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Strongly Hollow R - Annihilator Lifting Modules and Strongly R - Annihilator (Hollow- Lifting) Modules

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Abstract

Let R be a commutative ring with unity. Let W be an R-module, for K≤F, where F is a submodule of W and K is said to be R-annihilator coessential submodule of F in W (briefly R-a-coessential) if $\frac{K}{F} \ll^a \frac{W}{F}$ (denoted by $K \leq_{ace} F$ in W). An R-module W is called strongly hollow -R-annihilator -lifting module (briefly, strongly hollow-R-a-lifting), if for every submodule F of W with $\frac{W}{F}$ hollow, there exists a fully invariant direct summand K of W such that $K \leq_{ace} F$ in W. An R - module W is called strongly R - annihilator - (hollow - lifting) module (briefly strongly R - a - (hollow - lifting) module), if for every submodule F of W with $\frac{W}{F}$ R - a - hollow, there exists a fully invariant direct summand K of W such that $K \leq_{ace} F$ in W.

Keywords: R-annihilator submodule, coessential submodule, hollow-lifting module, strongly hollow lifting module.

مقاسات الرفع من النمط R المجوفه القويه ومقاسات الرفع المجوفه من النمط تالف R القويه

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الخلاصة

لتكن R حلقة ابداليه ذات عنصر محايد, وليكن W مقاسا معرفا على R , و K≤F, حيث Fهو مقاس جزئي من W. يدعى Kمقاس جزئي ضد جوهري تالف من النمط R من F في Wاذا كان $\frac{W}{F} \gg \frac{K}{F}$ ويرمز له $K \leq_{ace} F$ في W. يدعى المقاس W بانه مقاس رفع من النمط تالف R المجوف قوي اذا كان لكل مقاس جزئي F من W بحيث $\frac{W}{F}$ مجوف, يوجد جمع مباشر ثابت تام Kمن W بحيث $K \leq_{ace} F$ والمقاس W يدعى مقاس رفع مجوف من النمط تالف R قوي اذا كان لكل مقاس جزئي F من W بحيث $\frac{W}{F}$ مجوف من النمط تالف R , يوجد جمع مباشر ثابت تام K من W بحيث $K \leq_{ace} F$.

1.Inroduction

Throughout this paper, all rings are commutative rings with identity and all modules are unital. A submodule F of an R-module W is called R-annihilator -small (R-a-small) submodule if whenever F+T = W, where T is a submodule of W, implies that ann(T)=0, where ann(T)={rϵR:r.T =0}, denoted by $F \ll^a W$ [1].

An earlier work [2] introduced the concept of R-annihilator -hollow module where a nontrivial module W is called R-annihilator-hollow (briefly R-a-hollow) if every proper submodule of W is R-a-small. Another work [3] reported that an R-module W is called hollow -R- annahilitor-lifting if for every submodule A of W with $\frac{W}{A}$ hollow there exists a direct summand B of W such that B⊆ A and $\frac{A}{B} \ll^a \frac{W}{B}$. Also the concept of R-annihilator –(hollow-lifting)module was introduced where an R-module W is

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R-annihilator (hollow-lifting) module (briefly, R-a-(hollow-lifting)) if for every submodule A of W with $\frac{W}{A}$ is R-a-hollow, there exists a direct summand B of W such that B⊆ A and $\frac{A}{B} \ll^{a} \frac{W}{B}$ [4]. It was also shown that an R- module W is called strongly hollow-lifting if for every submodule N of W with $\frac{W}{N}$ hollow, there exists a stable (fully invariant) direct summand K of W such that $\frac{N}{K} \ll \frac{W}{K}$ [5].

In this paper, we introduce a new concept of strongly hollow-R-annihilator-lifting modules and strongly R-annihilator –(hollow-lifting) modules as a generalization of strongly hollow -lifting modules.

2 . Strongly Hollow - R - Annihilator - Lifting Modules

In this section, we introduce the new concept of strongly hollow - R - annihilator - lifting modules as a generalization of strongly hollow - lifting modules and as a proper stronger module of hollow - R - a - lifting modules.

