DOI: 10.24996/ijs.2020.SI.1.11

ISSN: 0067-2904

Essential T- Weak Supplemented Modules

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Received: 19/11/ 2019 Accepted: 15/ 3/2020

Abstract

An R-module M is called ET-H-supplemented module if for each submodule X of M, there exists a direct summand D of M, such that T⊆X+K if and only if T⊆D+K, for every essential submodule K of M and T≤ M. Also, let T, X and Y be submodules of a module M , then we say that Y is ET-weak supplemented of X in M if T⊆X+Y and (X∩Y) $\ll_{ET}$M. Also, we say that M is ET-weak supplemented module if each submodule of M has an ET-weak supplement in M. We give many characterizations of the ET-H-supplemented module and the ET-weak supplement. Also, we give the relation between the ET-H-supplemented and ET-lifting modules, along with the relationship between the ET weak -supplemented and ET-lifting modules.

Keywords: ET-small submodule, ET-lifting module, ET-H-supplemented, ET-weak – supplemented

المقاسات التكميلية الضعيفة الجوهرية من النمط –T–

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الخلاصه

ليكن M مقاسات على الحلقة الابدالية R ذات عنصر محايد ، في هذا البحث سنعرف نوعا جديدا في المقاسات يدعى مقاسا تكميليا جوهريا من النمط –T–H– (ET– H– Supplemented) كذلك المقاس التكميلي الضعيف الجوهري من النمط T . يدعى المقاس بأنه التكميلي جوهري من النمط–T–H– اذا كان لكل مقاس جزئي H في M يوجد مقاس جزئي $D\leq_{\oplus} M$ بحيث T⊆ X+K اذا وفقط اذا كان T⊆ D+K لكل مقاس جزئي جوهري K في M . كذلك Y يكون مقياس تكميلي جوهري ضعيف للمقياس الجزئي X في M اذا كان T⊆ X+Y و M $\ll_{ET}$ (X∩ Y) . سوف نعطي تعاريف مكافئة لهذا النوع من المقاسات وندرس العلاقة بين هذه المقاسات ومقاسات اخرى .

1. Introduction

Let R be a commutative ring with identity and M be an arbitrary R-module. A submodule H of M is called small (H≪ M), if for all submodule B of M , B ≤ M, such that H+ B = M implies that B= M [1] . A submodule H of M is essential (H $\leq_e$M) if for all B≤ M such that H∩B= 0, then B= 0[2] . A submodule H of M is closed (H $\leq_c$M) if H has no proper essential extensions inside M. That is, if H $\leq_e$K $\leq_e$ M then H=K[3]. A submodule H of M is called an essential- small (H $\ll_e$M)(E-small) submodule of M , if for all essential submodule B of M such that M = N + B implies that B = M [4]. Let T be a submodule of an R-module M .A submodule N of M is called T-small submodule of M

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(denoted by $N\ll_T M$), in case for any submodule X of M , T⊆N+X implies that T⊆ X . In a previous publication [5], we defined ET-small submodule of M; Let T ≤ M and A submodule H of M is “ET-small submodule of M” ($H\ll_{ET} M$), if for all $K \leq_e M$ such that T ⊆ H+K , then T ⊆ K. Let T be a submodule of a module M. Recalled that M is called T-H-supplemented module if for each submodule X of M, there exists a direct summand D of M, such that T⊆X+K if and only if T⊆D+K, for every submodule K of M [6]. Let T, X and Y be submodules of a module M, we say that Y is T-weak supplemented of X in M if T⊆X+Y and $X\cap Y\ll_T M$. We say that M is T-weak supplemented module if each submodule of M has a T-weak supplement in M [6]. In this work, we define the essential T-H-supplemented module (ET-H-supplemented module) and essential T- weak – supplemented module. We also provide some properties of these types of modules.

2. ET-H-supplemented module

Definition 2.1: Let T be a submodule of a module M. We say that M is ET-H-supplemented module if for each submodule X of M , there exists a direct summand D of M, such that T⊆X+K if and only if T⊆D+K, for every essential submodule K of M .

