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Some Properties of Fuzzy Anti-Inner Product Spaces

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Abstract:
In this paper, the definition of fuzzy anti-inner product in a linear space is introduced. Some results of fuzzy anti-inner product spaces are given, such as the relation between fuzzy inner product space and fuzzy anti-inner product. The notion of minimizing vector is introduced in fuzzy anti-inner product settings.

Keywords: fuzzy anti-inner product, fuzzy anti-norm, fuzzy inner product.

بعض خصائص الفضاءات ضد الجداء الداخلي الضبابية

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I. Introduction

Kohli and Kumar, in 1993 [1], introduced the definition of the fuzzy inner product space and fuzzy co-inner product space. In 1997, Alsina et al [2] introduced the ideal of probabilistic inner product space. After that, in 2010, Hasankhain *et al*. [3] introduced some properties of fuzzy Hilbert spaces and norm of operators. In 2013, the fuzzy real inner product space and its properties were proved by Mukherjee and Bag [4]. Finally, a note on fuzzy Hilbert spaces was introduced by Daraby *et al*. in 2016 [5].

II. Preliminaries

This section consists of some definitions and results that will be needed later in this paper.

Definition (2.1) [6]

Assume that V is a linear space over the field C of complex numbers. A mapping $M^*: V^2 \text{x}\, C \rightarrow I$ satisfies the following conditions for all x, y, z in V and t, s in C:

(FIP1) $M^*(x+y, z, |t|+|s|) \geq \min\{M^*(x, z, |t|), M^*(y, z, |s|)\}$

(FIP2) $M^*(x, y, |ts|) \geq \min\{M^*(x, x, |t|^2), M^*(y, y, |s|^2)\}$

(FIP3) $M^*(x, y, t) = M^*(y, x, \bar{t})$

(FIP4) $M^*(\alpha x, y, t) = M^*\left(x, y, \frac{t}{\alpha}\right)$,$0 \neq \alpha \in C$

(FIP5) for all $t \in C \backslash R^+, M^*(x, x, t) = 0$

(FIP6) $\forall\, t > 0, M^*(x, x, t) = 1$ if and only if $x = \underline{0}$

(FIP7) $M^*(x, x, .): R \rightarrow I$ is a monotonic non-decreasing function of R and $\lim_{t\rightarrow\infty} M^*(x, x, t) = 1$,

where M^* is called a fuzzy inner product function on V and (V, M^*) is called a fuzzy inner product space.

Definition (2.2) [7]

Let V be a linear space over a field F. A fuzzy set $\mathcal{N}: V \text{ x } R \longrightarrow I$ such that the following holds for all u, v in V and c in F :

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($\mathcal{N}$1) for all $t \in R$ with $t \leq 0$, $\mathcal{N}(u,t) = 1$;
($\mathcal{N}$2) for all $t \in R$ with $t > 0$, $\mathcal{N}(u,t) = 0$ if and only if $u = \underline{0}$;
($\mathcal{N}$3) for all $t \in R$ with $t > 0$, $\mathcal{N}(cu,t) = \mathcal{N}(u, \frac{t}{|c|})$ if $0 \neq c \in F$;
($\mathcal{N}$4) for all $s,t \in R$, $\mathcal{N}(u+v, s+t) \leq \max\{\mathcal{N}(u,s), \mathcal{N}(v,t)\}$;
($\mathcal{N}$5) $\mathcal{N}(u,t)$ is a decreasing function of $t \in R$ and $\text{Lim}_{t\to\infty} \mathcal{N}(u,t) = 0$,
where $\mathcal{N}$ is said to be a fuzzy anti-norm on V and $(V, \mathcal{N})$ is called a fuzzy anti-normed linear space.
Later on, the following condition of fuzzy norm $\mathcal{N}$ will be required:
($\mathcal{N}$6) for all $t \in R$ with $t > 0$, $\mathcal{N}(u,t) < 1$ implies $u = \underline{0}$.

Definition (2.3) [7]

Le $\mathcal{N}$ be a fuzzy anti-norm on V satisfying ($\mathcal{N}$6). Define
$\|u\|_\alpha^* = \inf\{ t > 0 : \mathcal{N}(u,t) < \alpha, \alpha \in (0,1]\}$.