We start with the following definition:

Definition (2-1)

An R - module W is called strongly hollow - R - annihilator - lifting module (briefly, strongly hollow - R - a - lifting) if for every submodule F of W with $\frac{W}{F}$ hollow, there exists a fully invariant direct summand K of W such that K $\leq_{\mathbf{ace}}$ F in W.

Proposition (2-2)

Every strongly hollow - R - a - lifting module W is hollow - R - a - lifting module.

Proof

Let F be a submodule of W such that $\frac{W}{F}$ hollow. Since W is strongly hollow - R - a - lifting, then there exists a fully invariant direct summand K of W such that K $\leq_{\mathbf{ace}}$ F in W.

But the converse of the previous proposition is not true in general (we have no example yet).

The following proposition shows that the converse is true if the module is indecomposable.

Proposition (2-3)

Let W be an indecomposable R - module. Then W is strongly hollow - R - a - lifting module if and only if W is hollow - R - a - lifting module.

Proof

From proposition (2-2), W is a hollow - R - a - lifting module.

Conversely, Let W be a hollow - R - a - lifting and A be a submodule of W with $\frac{W}{A}$ hollow. Since W is hollow - R - a - lifting module, then there exists a direct summand B of W such that B ≤ A and $\frac{A}{B} <<^{\mathbf{a}} \frac{W}{B}$. But W is an indecomposable, therefore B = (0), so $\frac{A}{(0)} <<^{\mathbf{a}} \frac{W}{(0)}$ such that B = (0) is a fully invariant direct summand of W. Thus W is strongly hollow - R - a - lifting.

Example (2-4)

Z_4 as Z - module is indecomposable and hollow - Z - a - lifting, thus by the last proposition, Z_4 as Z - module is strongly hollow - Z - a - lifting.

An earlier work [5, Examples and Remarks (3.1.2)(7), P. 50] proved that if W is indecomposable, then W is strongly hollow - lifting if and only if W is hollow - lifting. Thus we provide the following remark.

Remark (2-5)

Strongly hollow - R - a - lifting needs not to be strongly hollow - lifting module. For example, consider that W = Z as Z - module is indecomposable. Then by [6, Examples and Remarks (2.2.2)(3), P. 34] Z as Z - module is not hollow - lifting and by [5, Examples and Remarks (3.1.2)(7), P. 50] Z as Z - module is not strongly hollow - lifting module.

But Z as Z - module is hollow - Z - a – lifting, thus by proposition (2-3), Z as Z - module is strongly hollow - Z - a - lifting module.

We think that strongly hollow - lifting is not strongly hollow - R - a - lifting.

Remark (2-6)

Every R - a - hollow module is strongly hollow - R - a - lifting module. But the converse is not true. For example, Z_4 as Z - module is strongly hollow - Z - a - lifting but it is not Z - a - hollow.

Proposition (2-7)

If W is strongly hollow - R - a - lifting module, then every R - a - coclosed submodule L of W with $\frac{W}{L}$ hollow is a fully invariant direct summand.

Proof

Let W be strongly hollow - R - a - lifting module and let L be R - a - coclosed submodule of W such that $\frac{W}{L}$ hollow. Thus there is a fully invariant direct summand B of W such that B ≤ L and $\frac{L}{B} \ll^{a} \frac{W}{B}$. But L is R - a - coclosed of W, so L = B and hence L is a fully invariant direct summand of W.

Before we give the next proposition, we need the following lemma.

Lemma (2-8) [7]

Let W be an R - module. If $W = W_1 \oplus W_2$, then $\frac{W}{A} = \frac{A \oplus W_1}{A} \oplus \frac{A \oplus W_2}{A}$, for every fully invariant submodule A of W.