Remarks and Examples 2.2

1) Let M be an R- module and T=M . Then a module M is ET-H-supplemented if and only if M is an e-H-supplemented module.

2) Let M be an R-module and T=0. Then a module M is ET-H-supplemented.

3) Consider Z_4 as Z-module and $T=\{\bar{0},\bar{2}\}$, then Z_4 is ET-H-supplemented module. To prove that, let $X=\{\bar{0},\bar{2}\}$. Take $D=Z_4$. It is clear that $\{\bar{0},\bar{2}\}\subseteq X+K$, where $K \leq_e M$ if and only if $\{\bar{0},\bar{2}\}\subseteq D+K$ .Thus Z_4 is ET-H-supplemented module.

Proposition 2.3: Let T be a submodule of a module M .Then the following statements are equivalent:

1. M is ET-H-supplemented module.

2. For each submodule X of M, there exists a direct summand D of M such that for each essential submodule A of M with T⊆X+D+A, then T⊆X+A and T⊆D+A.

3. For each closed submodule X of M, there exists $D \leq_{\oplus} M$ such that $\frac{X+D}{X} \ll_{E(\frac{T+X}{X})} \frac{M}{X}$ and $\frac{X+D}{D} \ll_{E(\frac{T+D}{D})} \frac{M}{D}$.

Proof: 1⇒2

Let M be an ET-H-supplemented module and X be a submodule of M. Then there exists $D\leq_{\oplus}M$, which satisfies statement (**1**). Now, let A be an essential submodule of M such that T⊆ X+D+A. By statement (**1**), T⊆X+(X+A) = X+A and T⊆D+(D+A) = D+A .

2⇒1 Let X be a submodule of M. Then there exists $D\leq_{\oplus}M$, which satisfies statement (**2**). Let K be a submodule of M such that T⊆X+K. Then T⊆X+D+K . By (**2**), T⊆ D+K .
Now, let H be essential submodule of M such that T⊆D+H .Then T⊆X+D+H. By (**2**), T⊆X+H. Thus M is ET-H-supplemented module.

2⇒3 Let X be a submodule of M .Then there exists $D\leq_{\oplus}M$, which satisfies (**2**) . To show that $\frac{X+D}{X} \ll_{E(\frac{T+X}{X})} \frac{M}{X}$, let $\frac{A}{X} \leq_e \frac{M}{X}$, where X ≤A ≤ M such that $\frac{T+X}{X} \subseteq \frac{X+D}{X} + \frac{A}{X} = \frac{X+D+A}{X}$. Then T⊆T+X⊆X+D+A .By (**2**), T⊆X+A=A . Thus, $\frac{T+X}{X} \subseteq \frac{A}{X}$. By the same way, $\frac{X+D}{D} \ll_{E(\frac{T+D}{D})} \frac{M}{D}$.

3⇒2 Let X be a closed submodule of M, then there exists $D\leq_{\oplus}M$ such that $\frac{X+D}{X} \ll_{E(\frac{T+X}{X})} \frac{M}{X}$ and $\frac{X+D}{D} \ll_{E(\frac{T+D}{D})} \frac{M}{D}$. Let A be an essential submodule of M such that T⊆X+D+A . Now $\frac{T+X}{X} \subseteq \frac{X+D+A}{X} = \frac{X+D}{X} + \frac{A+X}{X}$. Since $\frac{X+D}{X} \ll_{E(\frac{T+X}{X}} \frac{M}{X}$, then $\frac{T+X}{X} \subseteq \frac{A+X}{X}$ and hence T⊆X+A . By the same way, T⊆D+A.

Recall that M is said to be an ET-lifting module if for all sub-module H of M, there exists a direct summand K of M and $L\ll_{ET}M$ such that H=K+L, where T ≤ M [7] .

Proposition 2.4: Let T be a submodule of M .Consider the following statements:

1. M is ET-lifting module.

2. For each submodule X of M, there exists a decomposition M = D⊕D,` such that D⊆X and

a. $\frac{X}{D} \ll_{E(\frac{T+D}{D})} \frac{M}{D}$ and

b. whenever T+D⊆ L+D, for some L ≤ M, then T⊆L .

