DOI: 10.24996/ijs.2020.SI.1.12

ISSN: 0067-2904

Complex of Characteristic Zero in the Skew-Shape (8, 6, 3) / (u,1) where u = 1 and 2

Shaymaa N. Abd-Alridah, Haytham R. Hassan
Department of Mathematics, College of Science, Mustansiriyah University, Baghdad, Iraq

Received: 23/11/ 2019 Accepted: 15/ 3/2020

Abstract

In this work, we find the terms of the complex of characteristic zero in the case of the skew-shape (8,6, 3)/(u,1), where u = 1 and 2. We also study this complex as a diagram by using the mapping Cone and other concepts.

Keywords: Weyl module, Place polarization, Complex, Characteristic Zero

معقدة المميز الصفري في حالة شبه الشكل المنحرف(u,1)/(8,6,3) عندما u=1و2

شيماء نوري عبدالرضا،هيثم رزوقي حسن
قسم الرياضيات ،كلية العلوم ،الجامعة المستنصرية ،العراق ، بغداد

الخلاصة

في هذا العمل وجدنا الحدود المعقدة للمميز الصفري في حالة شبه الشكل المنحرف و u=1 , 2 عندما (8,6,3)/(u,1) ودرسنا هذه المعقدة كمخططات ايضا للتجزئة ذاتها وذلك باستخدام تطبيق كون وغيرها من المفاهيم

1. Introduction

Let R be commutative ring with identity, F is a free R-module, and D_iF is the divided power algebra of degree i.

The complex of characteristic zero in the case of the partitions (2,2,2), (3,3,3) and (4,4,3) was illustrated by other authors [1,2,3],, while others [4] presented the diagram of the complex of characteristic zero in the case of the partition (8,7,3). Other articles [5,6] found the resolution of Weyl module for characteristic zero in the case of the partition (8,7,3) by using the mapping Cone [7].

In this work, we used the same idea where we consider the complex of skew-shape (8,6,3)/(u,1) where u = 1 and 2 as well as the diagram of the complex of characteristic zero in skew-shape (8,6,3)/(u,1) where u = 1 and 2, using the mapping Cone after we illustrate the terms of that complex. The map $\partial_{ij}^{(f)}$ means the divided power of the place polarization ∂_{ij} where j must be less than i, with its Capelli identities [8]. So we need the identities below

$$\partial_{21}^{(u)} \circ \partial_{32}^{(V)} = \sum_{e\geq 0} \quad (-1)^e \partial_{32}^{(V-e)} \circ \partial_{21}^{(u-e)} \circ \partial_{31}^{(e)} \qquad \dots(1.1)$$

$$\partial_{32}^{(V)} \circ \partial_{21}^{(u)} = \sum_{e\geq 0} \quad \partial_{21}^{(u-e)} \circ \partial_{32}^{(V-e)} \circ \partial_{31}^{(e)} \qquad \dots(1.2)$$

2. Complex of characteristic zero for the skew-shape (8,6,3)/(1,1)

2.1 The terms

To find the terms of our case (p,q,r,t_1,t_2), we used the following [7]:

*Email: sh_8079@uomustansiriyah.edu.iq

$$0 \to \langle (p+|t|+2)|(q)|(r-|t|-2)\rangle \xrightarrow{\partial_3} \begin{matrix} \langle (p+|t|+2)|(q-t_1-1)|(r-t_2-1)\rangle \\ \oplus \\ \langle (p+t_1+1)|(q+t_2+1)|(r-|t|-2)\rangle \end{matrix} \xrightarrow{\partial_2} \begin{matrix} \langle (p)|(q+t_2+1)|(r-t_2-1)\rangle \\ \oplus \\ \langle (p+t_1+1)|(q+t_1-1)|(r)\rangle \end{matrix} \xrightarrow{\partial_1} \langle (p)\,|(q)|(r)\rangle$$

where $|t| = t_1 + t_2$.