We need the following proposition

Proposition (2-9) [8]

Let W_1 and W_2 be R-modules and let $f: W_1 \to W_2$ be an epimorphism if $K<L<W_2$ and $K \leq_{ace} L$ in W_2, then $f^{-1}(K) \leq_{ace} f^{-1}(L)$ in W_1

The following proposition shows the case when the quotient module be strongly hollow - R - a - lifting.

Proposition (2-10)

Let W be an R - module. If W is strongly hollow - R - a - lifting, then $\frac{W}{F}$ is strongly hollow - R - a - lifting for every fully invariant submodule F of W.

Proof

Let $\frac{A}{F}$ be a submodule of $\frac{W}{F}$ such that $\frac{W/F}{A/F}$ is hollow. So, by the third isomorphism theorem, $\frac{W/F}{A/F} \cong \frac{W}{A}$, then $\frac{W}{A}$ is hollow. Now, since W is strongly hollow - R - a - lifting, then there exists a fully invariant direct summand L of W such that $\frac{A}{L} \ll^{a} \frac{W}{L}$. Let $W = L \oplus H$, where H is a submodule of W. Now, clearly $L + F \leq A$ and hence $\frac{L+F}{F} \leq \frac{A}{F}$. Let $f : \frac{W}{L+F} \to \frac{W}{L}$ be a mapping defined by $f(w + (L + F)) = w + L$, for all $w \in W$. One can easily find that f is an isomorphism. Then by proposition (2-9), we have $f^{-1}(\frac{A}{L}) \ll^{a} \frac{W}{L+F}$, and hence $f^{-1}(\frac{A}{L}) = \frac{A}{L+F} \ll^{a} \frac{W}{L+F}$. Since F is fully invariant submodule of W, thus by lemma (2-8), we have $\frac{W}{F} = \frac{L+F}{F} \oplus \frac{H+F}{F}$ and $\frac{L+F}{F}$ is a fully invariant direct summand of $\frac{W}{F}$. Thus $\frac{W}{F}$ is strongly hollow - R - a - lifting.

Recall that an R-module W is SS - module if every direct summand D of W is stable. Equivalently, W is SS - module if and only if every direct summand D is fully invariant [9].

Proposition (2-11)

Let W be an R - module. If W is SS - module, then W is strongly hollow - R - a - lifting if and only if W is hollow - R - a - lifting.

Proof: Clear.

Corollary (2-12):

Let W be duo (fully invariant) module. Then W is strongly hollow - R - a - lifting if and only if W is hollow - R - a - lifting.

Proposition (2-13)

If W is strongly hollow - R - a - lifting module, then for every submodule A of W with $\frac{W}{A}$ is hollow, there exists a fully invariant direct summand B of W with B ≤ A such that $W = B \oplus L$ and $A \cap L \ll^{a} L$.

Proof

Suppose that W is strongly hollow - R - a - lifting, and let A be a submodule of W such that $\frac{W}{A}$ is hollow. Then by our assumption, there exists a fully invariant direct summand B of W such that B ≤ A

and $\frac{A}{B} \ll^{a} \frac{W}{B}$. Let W = B ⊕ L, where L is a submodule of W. We must show that $A \cap L \ll^{a} L$. To show that, let L = A ∩ L + X, where X ≤ L. By the modular law, A = A ∩ W = A ∩ (B ⊕ L) = B ⊕ (A ∩ L), then W = B ⊕ L = B + (A ∩ L) + X = A + X. We get $\frac{W}{B} = \frac{A+X}{B} = \frac{A}{B} + \frac{X+B}{B}$, but $\frac{A}{B} \ll^{a} \frac{W}{B}$ implies that ann ($\frac{X+B}{B}$) = 0. One can easily show that ann (X) = 0. Hence $A \cap L \ll^{a} L$.
Now, we need the following.

Lemma(2-14) [1]

Let L and H be submodules of an R-module W such that L≤ H. If $\frac{H}{L}$ is R-a-small in $\frac{W}{L}$, then H is R-a-small submodule of W.