Then **1⟹2(a)** and **2⟹1**

Proof: 1⟹2(a)

Suppose that M is ET-lifting module. Let X be submodule of M, then X=D+H, where D $\leq_{\oplus}$M and H$\ll_T$ M. Let M = D⨁D\`. To show that $\frac{X}{D} \ll_{E(\frac{T+D}{D})} \frac{M}{D}$, let $\frac{K}{D} \leq_e \frac{M}{D}$ such that $\frac{T+D}{D} \subseteq \frac{X}{D} + \frac{K}{D}$. Then T⊆T+D⊆X+K and hence T⊆D+H+K. Since H$\ll_{ET}$M, then T⊆D+K .Therefore, $\frac{T+D}{D} \subseteq \frac{K}{D}$. Thus, $\frac{X}{D} \ll_{E(\frac{T+D}{D})} \frac{M}{D}$.

2⟹1 Let X be a submodule of M, then there exists a decomposition M = D⨁D\` such that D⊆X and D $\subseteq_{\mathbf{ET.ce}}$ X in M. Now, to show that (X⋂D\`) $\ll_{ET}$M, let K $\leq_e$ M such that T⊆(X⋂D\`)+K .Then $\frac{T+D}{D} \subseteq \frac{(X\cap D`)+D}{D} + \frac{K+D}{D}$. By **2**.(**a**) , then $\frac{T+D}{D} \subseteq \frac{K+D}{D}$ and hence T+D⊆K+D. By **2**.(**b**), then T⊆K. Thus M is ET-lifting module, as previously published [7] .

Lemma 2.5: Let M be a ET-H-supplemented module such that for every direct summand D of M and A$\leq_c$M, $\frac{D+A}{A}$ is direct summand of $\frac{M}{A}$ .Then $\frac{M}{A}$ is E($\frac{T+A}{A}$) -H-supplemented module.

Proof*:* Assume that M is ET-H-supplemented module. Let $\frac{X}{A}$ be a submodule of $\frac{M}{A}$, then there exists a direct summand D$\leq_{\oplus}$M such that T⊆X+K if and only if T⊆D+K, for every K submodule of M. By our assumption, $\frac{D+A}{A} \leq_{\oplus} \frac{M}{A}$ since A$\leq_c$M, then $\frac{K}{A}$ be an essential submodule of $\frac{M}{A}$ such that $\frac{T+A}{A} \subseteq \frac{X}{A} + \frac{K}{A} = \frac{X+K}{A}$, then T⊆T+A⊆X+K . Since M is ET-H-supplemented, then T⊆D+K. So $\frac{T+A}{A} \subseteq \frac{D+K}{A} = \frac{D+A}{A} + \frac{K}{A}$. By the same way, we can show that $\frac{T+A}{A} \subseteq \frac{D+A}{A} + \frac{K}{A}$ implies that $\frac{T+A}{A} \subseteq \frac{X}{A} + \frac{K}{A}$, for every submodule $\frac{K}{A}$ of $\frac{M}{A}$. Thus, $\frac{M}{A}$ is E($\frac{T+A}{A}$) -H-supplemented module.

Corollary ***2.6:*** Let A be a closed submodule of a distributive module M. If M be a ET-H-supplemented module, then $\frac{M}{A}$ is E($\frac{T+A}{A}$) -H-supplemented module.

Proof: Assume that M is ET-H-supplemented module. Let A be a closed submodule of M and D be a direct summand of M .Then M=D⨁D\` and hence $\frac{M}{A} = \frac{D+A}{A} + \frac{D`+A}{A}$. Since M is a distributive module, then A=A+(D⋂D\`)=(A+D)⋂(A+D\`)= $\frac{D+A}{A} \cap \frac{D`+A}{A} = \frac{A}{A}$ =A . Therefore, $\frac{M}{A} = \frac{D+A}{A} \oplus \frac{D`+A}{A}$. Thus, $\frac{M}{A}$ is E($\frac{T+A}{A}$)-H-supplemented module, by Lemma (2.5) .