In our case, i.e. (7,5,3;0,1), the complex of characteristic zero has the following terms:

$$0 \to D_{10}F \otimes D_5F \otimes D_0F \to \begin{matrix} D_{10}F \otimes D_4F \otimes D_1F \\ \oplus \\ D_8F \otimes D_7F \otimes D_0F \end{matrix} \to \begin{matrix} D_7F \otimes D_7F \otimes D_1F \\ \oplus \\ D_8F \otimes D_4F \otimes D_3F \end{matrix} \to D_7F \otimes D_5F \otimes D_3 \boldsymbol{F}$$

2.2 The diagram

Consider the following diagram**:**

$$\begin{array}{ccccc} D_{10}F \otimes D_5F \otimes D_0F & \xrightarrow{f_1} & D_{10}F \otimes D_4F \otimes D_1F & \xrightarrow{f_2} & D_8F \otimes D_4F \otimes D_3F \\ \downarrow h_1 & A & \downarrow h_2 & B & \downarrow h_3 \\ D_8F \otimes D_7F \otimes D_0F & \xrightarrow{g_1} & D_7F \otimes D_7F \otimes D_1F & \xrightarrow{g_2} & D_7F \otimes D_5F \otimes D_3F \end{array}$$

Where

$f_1(v): D_{10}F \otimes D_5F \otimes D_0F \to D_{10}F \otimes D_4F \otimes D_1F$, such that

$$f_1(v) = \partial_{32}(v)\,; v \in D_{10}F \otimes D_5F \otimes D_0F$$

$h_1(v): D_{10}F \otimes D_5F \otimes D_0F \to D_8F \otimes D_7F \otimes D_0F$, such that $h_1(v) = \partial_{21}^{(2)}(v)$; $v \in D_{10}F \otimes D_5F \otimes D_0F$

$g_2(v): D_7F \otimes D_7F \otimes D_1 \to D_7F \otimes D_5F \otimes D_3F$, such that

$$g_2(v) = \partial_{32}^{(2)}(v); v \in D_7F \otimes D_7F \otimes D_1F$$

$h_2(v): D_{10}F \otimes D_4F \otimes D_1F \to D_7F \otimes D_7F \otimes D_1F$, such that

$$h_2(v) = \partial_{21}^{(3)}(v); v \in D_{10}F \otimes D_4F \otimes D_1F$$

$h_3(v): D_8F \otimes D_4F \otimes D_3F \to D_7F \otimes D_5F \otimes D_3F$, such that

$$h_3(v) = \partial_{21}(v); v \in D_8F \otimes D_4F \otimes D_3F$$

And we define $g_1(v): D_8F \otimes D_7F \otimes D_0F \to D_7F \otimes D_7F \otimes D_1F$ by $g_1(v) = \frac{1}{3}\,\partial_{32} \circ \partial_{21} - \partial_{31}$; $v \in D_8F \otimes D_7F \otimes D_0F$

Proposition (2.1): The diagram A is commutative.

Proof: We must prove that $(h_2 \circ f_1)(v) = (g_1 \circ h_1)(v)$

$(h_2 \circ f_1)(v) = \partial_{21}^{(3)} \circ \partial_{32}(v) = \partial_{32} \circ \partial_{21}^{(3)} - \partial_{21}^{(2)} \circ \partial_{31}$, and

$$(g_1 \circ h_1)(v) = \left(\frac{1}{3}\,\partial_{32} \circ \partial_{21} - \partial_{31}\right) \circ \partial_{21}^{(2)} = \partial_{32} \circ \partial_{21}^{(3)} - \partial_{21}^{(2)} \circ \partial_{31}$$

Implies that $(h_2 \circ f_1)(v) = (g_1\, h_1)(v)$. ■

Now we define

$f_2(v): D_{10}F \otimes D_4F \otimes D_1F \to D_8F \otimes D_4F \otimes D_3F$ by

$$f_2(v) = \frac{1}{3}\,\partial_{21}^{(2)} \circ \partial_{32}^{(2)} + \frac{1}{2}\,\partial_{21} \circ \partial_{32} \circ \partial_{31} + \partial_{31}^{(2)}$$

Proposition (2.2): The diagram B is commutative.