Theorem (2.4) [7]

Let $(V, \mathcal{N})$ be a fuzzy anti-normed linear space. Then $\{\|u\|_\alpha^* : \alpha \in (0,1]\}$ is a decreasing family of norms on V.

III. Fuzzy anti-inner product space

The definition of fuzzy anti-inner product space on a complex linear space is introduced and some of its results are investigated.

Definition (3.1) [8]

Assume that V is a linear space over the filed C of complex numbers. Define
$M^\circ : V^2 \text{ x } C \to I$ to be a mapping such that the following holds for all x , y in V and t, s in C:
Fa-IP1) $M^\circ(x+y, z, |t|+|s|) \leq \max\{M^\circ(x,z,|t|), M^\circ(y,z,|s|)\}$ (
Fa-IP2) $M^\circ(x,y,|ts|) \leq \max\{ M^\circ(x,x,|t|^2), M^\circ(y,y,|s|^2)\}$ (
Fa-IP3) $M^\circ(x,y,t) \leq M^\circ(y,x,\bar{t})$ (
Fa-IP4) $M^\circ(\alpha x, y, t) \leq M^\circ\left(x, y, \frac{t}{|\alpha|}\right)$, $0 \neq \alpha \in C$ (
Fa-IP5) $M^\circ(x,x,t) = 1 \quad \forall\, t \in C \setminus R^+$ (
Fa-IP6) $\forall\, t > 0$, $M^\circ(x,x,t) = \underline{0}$ if and only if $x = \underline{0}$ (
(Fa-IP7) $M^\circ(x,x,.) : R \to I$ is a monotonic non-increasing function of R
and $\lim_{t\to\infty} M^\circ(x,x,t) = 0$.
Where M° is called a fuzzy anti-inner product function on V and (V, M°) is called a fuzzy anti-inner product space.

Example (3.2)

Assume that $(V, <,>)$ is an inner product space over C. A function $M^\circ : V^2 \text{ x } C \to I$ is defined by

$$M^\circ(x,y,t) = \begin{cases} 1 & \text{if } t \leq |<x,y>| \\ 0 & \text{if } t > |<x,y>| \\ 1 & \forall\, t \in C \setminus R \end{cases}$$

Then M° is a fuzzy anti- inner product space on V.

Proof:

(Fa-IP1) Consider the following cases:
Case (i) if one of $|t| \leq |<x,z>|$, $|s| \leq |<y,z>|$ holds, then
$\max\{ M^\circ(x,z,|t|), M^\circ(y,z,|s|)\} = 1$ and obviously
$M^\circ(x+y,z,|t|+|s|) = 0 \leq \max\{ M^\circ(x,z,|t|), M^\circ(y,z,|s|)\}$
Case (ii) let $|t| > |<x,z>|$ and $|s| > |<y,z>|$
$\Rightarrow |t| + |s| > |<x+y, z>|$
$\therefore M^\circ(x+y,z,|t|+|s|) = 0 \leq \max\{ M^\circ(x,z,|t|), M^\circ(y,z,|s|)\}$
(Fa-IP2) We observe that $|s|^2 > |<x,x>|$ and $|t|^2 > |<y,y>|$
$\Rightarrow |s|^2.|t|^2 > |<x,x>| . |<y,y>| = \|x\|^2 . \|y\|^2$
$|s|.|t| > \|x\| . \|y\| \quad \Rightarrow$
$|st| > \|x\| . \|y\| \quad \Rightarrow$
so (Fa-IP2) follows.
Next (Fa-IP3) (Fa-IP5) and (Fa-IP7) hold obviously.
(Fa-IP4) If $t \in (C \setminus R^+)$, then the result is obvious.

For $t \in R^+, 0 \neq \alpha \in C$, then the property follows from the fact that

$$|< \alpha x, y >| = |\alpha||< x, y >|$$

(Fa-IP6) If $x = 0 \Rightarrow < x, x > = 0 \Rightarrow \forall\, t > 0\,, |< x, x >| > t$

$$\Rightarrow M^\circ(x, x, t) = 0$$

Conversely, if $\forall\, t > 0,\ M^\circ(x, x, t) = 0 \Rightarrow \forall\, t > 0\,, |< x, x >| > t$

$\Rightarrow\ < x, x > = 0 \Rightarrow x = 0.$

This completes the proof.