Proposition 2.7: Let M be a finitely generated, faithful and multiplication R-module .Then M is ET-H-supplemented module if and only if R is E[T:M]-H-supplemented .

Proof: Assume that M is ET-H-supplemented module. Let I be an ideal of R . Since M is ET-H-supplemented module, then there exists D $\leq_{\oplus}$M such that T⊆IM+N if and only if T⊆D+N, for every an essential submodule N of M. Since M is a multiplication module, then there exists ideals S, J and K of R such that T=SM, D=JM and N=KM . Hence, SM⊆IM+KM=(I+K)M if and only if SM⊆JM+KM=(J+K)M. But M is finitely generated, faithful and multiplication module, therefore M is a cancellation module , as previously published [8] .Thus S⊆I+K if and only if S⊆J+K, for every ideal K of R . We claim that J$\leq_{\oplus}$R. To show that, let M=D⨁D\` and D\`=J\`M, for some ideal J\` of R. Hence, RM=M=JM⨁J\`M=(J+J\`)M . But M is a cancellation module, therefore R=J+J\` .

To show that J⋂J\`=0 ; Since M is a finitely generated, faithful multiplication module, then 0=JM⋂J\`M=(J⋂J\`)M and hence J⋂J\`=0 . Thus J$\leq_{\oplus}$R . Thus R is E[T:M]-H-supplemented.

Conversely, assume that R is E[T:M]-H-supplemented and let X be a submodule of M .Since M is a multiplication module, then there exists an ideal I of R such that X=IM. Then there exists J$\leq_{\oplus}$R such that S⊆I+K if and only if S⊆J+K, for every ideal K of R . Hence, SM⊆(I+K)M=IM+KM if and only if SM⊆(J+K)M=JM+KM, for every submodule KM of M . We claim that JM $\leq_{\oplus}$M. To show that, let R=J⨁J\`, for some ideal J\` of R and hence M=RM=(J+J\`)M=JM+J\`M . Since M is a finitely generated, faithful and multiplication module, then JM⋂J\`M=(J⋂J\`)M=0M=0 .Thus, JM $\leq_{\oplus}$M . Thus, M is E[T:M]-H-supplemented module.

3.ET-weak supplemented modules

Definition 3.1: Let T, X and Y be submodules of a module M . We say that Y is ET-weak supplemented of X in M if T⊆X+Y and X⋂Y$\ll_{ET}$M.

We say that M is ET-weak supplemented module if each submodule of M has an ET-weak supplement in M.

Remarks and Examples 3.2

1. Consider Z_6 as Z-module and let T $=\{\bar{0},\bar{3}\}$,X $=\{\bar{0},\bar{2},\bar{4}\}$ and Y $=\{\bar{0},\bar{3}\}$. It is clear that T⊆X+Y and X⋂Y=0$\ll_{ET}$$Z_6$. Thus, Y is ET-weak supplement of X in Z_6 .One can easily show that Z_6 is ET-weak supplemented module.

2. Consider Z as Z-module and let T=2Z . For each integer n>0, 2Z⊆nZ+ (n+2)Z . But 2Z⊈(n+2)Z . So 0 is the only ET-small submodule of Z . One can easily show that Z is not ET-weak supplemented module.

Now let T = 0, X = 2Z and Y = 3Z, then T⊆2Z+3Z and 2Z⋂3Z=6Z$\ll_{ET}$Z, by [5] . Therefore, Y is ET-weak supplement of X in Z. But Y is not weak supplement of X in M, where 0 is the only small submodule of Z.

3. A module M is EM-weak supplemented module if and only if M is an E-weak supplemented module.

4. Every module M is E(0)-weak supplemented module.

5. If M is a uniform module, then M is EM-weak supplemented module if and only if M is a T-weak supplemented module.

Proposition 3.3: Let T, X and Y be submodules of a module M such that Y is ET-weak supplement of X in M. If T⊆ K+Y , for some submodule K of X, then Y is an ET-weak supplement of K in M.