$$(h_3 \circ f_2)(v) = \partial_{21}\left(\frac{1}{3}\,\partial_{21}^{(2)} \circ \partial_{32}^{(2)} + \frac{1}{2}\,\partial_{21} \circ \partial_{32} \circ \partial_{31} + \partial_{31}^{(2)}\right)$$
$$= \partial_{21}^{(3)} \circ \partial_{32}^{(2)} + \partial_{21}^{(2)} \circ \partial_{32} \circ \partial_{31} + \partial_{21} \circ \partial_{31}^{(2)}$$

Where

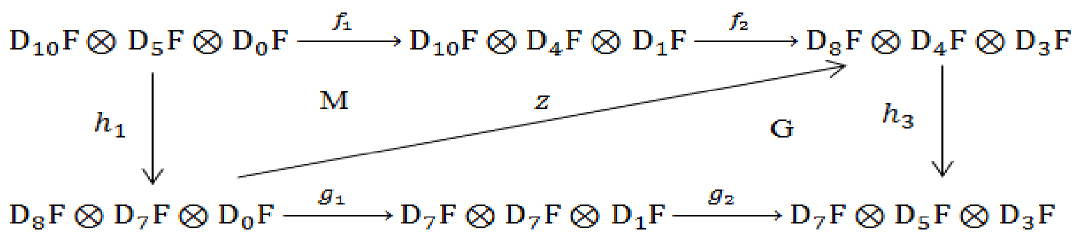
$$(g_2 \circ h_2)(v) = \partial_{32}^{(2)} \circ \partial_{21}^{(3)}$$

By Capelli identity (1.2), we get

$$(g_2 \circ h_2)(v) = \partial_{21}^{(3)} \circ \partial_{32}^{(2)} + \partial_{21}^{(2)} \circ \partial_{32} \circ \partial_{31} + \partial_{21} \circ \partial_{31}^{(2)}$$

Implies that $(h_3 \circ f_2)(v) = (g_2 \circ h_2)(v)$.

Now consider the following diagram:



Define $z(v)$: $D_8F \otimes D_7F \otimes D_0F \to D_8F \otimes D_4F \otimes D_3F$ *by*

$$z(v) = \partial_{32}^{(3)} \text{ where } v \in D_8F \otimes D_7F \otimes D_0F$$

Proposition (2.3): The diagram M is commutative.

$$(f_2 \circ f_1)(v) = \left(\frac{1}{3}\, \partial_{21}^{(2)} \circ \partial_{32}^{(2)} + \frac{1}{2}\, \partial_{21} \circ \partial_{32} \circ \partial_{31} + \partial_{31}^{(2)}\right) \circ \partial_{32}(v)$$

$$= \partial_{21}^{(2)} \circ \partial_{32}^{(2)} + \partial_{21} \circ \partial_{31} \circ \partial_{32}^{(2)} + \partial_{31}^{(2)} \circ \partial_{32}$$

$$(z \circ h_1)(v) = \partial_{32}^{(3)} \circ \partial_{21}^{(2)}(v)$$

And from (1.2)

$$= \partial_{21}^{(2)} \circ \partial_{32}^{(2)} + \partial_{21} \circ \partial_{31} \circ \partial_{32}^{(2)} + \partial_{31}^{(2)} \circ \partial_{32}$$

Which implies that $(f_2 \circ f_1)(v) = (z \circ h_1)(v)$, which means that the diagram M is commutative

Proposition (2.4): The diagram G is commutative.

Proof: From (1.1), we get

$(h_3 \circ z)(v) = \partial_{21} \circ \partial_{32}^{(3)}(v) = \partial_{32}^{(3)} \circ \partial_{21} - \partial_{32}^{(2)} \circ \partial_{31}$

But $(g_2 \circ g_1)(v) = \partial_{32}^{(2)} \circ \left(\frac{1}{3}\, \partial_{32} \circ \partial_{21} - \partial_{31}\right) = \partial_{32}^{(3)} \circ \partial_{21} - \partial_{32}^{(2)} \circ \partial_{31}$

Which implies that $(h_3 \circ z)(v) = (g_2 \circ g_1)(v)$, which means that the diagram G is commutative