Proposition (3.3) [8]

Let (V, M°) be a fuzzy anti-inner product space. Then for x, y, z in V and s, t in C

(i) $M^\circ(x, y + z, |t| + |s|) \leq M^\circ(x, y, |t|) \vee M^\circ(x, z, |s|)$

(ii) For $\alpha \in C$ and $\alpha \neq 0$, $M^\circ(\alpha x, y, t) = M^\circ(x, \alpha y, t)$

(iii) $\forall\, t \in R$ and $t > 0$, $M^\circ(\underline{0}, \underline{0}, t) \leq M^\circ(x, y, t)$

Note (3.4) [8]

Assume that M° satisfies the condition:

(Fa-IP8) $\forall\, t > 0\,, M^\circ(x, x, t^2) < 1 \Longrightarrow x = \underline{0}$

Let (V, M°) be a fuzzy anti-inner product space, satisfying (Fa-IP8).Then $\forall\, \alpha \in (0,1)$,
$\|x\|_\alpha^* = \wedge\ \{ t > 0 : M^\circ(x, x, t^2) \leq 1 - \alpha \}$ is a crisp norm on V, called the
α –anti norm on V generated from M°.

In the sequel we shall consider the following condition:

(Fa-IP9) $\forall\, x, y$ in V and p, q in R,

$M^\circ(x + y, x + y, 2\, q^2) \vee M^\circ(x - y, x - y, 2\, p^2) \leq M^\circ(x, x, p^2) \vee M^\circ(y, y, q^2)$

Theorem (3.5) [8]

Let M° be a fuzzy anti-inner product on the V defined function $\mathcal{N}$, as follows:

$\mathcal{N}(x, t) = M^\circ(x, x, t^2) \quad \forall\, t \in R$ and $t > 0$

$= 1 \qquad\qquad \forall\, t \in R$ and $t \leq 0$

Then $\mathcal{N}$ is a fuzzy anti-norm on V.

From now on, if (Fa-IP8) and (Fa-IP9) hold for each $\alpha \in (0,1)$, then
$\|x\|_\alpha^* = \wedge \{ t > 0 : M^\circ(x, x, t^2) \leq 1 - \alpha \}$ is an ordinary anti-norm on V satisfying parallelogram law.

So, by using polarization identity, one can get an ordinary inner product, called the α-anti-inner product, as follows:

$$< x, y >_\alpha^* = X_\alpha^* + i\, Y_\alpha^*$$

where $\quad X_\alpha^* = \frac{1}{4}(\|x + y\|_\alpha^{*2} - \|x - y\|_\alpha^{*2})$

and $\quad Y_\alpha^* = \frac{1}{4}(\|x + iy\|_\alpha^{*2} - \|x - iy\|_\alpha^{*2})$, where $\alpha \in (0,1)$.

Definition (3.6)

V is said to be anti-level complete (AL-complete). If (V, M°) is a fuzzy anti-inner product space satisfying (Fa-IP8) for any $\in (0,1)$, then every Cauchy sequence converges in V w.r.t the α-anti-norm $\|x\|_\alpha^*$ generated by the fuzzy anti-norm $\mathcal{N}$ which is induced by fuzzy anti-inner product M°.

Theorem (3.7) (Minimizing vector)

Let (V, M°) be a fuzzy anti-inner product space satisfying (Fa-IP8) and (Fa-IP9), let M ($\neq \emptyset$) be the convex subset of V which is anti-level complete, and let $x \in V$. Then for each $\alpha \in (0,1)$, $\exists$ y_α^* in M such that

$m_{y_\alpha^*}^{(\alpha)^*} = \inf_{y \in M} \{ m_y^{(\alpha)^*} \}$,where

$$m_y^{(\alpha)^*} = \wedge \{ t \in R^+, \mathcal{N}(x - y, t) \leq 1 - \alpha \}$$

M°. $\mathcal{N}$ is the fuzzy anti-norm induced by fuzzy anti-inner

Proof:

We note that if (V, M°) is a fuzzy anti-inner product space, then for each $\alpha \in (0,1)$, $(V, \|.\|_\alpha^*)$ is a crisp anti- normed linear space satisfying the parallelogram law. Again

$m_y^{(\alpha)^*} = \|x - y\|_\alpha^*.$

Hence the result follows from the corresponding crisp minimization vector theorem in $(V, \|.\|_\alpha^*)$.