Proof: Assume that Y is ET-weak supplement of X in M and let K be submodule of X such that T⊆K+Y. Since K⋂Y⊆X⋂Y$\ll_{ET}$M, then K⋂Y$\ll_{ET}$M, by [5]. Thus, Y is ET-weak supplement of K in M.

Proposition *3.4:* Let T, X and Y be submodules of a module M such that Y is ET-weak supplement of X in M . If T is finitely generated , then Y is containing a finitely generated ET-weak supplement of X in M .

Proof: Let T=$Rt_1+Rt_2+\ldots+Rt_n$,for some t_i ∈T, $\forall i=1,\ldots,n$. Since T⊆X+Y , then for each $1\leq i\leq n$ we have $t_i = a_i + b_i$, where a_i ∈X and b_i ∈Y . Now, let Y\`=$Rb_1+Rb_2+\ldots+Rb_n$ . It is clear that Y\`⊆Y . We claim that T⊆X+Y\` . To show that , let $t$ ∈T then $t$ $=r_1t_1+r_2t_2+\ldots+r_nt_n$ ⊆ $r_1a_1+r_2a_2+\ldots+r_na_n+r_1b_1+r_2b_2+\ldots+r_nb_n$ , for some $r_1,r_2,\ldots,r_n$∈R . So $t$ ∈X+Y\` . Since X⋂Y\`⊆X⋂Y and X⋂Y$\ll_{ET}$ M , then by [5], X⋂Y\`$\ll_{ET}$ M . Thus Y\` is ET-weak supplement of X in M .

Proposition 3.5: Let T, X and Y be submodules of a module M such that Y is ET-weak supplement of X in M . If L⊆Y and L$\ll_{ET}$M , then Y is a ET-weak supplement of X+L .

Proof: Let Y be a ET-weak supplement of X in M, L⊆Y and L$\ll_{ET}$ M. Then T⊆X+Y⊆X+Y+L and (X⋂Y) $\ll_{ET}$M. To show that Y⋂(X+L) $\ll_{ET}$ M, let K be an essential submodule of M such that T⊆(Y⋂(X+L))+K. By the modular Law T⊆(X⋂Y)+L+K, since K$\ll_e$M, then (L+K) $\leq_e$M [2], since (X⋂Y) $\ll_{ET}$M therefore T⊆L+K. But L$\ll_{ET}$ M, therefore T⊆K. Thus Y is an ET-weak supplement of X+L .

Proposition 3.6*:* Let T, X, Y and L be submodules of a module M such that Y is ET-weak supplement of X in M and L⊆X . Then $\frac{Y+L}{L}$ is E($\frac{T+L}{L}$) -weak supplement of $\frac{X}{L}$ in $\frac{M}{L}$.

Proof: Since T⊆X+Y , then $\frac{T+L}{L}\subseteq\frac{X}{L}+\frac{Y+L}{L}$. To show that ($\frac{X}{L}\cap\frac{Y+L}{L}$) $\ll_{E(\frac{T+L}{L})}\frac{M}{L}$, let $\frac{K}{L}$ be an essential submodule of $\frac{M}{L}$ such that $\frac{T+L}{L}\subseteq$($\frac{X}{L}\cap\frac{Y+L}{L}$) + $\frac{K}{L}\subseteq\frac{X\cap(Y+L)}{L}+\frac{K}{L}$. By the modular Law $\frac{T+L}{L}\subseteq\frac{(X\cap Y)+L}{L}+\frac{K}{L}$, therefore T⊆T+L⊆(X⋂Y)+L+K . Since K$\ll_e$M then (L+K) $\leq_e$M and since (X⋂Y)$\ll_{ET}$M, then T⊆L+K . Therefore, $\frac{T+L}{L}\subseteq\frac{K}{L}$. Thus $\frac{Y+L}{L}$ is E($\frac{T+L}{L}$) -weak supplement of $\frac{X}{L}$ in $\frac{M}{L}$.

Proposition 3.7: Let M and N be R-modules and let f : M → N be an epimorphism . If M is ET-weak supplemented module, then N is Ef (T)-weak supplemented module.