Eventually, we define the maps σ_1, σ_2 and σ_3 where:

$$\sigma_3 : \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_5\mathcal{F} \otimes \mathcal{D}_0\mathcal{F} \longrightarrow \begin{array}{c} \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_4\mathcal{F} \otimes \mathcal{D}_1\mathcal{F} \\ \oplus \\ \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_0\mathcal{F} \end{array}$$

$$\sigma_3(x) = \big(f_1(x), h_1(x)\big); \forall\, x \in D_{10}F \otimes D_5F \otimes D_0F$$

$$\sigma_2 : \begin{array}{c} \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_4\mathcal{F} \otimes \mathcal{D}_1\mathcal{F} \\ \oplus \\ \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_0\mathcal{F} \end{array} \longrightarrow \begin{array}{c} \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_4\mathcal{F} \otimes \mathcal{D}_3\mathcal{F} \\ \oplus \\ \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_1\mathcal{F} \end{array}$$

$$\sigma_2((x_1, x_2)) = \big(f_2(x_1) - z(x_2), g_1(x_2) - h_2(x_1)\big);$$

$\forall\, x \in D_{10}F \otimes D_4F \otimes D_1F \quad D_8F \otimes D_7F \otimes D_0F$

$$\sigma_1 : \begin{array}{c} \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_4\mathcal{F} \otimes \mathcal{D}_3\mathcal{F} \\ \oplus \\ \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_1\mathcal{F} \end{array} \longrightarrow \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_5\mathcal{F} \otimes \mathcal{D}_3\mathcal{F}$$

$$\sigma_1((x_1, x_2)) = \big(h_3 + g_2(x_2)\big)$$

$\forall\, x \in D_8F \otimes D_4F \otimes D_3F \quad D_7F \otimes D_7F \otimes D_1F$

Proposition (2.5):

$$0 \to D_{10}F \otimes D_5F \otimes D_0F \xrightarrow{\sigma_3} \begin{matrix} D_{10}F \otimes D_4F \otimes D_1F \\ \oplus \\ D_8F \otimes D_7F \otimes D_0F \end{matrix} \xrightarrow{\sigma_2} \begin{matrix} D_8F \otimes D_4F \otimes D_3F \\ \oplus \\ D_7F \otimes D_7F \otimes D_1F \end{matrix}$$
$$\xrightarrow{\sigma_1} D_7F \otimes D_5F \otimes D_3\, F$$

is complex.

Proof: From the definition of place polarization, we have ∂_{21} and ∂_{32} are injectives [9], and we get σ_3 is injective.

Now

$$(\sigma_2 \circ \sigma_3)(x) = \sigma_2\big(f_1(x), h_1(x)\big)$$
$$= \sigma_2\left(\partial_{32}(x), {\partial_{21}}^{(2)}(x)\right)$$
$$= \left(f_2\big(\partial_{32}(x)\big) - z\left({\partial_{21}}^{(2)}(x)\right), g_1\left({\partial_{21}}^{(2)}(x)\right) - h_2\big(\partial_{32}(x)\big)\right)$$

So

$f_2\big(\partial_{32}(x)\big) - z\left(\partial_{21}^{(2)}(x)\right)$

$= \left(\frac{1}{3}\,\partial_{21}^{(2)} \circ \partial_{32}^{(2)} + \frac{1}{2}\,\partial_{21} \circ \partial_{32} \circ \partial_{31} + \partial_{31}^{(2)}\right) \circ \partial_{32}(x) - \partial_{32}^{(3)} \circ {\partial_{21}}^{(2)}(x)$

$= ({\partial_{21}}^{(2)} \circ \partial_{32}^{(3)} + \partial_{21} \circ \partial_{32} \circ^{(2)} \partial_{31} + \partial_{32}^{(2)} \circ \partial_{31} - \partial_{32}^{(3)} \circ {\partial_{21}}^{(2)})(x)$

$= ({\partial_{21}}^{(2)} \circ \partial_{32}^{(3)} + \partial_{21} \circ \partial_{32}^{(2)} \circ \partial_{31} + \partial_{32}^{(2)} \circ \partial_{31} - \partial_{21}^{(2)} \circ \partial_{32}^{(3)} - \partial_{21} \circ \partial_{32}^{(2)} \circ \partial_{31} -$
$\partial_{32}^{(2)} \circ \partial_{31})(x) = 0$