Theorem (3.8)

$M°$ is a fuzzy anti-inner product space on V if and only if $1 - M°$ is a fuzzy inner product space on V .

Proof: For x, y, z in V and t, s in C

(Fa-IP1) $\max\{ M°(x, z, |t|) , M°(y, z, |s|)\}$

$= 1 - \max \{ M°(x, z, |t|) , M°(y, z, |s|)\}$

$= \min\{ M^*(x, z, |t|) , M^*(y, z, |s|)\}$

$\leq M^*(x + y, z, |t| + |s|)$

$$\geq 1 - M^*(x + y, z, |t| + |s|) = M°(x + y, z, |t| + |s|)$$

(Fa-IP2) $\max \{ M°(x, x, |s|^2) , M°(y, y, |t|^2)\}$

$= 1 - \max \{M°(x, x, |s|^2) , M°(y, y, |t|^2)\}$

$= \min \{M^*(x, x, |s|^2) , M^*(y, y, |t|^2)\}$

$\leq M^*(y, z, |st|)$

$\geq 1 - M^*(y, z, |st|) = M°(y, z, |st|)$

(Fa-IP3) $M°(x, y, t) = 1 - M^*(x, y, t)$

$= 1 - M^*(y, x, \bar{t}) = M°(y, x, \bar{t})$

(Fa-IP4) $M°(\alpha x, y, t) = 1 - M^*(\alpha x, y, t)$ for $0 \neq \alpha \in C$

$= 1 - M^* \left(x, y, \frac{t}{|\alpha|}\right) = M° \left(x, y, \frac{t}{|\alpha|}\right)$

(Fa-IP5) $M°(x, x, t) = 1 - M^*(x, x, t) = 1 - 0 = 1 \quad \forall t \in C \backslash R^+$

(Fa-IP6) $M°(x, x, t) = 1 - M^*(x, x, t) \Leftrightarrow x = 0$

$= 1 - 1 \Leftrightarrow x = 0$

$$= 0 \Leftrightarrow x = 0$$

(Fa-IP7) We have $M^*(x, x, .): R \rightarrow I$ is a monotonic non-decreasing function and $\lim_{t\rightarrow\infty} M^*(x, x, t) = 1$

then $M°(x, x, .): R \rightarrow I$ is a monotonic non-increasing function and

$$\lim_{t\rightarrow\infty} M°(x, x, t) = 1 - \lim_{t\rightarrow\infty} M^*(x, x, t) = 1 - 1 = 0$$

Hence $(V, M°)$ is a fuzzy anti-inner product space.

Theorem (3.9)

Let $(V, \mathcal{N})$ be a fuzzy anti-normed linear space. Suppose that for x, y, z in V and t, s, r in C,

$$\max\{\mathcal{N}(x, |st|), \mathcal{N}(y, |st|)\} \leq \max\{\mathcal{N}(x, |s|^2), \mathcal{N}(y, |t|^2)\}$$

Define $M°: V^2 \text{x } C \rightarrow I$ as $M°(x, y, s + t) = 1$

if $x = y$ and $s + t \in C \backslash R^+$ and elsewhere as

$$M°(x, y, s + t) = \mathcal{N}(x, |s|) \wedge \mathcal{N}(y, |t|)$$

Then $M°$ is a fuzzy anti-inner product on V.