Proof: Let f : M → N be an epimorphism and M be an ET-weak supplemented. Let K be essential submodule of N. Since M is ET-weak supplemented, then there is a submodule L of M such that T⊆L+ f^{-1}(K) and f^{-1}(K)⋂L $\ll_{ET}$M .Therefore, f (T) ⊆ f (L+ f^{-1}(K)) and hence f (T) ⊆ f (L)+K. Since f^{-1}(K)⋂L$\ll_{ET}$M, then K⋂ f (L)= f (f^{-1}(K)⋂L) $\ll_{Ef(T)}$ f (M) [5]. Thus f (L) is Ef (T)-weak supplement of K in N .

Lemma *3.8:* Let M be a ET-lifting module and let Y be submodule of M and $X \leq_e M$ such that $T \subseteq X+Y$. Then, there exists $D \leq_{\oplus} M$ such that $T \subseteq X+D$ and $D \subseteq Y$.

Proof: Assume that M is an ET-lifting module and let Y be submodule of M such that $T \subseteq X+Y$, then $Y=D+H$, where $D \leq_{\oplus} M$ and $H \ll_{ET} M$. Since $T \subseteq X+Y$, then $T \subseteq X+D+H$. Since $X \leq_e M$ then $X+D \leq_{\square}$. But $H \ll_{\square\square}$ M, therefore $T \subseteq X+D$ and $D \subseteq Y$.

Recall that a submodule X of a module M is called a projective invariant, if for every $P=P^2 \in End$ (M), $P(X) \leq X$ [9].

Proposition *3.9:* Let M be an ET-lifting module such that every ET-small submodule of M is projective invariant . If $T \subseteq X+Y$, where X and Y are submodules of M, then Y contains a ET-weak supplement of X.

Proof*:* Assume that $T \subseteq X+Y$. By Lemma (2.8), there exists $D \leq_{\oplus} M$ such that $T \subseteq X+D$ and $D \subseteq Y$. Since M is ET-lifting module and $X \cap Y$ is a submodule of M , then there exists a decomposition $M=D_1 \oplus D_1`$ such that $D_1 \subseteq X \cap Y$ and $(X \cap Y) \cap D_1` \ll_{\square\square} M$ [7]. By the Modular Law:

$Y=Y \cap M=Y \cap (D_1 \oplus D_1`)=D_1+(Y \cap D_1`)$.

So $T \subseteq X+Y=X+D_1+(Y \cap D_1`)=X+(Y \cap D_1`)$. Thus $Y \cap D_1`$ is ET-weak supplement of X in M.

Proposition *3.10*: Let M be a ET-lifting module and Y be a ET-weak supplement of X in M .Then Y contains an ET-weak supplemented of X, which is a direct summand of M.

Proof: Assume that M is ET-lifting . Let Y be a ET-weak supplemented of X in M, then $T \subseteq X+Y$ and $X \cap Y \ll_{ET} M$. Since M is ET-lifting, then $Y=D+H$, where $D \leq_{\oplus} M$ and $H \ll_{ET} M$. Since $T \subseteq X+Y$, then $T \subseteq X+D+H$, since $X \leq_e M$ then $X+D \leq_e M$, but $H \ll_{ET} M$ then $T \subseteq X+D$. Now $X \cap D \subseteq X \cap Y \ll_{ET} M$. Then $X \cap D \ll_{ET} M$, by [5]. Thus D is an ET-weak supplemented of X in M.

Proposition 3.11*:* Let M be an ET-lifting module such that every ET-small submodule of M is projective invariant .Then M is ET-weak supplemented module.

Proof: Assume that M is ET-lifting module and let X be submodule of M .Then $X=D+H$, where $D \leq_{\oplus} M$ and $H \ll_{ET} M$. Then $M=D \oplus D`=X+D`$ and hence $T \subseteq X+D`$. Since M is ET-lifting, then $X \cap D` \ll_{ET} M$, by [7]. Hence, D` is ET-weak supplemented of X in M. Thus, M is ET-weak supplemented module.

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