$g_1\left(\partial_{21}^{(2)}(x)\right) - h_2\big(\partial_{32}(x)\big)$

$= \left(\frac{1}{3}\partial_{32} \circ \partial_{21} - \partial_{31}\right) \circ {\partial_{21}}^{(2)}(x) - \partial_{21}^{(3)} \circ \partial_{32}(x)$

$= (\partial_{32} \circ \partial_{21}^{(3)} - \partial_{31} \circ {\partial_{21}}^{(2)} - \partial_{21}^{(3)} \circ \partial_{32})(x)$

By using (1.2) again

$= (\partial_{21}^{(3)} \circ \partial_{32} + {\partial_{21}}^{(2)} \circ \partial_{31} - \partial_{31} \circ {\partial_{21}}^{(2)} - \partial_{21}^{(3)} \circ \partial_{32})(x)$

$= 0$

So, $(\sigma_2 \circ \sigma_3)(x) = 0$.

And

$(\sigma_1 \circ \sigma_2)(x_1, x_2) = \sigma_1(f_2(x_1) - z(x_2), g_1(x_2) - h_2(x_1))$

$= \sigma_1(\frac{1}{3}\,\partial_{21}^{(2)} \circ \partial_{32}^{(2)} + \frac{1}{2}\,\partial_{21} \circ \partial_{32} \circ \partial_{31} + \partial_{31}^{(2)})(x_1) - \partial_{32}^{(3)}(x_2), (\frac{1}{3}\partial_{32} \circ \partial_{21} - \partial_{31})(x_2) -$
$\partial_{21}^{(3)}(x_1))$

$= \partial_{21} \circ (\frac{1}{3}\,\partial_{21}^{(2)} \circ \partial_{32}^{(2)} + \frac{1}{2}\,\partial_{21} \circ \partial_{32} \circ \partial_{31} + \partial_{31}^{(2)})(x_1) - \partial_{32}^{(3)}(x_2)) + {\partial_{32}}^{(2)} \circ$

$$((\frac{1}{3}\,\partial_{32} \circ \partial_{21} - \partial_{31}(x_2) - \partial_{21}^{(3)}(x_1))$$

$= \left(\partial_{21}^{(3)} \circ \partial_{32}^{(2)} + \partial_{21}^{(2)} \circ \partial_{32} \circ \partial_{31} + \partial_{21} \circ \partial_{31}^{(2)} - \circ\, \partial_{32}^{(2)} \circ \partial_{21}^{(3)}\right)(x_1) +$
$\left(\partial_{32}^{(3)} \circ \partial_{21} - \partial_{32}^{(2)} \circ \partial_{31} - \partial_{21} \circ \partial_{32}^{(3)}\right)(x_2)$

$= (\partial_{21}^{(3)} \circ \partial_{32}^{(2)} + \partial_{21}^{(2)} \circ \partial_{32} \circ \partial_{31} + \partial_{21} \circ {\partial_{31}}^{(2)} - \partial_{21}^{(3)} \circ \partial_{32}^{(2)} - \partial_{21}^{(2)} \circ \partial_{32} \circ \partial_{31} - \partial_{21} \circ$
$\partial_{31}^{(2)})\,(x_1)$
$+\left(\partial_{21} \circ \partial_{32}^{(3)} + \partial_{32}^{(2)} \circ \partial_{31} - \partial_{32}^{(2)} \circ \partial_{31} - \partial_{21} \circ \partial_{32}^{(3)}\right)(x_2) = 0$ ■

3. Complex of characteristic zero for the skew-shape (8,6,3)/(2,1)

3.1 The terms

The authors in a previous work [7] gave the terms in the general case (p,q,r,t_1,t_2), as follows

$$0 \to \langle (p+|t|+2)\,|(q-t_1-1))|(r-t_2-1\rangle \to \begin{matrix} \langle (p+t_1+1)\,|(q-t_1-1)|(r)\rangle \\ \oplus \\ \langle (p)\,|(q+t_2+1)|(r-t_2-1)\rangle \end{matrix} \to$$

$\langle (p)\,|(q)|(r)\rangle$

Where $|t| = t_1 + t_2$.