Proof: For x, y, z in V and t, s in C,

(Fa-IP1) $M°(x + y, z, |s| + |t|) = M°(x + y, z, |s| + |t| + 0)$

$= \mathcal{N}(x + y, |s| + |t|) \wedge \mathcal{N}(z, 0)$

$= \mathcal{N}(x + y, |s| + |t|)$

$\leq \max\{ \mathcal{N}(x, |s|) , \mathcal{N}(y, |t|)\}$

$= \max\{ M°(x, z, |s|) , M°(y, z, |t|)\}$

Fa-IP2) $M°(x, y, |st|) = \mathcal{N}(x, |st|) \wedge \mathcal{N}(y, |st|)($

$= \max \{\mathcal{N}(x, |st|), \mathcal{N}(y, |st|)\}$

$\leq \max \{\mathcal{N}(x, |s|^2), \mathcal{N}(y, |t|^2)\}$

$= \max \{ M°(x, x, |s|^2), M°(y, y, |t|^2)\}$

(Fa-IP3) $M°(x, y, t) = \mathcal{N}(x, |t|) = \mathcal{N}\left(x, \overline{|t|}\right)$

$= M°(x, y, \bar{t}) = \mathcal{N}\left(y, \overline{|t|}\right)$

$$= M°(y, x, \bar{t})$$

(Fa-IP4) For $\alpha \neq 0$

$$M°(\alpha x, y, t) = \mathcal{N}(\alpha x, |t|) = \mathcal{N}\left(x, \frac{|t|}{|\alpha|}\right) = M°\left(x, y, \frac{|t|}{|\alpha|}\right)$$

(Fa-IP5) By definition $\forall t \in C\backslash R^+,\ M^\circ(x,y,t)=1$
(Fa-IP6) $M^\circ(x,x,t)=0 \quad \forall t>0$
$\Leftrightarrow \mathcal{N}(x,|t|)=0 \quad \forall t>0$
$\Leftrightarrow x=0$
Hence (V,M°) is a fuzzy anti-inner product space.

Theorem (3.10)

Let (V,M°) be a fuzzy anti-inner product space satisfying (Fa-IP8) and (Fa-IP9) and $<,>_\alpha$ be α-anti-inner product $\forall\, \alpha \in (0,1)$.
Define a function
$M^\circ : V^2 \mathrm{x}\, C \to I$ as $M^\circ(x,y,s+t)=1$
if $x=y$ and $t \in C\backslash R^+$ and elsewhere as

$$M^\circ(x,y,t)=\wedge\ \{\ \alpha \in (0,1): |<,>_\alpha| \geq |t|\}$$

Then M° is a fuzzy anti-inner product on V if $|<,>_\alpha|$ is a decreasing function of R.
Proof: For $x,y,z,$ in V and $t,s,$ in C,
(Fa-IP1) To prove that $M^\circ(x+y,z,|s|+|t|) \leq \max\{M^\circ(x,z,|s|), M^\circ(y,z,|t|)\}$
Let $p=M^\circ(x,z,|s|)$ and $q=M^\circ(y,z,|t|)$.
Without loss of generality, assume that $p \leq q$ and let $0<r<p \leq q$
Then $\exists\ 0<\alpha<r$ such that $|<x,z>_\alpha| > |s|$ and
$\exists\ 0<\beta<r$ such that $|<y,z>_\beta| > |t|$
Let $0<\gamma=\alpha \vee \beta < r$. Thus

$$|<x,z>_\gamma| > |<x,z>_\alpha| > |s|$$

and

$$|<y,z>_\gamma| > |<y,z>_\beta| > |t|$$

[Since $|<,>_\alpha|$ is a decreasing function]
Now $|<x+y,z>_\gamma| = |<x,z>_\gamma + <y,z>_\gamma|$
$\leq |<x,z>_\gamma| + |<y,z>_\gamma|$
$> |s|+|t|$
Therefore $M^\circ(x+y,z,|s|+|t|) \leq \gamma < r$, since $r>0$, thus

