution of the non-fuzzy initial value problem . The results of the calculations for all the above cases are given in table 5.1:

x	$\underline{y}$	$\overline{y}$	exact solution
0	4	4	4.0000003
0.1	3.8470	3.8470	3.8472
0.2	3.7960	3.7960	3.7998
0.3	3.8590	3.8590	3.8795
0.4	4.0480	4.0480	4.1171
0.5	4.3750	4.3750	4.5555
0.6	4.8520	4.8520	5.2537
0.7	5.4910	5.4910	6.2925
0.8	6.3040	6.3040	7.7818
0.9	7.3030	7.3030	9.8711
1	8.5000	8.5000	12.7628

Also, the lower bound of solution $\underline{y}$ and the upper bound of solution $\overline{y}$ for different α-levels (where $\alpha \in (0, 1]$) which represent the fuzzy solution y are presented in Fig. 5.1:

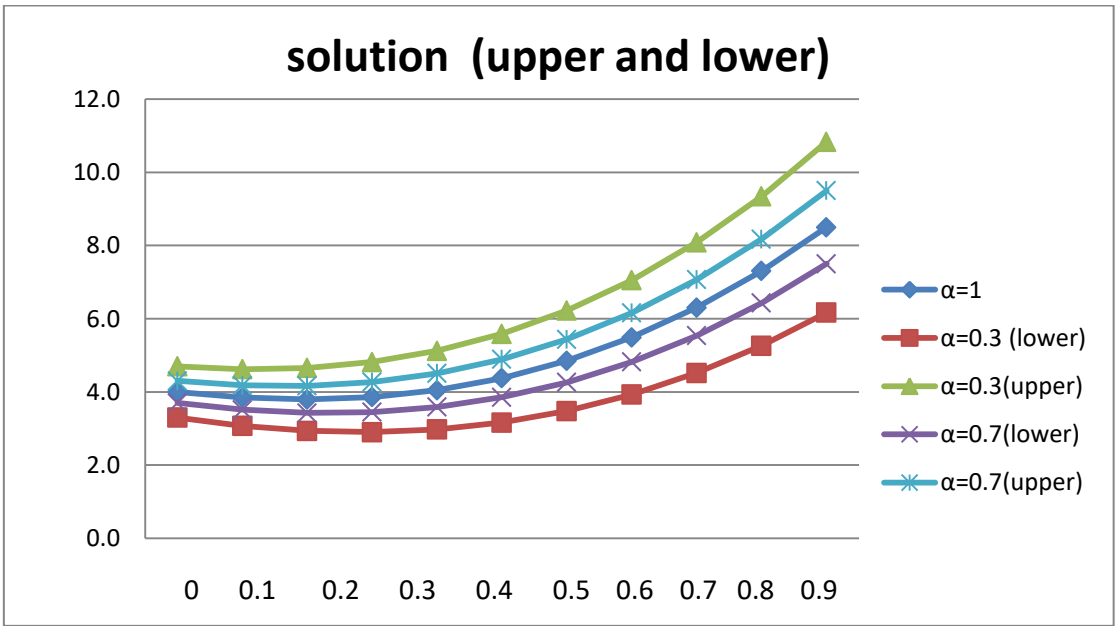

Figure 5.1-Upper and lower solutions of problem (5.1) –(5-2) for different values of α .

CONCLUSIONS

From the present study, we may conclude the following:

1. The accuracy of the results may be checked with $\alpha = 1$, in which the upper and lower solutions must be equal.
2. The crisp solution or the solution of the nonfuzzy boundary value problem is obtained from the fuzzy solution by setting $\alpha = 1$, and, therefore, the fuzzy boundary value problems may be considered as a generalization to the nonfuzzy boundary value problems.
3. The Residual Power Series Method proved its validity and accuracy in solving Fuzzy Deferential Equations.

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References

1. Chang S. L. and Zadeh L.A **1972**. On Fuzzy Mapping and Control, *IEEE Trans. System Man Cybernet*, **2**: 30-34.
2. Dubois, D. and Prade, H. **1988**. *Fuzzy Sets and Systems*; Theory and Applications , Academic Press, Inc.
3. Kaleva O. **1987**. Fuzzy differential equations. *Fuzzy Set Syst*., **24**: 301 - 317, doi: 10.1016/0165-0114(87)90029-7.
4. Seikkala S. **1987**. On the fuzzy initial value problem. *Fuzzy Set Syst*., **24**: 319 - 330, doi: 10.1016/0165-0114(87)90030-3.
5. Ibraheem R. H. **2015**. Solution of Seventh Order Boundary Value Problem by Differential Transformation Method., **1**: 299-320.
6. Rasheed M. A. , Laverty S. and Bannish B. **2019**. Numerical solutions of a linear age-structured population model, AIP Conference Proceedings, 2096, 020002 (2019). doi.org/10.1063/1.5097799.
7. Komashynska I., Al-Smadi M., Ateiwi A. and Al-Obaidy S. **2016**. Approximate analytical solution by residual power series method for system of Fredholm integral equations, Applied Mathematics and Information Sciences, **10**(2016): 975- 985. http://dx.doi.org/10. 18576/amis /100315.
8. Goetschel, R. and Voxman, W. **1983**. Topological properties of fuzzy numbers. *Fuzzy Set Syst*, **10**: 87 - 99.
9. Goetschel, R. and Voxman, W. **1986**. Elementary fuzzy calculus. *Fuzzy Set Syst*., **18**: 31 - 43.
10. Bede, B. and Gal, S.G. **2005**. Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations. *Fuzzy Set Syst*., **151**: 581 - 599, doi: 10.1016/j.fss.2004.08.001.
11. Chalco-Cano, Y. and Rom´an-Flores, H. **2008**. On new solutions of fuzzy differential equations. *Chaos Solitons Fract*. , **38**: 112 - 119.
12. Khastan A., Bahrami F., Ivaz K. ,New Results on multiple solutions for Nth-order fuzzy differential equations under generalized differentiability. Boundary Value Problems, 2009, Article ID 395714, 13 pages, doi: 10.1155/2009/395714.
13. Bede B., Rudas I.J., Bencsik A.L. **2007**. First order linear fuzzy differential equations under generalized differentiability. *Information Sciences*, **177**: 1648 - 1662.
14. Nguyen H.T. **1978**. A note on the extension principle for fuzzy set. *Journal Mathematical Analysis and Applications*, **64**: 369 - 380. 114(87)90030-3.doi: 10.1155/2009/395714.
15. Bassanezi R.C., de Barros L.C., Tonelli P.A. **2000**. Attractors and asymptotic stability for fuzzy dynamical systems. *Fuzzy Set Syst*, **113**: 473 - 483.
16. Abu Arqub O. **2013**. Series solution of fuzzy differential equations under strongly generalized differentiability, *Journal of Advanced Research in Applied Mathematics*, **5**(1): 31-52 doi: 10.5373/jaram.1447.051912.