In our case, i.e (6,5,3;1,1), the complex of characteristic zero has the terms as follow:

$$0 \to D_{10}F \otimes D_3F \otimes D_1F \to \begin{matrix} D_8F \otimes D_3F \otimes D_3F \\ \oplus \\ D_6F \otimes D_7F \otimes D_1F \end{matrix} \to D_6F \otimes D_5F \otimes D_3\, \boldsymbol{F}$$

3.2 Complex of characteristic zero as a diagram

Consider the following diagram

$$\begin{array}{ccc} D_{10}F \otimes D_3F \otimes D_1F & \xrightarrow{u_1} & D_8F \otimes D_3F \otimes D_3F \\ k_1 \downarrow & N & \downarrow k_2 \\ D_6F \otimes D_7F \otimes D_1F & \xrightarrow{u_2} & D_6F \otimes D_5F \otimes D_3F \end{array}$$

Where

$k_1(v): D_{10}F \otimes D_3F \otimes D_1F \to D_6F \otimes D_7F \otimes D_1F$, such that

$$k_1(v) = \partial_{21}^{(4)}(v) \quad ; v \in D_{10}F \otimes D_3F \otimes D_1F$$

$k_2(v): D_8F \otimes D_3F \otimes D_3F \to D_6F \otimes D_5F \otimes D_3F$, such that

$$k(v) = \partial_{21}^{(2)}(v)\, ; v \in D_8F \otimes D_3F \otimes D_3F$$

$$u_2(v): D_6F \otimes D_7F \otimes D_1 \to D_6F \otimes D_5F \otimes D_3F$$
$$u_2(v) = \partial_{32}^{(2)}(v) \quad ; v \in D_6F \otimes D_7F \otimes D_1F$$

Now we define $u_1(v): D_{10}F \otimes D_3F \otimes D_1F \to D_8F \otimes D_3F \otimes D_3F$ by

$$u_1(v) = \frac{1}{6}\,\partial_{21}^{(2)} \circ \partial_{32}^{(2)} + \frac{1}{3}\,\partial_{21} \circ \partial_{32} \circ \partial_{31} + \partial_{31}^{(2)}$$

Proposition (3.1): The diagram N is commutative.

Proof: $(u_2 \circ k_1)(v) = \partial_{32}^{(2)} \circ \partial_{21}^{(4)}(v)$

By using (1.1)

$= \partial_{21}^{(4)} \circ \partial_{32}^{(2)} + \partial_{21}^{(3)} \circ \partial_{32} \circ \partial_{31} + \partial_{21}^{(2)} \circ \partial_{31}^{(2)}$

We have

$$k_2 \circ u_1(v) = \partial_{21}^{(2)} \circ \left(\frac{1}{6}\,\partial_{21}^{(2)} \circ \partial_{32}^{(2)} + \frac{1}{3}\,\partial_{21} \circ \partial_{32} \circ \partial_{31} + \partial_{31}^{(2)} \right)$$
$$= \partial_{21}^{(4)} \circ \partial_{32}^{(2)} + \partial_{21}^{(3)} \circ \partial_{32} \circ \partial_{31} + \partial_{21}^{(2)} \circ \partial_{31}^{(2)}$$

which implies that $k_2 \circ u_1(v) = u_2 \circ k_1(v)$, which means that the diagram N is commutative. ■

Eventually, we define the maps φ_1 and φ_2, as follows:

$\varphi_2(x_1) = \big(-u_1(x_1), k_1(x_1)\big)\, ; \forall\, x_1 \in D_{10}F \otimes D_3F \otimes D_1F$

$\varphi_1((x_1, x_2)) = \big(k_2(x_1) + u_2(x_2)\big)\; ; \forall\, x_1 \in D_8F \otimes D_3F \otimes D_3F, x_2 \in D_6F \otimes D_7F \otimes D_1F$

Proposition (3.2):

$$0 \to D_{10}F \otimes D_3F \otimes D_1F \overset{\varphi_2}{\to} \begin{matrix} D_8F \otimes D_3F \otimes D_3F \\ \oplus \\ D_6F \otimes D_7F \otimes D_1F \end{matrix} \overset{\varphi_1}{\to} D_6F \otimes D_5F \otimes D_3\, \boldsymbol{F}$$

is complex.