$$M^\circ(x+y,z,|s|+|t|) \leq \max\{M^\circ(x,z,|s|), M^\circ(y,z,|t|)\}$$

(Fa-IP2) To prove that
$M^\circ(x,y,|st|) \leq \max\{M^\circ(x,y,|s|^2), M^\circ(x,y,|t|^2)\}$
Let $p=M^\circ(x,y,|s|^2)$ and $q=M^\circ(x,y,|t|^2)$.
Without loss of generality, assume that $p \leq q$ and let $0<r<p \leq q$
Then $\exists\ 0<\alpha<r$ such that $|<x,y>_\alpha| > |s|^2$ and
$\exists\ 0<\beta<r$ such that $|<x,y>_\beta| > |t|^2$
Let $0<\gamma=\alpha \vee \beta < r$. Thus

$$|<x,y>_\gamma| > |<x,y>_\alpha| > |s|^2$$

and

$$|<x,y>_\gamma| > |<x,y>_\beta| > |t|^2$$

[Since $|<,>_\alpha|$ is decreasing function]
Therefor $|<x,y>_\gamma|^2 > |s|^2.|t|^2 \Rightarrow |<x,y>_\gamma| > |st|$
therefore $M^\circ(x,y,|st|) \leq \gamma < r$, since $r>0$ is arbitrary, thus

$$M^\circ(x,y,|st|) \leq \max\{M^\circ(x,y,|s|^2), M^\circ(x,y,|t|^2)\}$$

(Fa-IP3) For $t \in C$, $M^\circ(x,y,t) = M^\circ(x,y,\bar{t}) = 1$
if $x=y$ and $\forall t \in C\backslash R^+$
Now let $t \in C$ and $x \neq y$, then

$$M^\circ(x,y,t)=\wedge\ \{\ \alpha \in (0,1): |<x,y>_\alpha| \geq |t|\}$$
$$=\wedge\ \{\ \alpha \in (0,1): |<x,y>_\alpha| \geq |\bar{t}|\}$$
$$=M^\circ(x,y,\bar{t})$$

(Fa-IP4) For $c \in C$,

$$M^\circ(cx,y,t)=\wedge\ \{\ \alpha \in (0,1): |<cx,y>_\alpha| \geq |t|\}$$

$$= \wedge \{ \alpha \in (0,1): |c|\, |< x , y >_\alpha| \geq |t| \}$$
$$= \wedge \{ \alpha \in (0,1): |c|\, |< x , y >_\alpha| \geq \frac{|t|}{|c|} \}$$
$$= M^\circ\left(x, y, \frac{t}{|c|}\right)$$

(Fa-IP5) By definition $M^\circ(x,x,t) = 1 \quad \forall\, t \in C \backslash R^+$

(Fa-IP6) $\forall\, t > 0,\ M^\circ(x,x,t) = 0$

$$\Leftrightarrow \wedge \{ \alpha \in (0,1): |< x, x >_\alpha| \geq |t| \} = 0$$

$\Leftrightarrow < x , x >_\alpha = 0$

$\Leftrightarrow x = \underline{0}$

(Fa-IP7) $\forall\, t > 0,\ M^\circ(x,x,t) = \wedge \{ \alpha \in (0,1): |< x , x >_\alpha| \geq |t| \}$

$$= \wedge \{ \alpha \in (0,1): {\|x\|_\alpha}^2 \geq |t| \}$$
$$= \wedge \{ \alpha \in (0,1): \|x\|_\alpha \geq \sqrt{t} \}$$

Now $t_1 < t_2 \Rightarrow \sqrt{t_1} < \sqrt{t_2}$

$$\Rightarrow \left\{ \alpha \in (0,1): \|x\|_\alpha \geq \sqrt{t_1} \right\} \subset \{ \alpha \in (0,1): \|x\|_\alpha \geq \sqrt{t_2} \}$$
$$\Rightarrow \wedge \left\{ \alpha \in (0,1): \|x\|_\alpha \geq \sqrt{t_1} \right\} \leq \wedge \left\{ \alpha \in (0,1): \|x\|_\alpha \geq \sqrt{t_2} \right\}$$

$\Rightarrow M^\circ(x,x,t_1) \leq M^\circ(x,x,t_2)$

Therefore $M^\circ(x,x,.): R^+ \to I$ is decreasing and $\lim_{t\to\infty} M^\circ(x,x,t) = 0$.

Thus M° is a fuzzy anti-inner product on V.

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