Proof: As previously shown [9], place polarizations ∂_{21} ∂_{32} ,and ∂_{31} as injectives, which implies that φ_2 is injective

$$\varphi_1 \circ \varphi_2(x_1) = \varphi_1\left(-\left(\frac{1}{6}\,\partial_{21}^{(2)} \circ \partial_{32}^{(2)} + \frac{1}{3}\,\partial_{21} \circ \partial_{32} \circ \partial_{31} + {\partial_{31}}^{(2)}\right)(x_1), \partial_{21}^{(4)}(x_1)\right)$$

$$= \left(\partial_{21}^{(2)}(x_1) + \partial_{32}^{(2)}(x_1)\right) \circ \left(\frac{1}{6}\,\partial_{21}^{(2)} \circ \partial_{32}^{(2)} + \frac{1}{3}\,\partial_{21} \circ \partial_{32} \circ \partial_{31} + {\partial_{31}}^{(2)}\right)(x_1), \partial_{21}^{(4)}(x_1))$$

$$= \partial_{21}^{(2)} \circ \left(\frac{1}{6}\,\partial_{21}^{(2)} \circ \partial_{32}^{(2)} + \frac{1}{3}\,\partial_{21} \circ \partial_{32} \circ \partial_{31} + {\partial_{31}}^{(2)}\right)(x_1) + \partial_{32}^{(2)} \circ \partial_{21}^{(4)}(x_1)$$

$$= -\,\partial_{21}^{(4)} \circ \partial_{32}^{(2)} - \partial_{21}^{(3)} \circ \partial_{32} \circ \partial_{31} - \partial_{21}^{(2)} \circ \partial_{31}^{(2)} + \partial_{21}^{(4)} \circ \partial_{32}^{(2)} + \partial_{21}^{(3)} \circ \partial_{32} \circ \partial_{31} + \partial_{21}^{(2)} \circ \partial_{31}^{(2)} = 0$$

■

Acknowledgments

The authors thank Mustansiriyah University / College of Science / Department of Mathematics for supporting this work.

References

1. David A.B. **1986**. *A characteristic-free realizations of the Giambelli and Jacoby-Trudi determinatal identities,* proc. of K.I.T. workshop on Algebra and Topology, Springer-Verlag.
2. Haytham R.H. **2006**. Application of the characteristic-free resolution of Weyl Module to the Lascoux resolution in the case (3,3,3), Ph. D. thesis, Universita di Roma "Tor Vergata".
3. Haytham R.H. **2012**. The Reduction of Resolution of Weyl Module From Characteristic-Free Resolution in Case (4,4,3), *Ibn Al-Haitham Journal for Pure and Applied Science*, **25**(3): 341-355.
4. Haytham, R.H. and Niran S.J. **2018**. A Complex of Characteristic Zero in the Case of the Partition (8,7,3). *Science International* (Lahore), **30**(4): 639-641.
5. Haytham, R.H. and Niran S.J. **2019**. Characteristic Zero Resolution of Weyl Module in the Case of the Partition (8,7,3), IOP Conf. Series*: Materials Science and Engineering*, **571**: 1-10, doi:10.1088/1757-899X/571/1/012039.
6. Niran S.J. **2019**. Application of the characteristic-free resolution of Weyl module to the Lascoux resolution in the case of partition (8,7,3), Ph.D. thesis, College of Science, Mustansiriyah University, 2019.
7. David A.B and Mari A. **2010**. Resolution of Three-Rowed Skew-and Almost Shew-Shapes in Characteristic Zero, *European Journal of Combinatorics*, **31**: 325-335.
8. David A.B. and Rota G.C. **2001**. Approaches to Resolution of Weyl Modules, *Adv. in applied Math.*, **27**: 82-191.
9. Giandomenico B. and David A. B. **2006**. *Threading Homology through Algebra:* Selected Patterns, Clarendon Press, Oxford.