Proposition (2-15) [1]

Let E and F be submodules of an R- module W such that E≤ F, then the following holds:

(1) If E is R-a-small submodule of F, then E is R-a-small submodule of W .

(2) If F is R-a-small submodule of W, then E is R-a-small submodule of W.

Proposition (2-16)

If W is strongly hollow - R - a - lifting, then every submodule A of W with $\frac{W}{A}$ is hollow can be written as A = B ⊕ S, with B is a fully invariant direct summand of W and $A \cap S \ll^{a} W$.

Proof

Suppose that W is strongly hollow - R - a - lifting, and let A ≤ W with $\frac{W}{A}$ is hollow. Then by our assumption, there exists a fully invariant direct summand B of W such that B ≤ A and $\frac{A}{B} \ll^{a} \frac{W}{B}$. Let W = B ⊕ L, where L is a submodule of W. By the modular law, A = A ∩ W = A ∩ (B ⊕ L) = B ⊕ (A ∩ L). We want to show that $A \cap L \ll^{a} W$, since $\frac{A}{B} \ll^{a} \frac{W}{B}$, thus by corollary (2-14), we get $A \ll^{a} W$. But A ∩ L ≤ A, thus by proposition (2-15) , we get $A \cap L \ll^{a} W$. By putting S = A ∩ L then A = B ⊕ S with B is a fully invariant direct summand of W and $S \ll^{a} W$.

Proposition (2-17)

If W is strongly hollow - R - a - lifting, then every submodule A of W with $\frac{W}{A}$ is hollow can be written as A = B + S, with B is a fully invariant direct summand of W and $S \ll^{a} W$.

Proof: Clear.

Corollary (2-18)

Let W be a projective SS - module. If W is a strongly hollow - R - a - lifting, then for every submodule A of W, such that $\frac{W}{A}$ is hollow, $\frac{W}{A}$ has a projective R - a - cover.

Proof

Suppose that W is a projective SS - module and W is a strongly hollow - R - a - lifting module, and let A be a submodule of W with $\frac{W}{A}$ is hollow, then by proposition (2-11), we have W is hollow - R - a – lifting, and by proposition (2-17), we have that $\frac{W}{A}$ has a projective R - a - cover.

3. Strongly R - Annihilator - (Hollow - Lifting) Modules

In this section, we introduce the new concept of strongly R - annihilator - (hollow - lifting) modules as a generalization of strongly hollow - lifting modules and as a proper stronger module of R - a - (hollow - lifting) modules.

We start with the following definition:

Definition (3-1)

An R - module W is called strongly R - annihilator - (hollow - lifting) module (briefly, strongly R - a - (hollow - lifting) module) if for every submodule F of W with $\frac{W}{F}$ R - a - hollow, there exists a fully invariant direct summand K of W such that $K \leq_{ace} F$ in W.

Proposition (3-2)

Every strongly R - a - (hollow - lifting) module W is R - a - (hollow - lifting) module.

Proof

Let F be a submodule of W such that $\frac{W}{F}$ R - a - hollow. Since W is strongly R - a - (hollow – lifting), then there exists a fully invariant direct summand K of W such that $K \leq_{ace} F$ in W.

But the converse of the previous proposition is not true in general (we have no example yet).

The following proposition shows that the converse is true if the module is indecomposable.

Proposition (3-3)

Let W be an indecomposable R - module. Then W is strongly R - a - (hollow - lifting) module if and only if W is R - a - (hollow - lifting) module.

Proof

From proposition (3-2), we have that W is R - a - (hollow - lifting).

Conversely, Let W be an R - a - (hollow - lifting) and A be a submodule of W with $\frac{W}{A}$ R - a - hollow. Since W is R - a - (hollow - lifting) module, then there exists a direct summand B of W such that B ≤ A and $\frac{A}{B} \ll^{a} \frac{W}{B}$. But W is an indecomposable, therefore B = (0), so $\frac{A}{(0)} \ll^{a} \frac{W}{(0)}$ such that B = (0) is a fully invariant direct summand of W. Thus W is strongly R - a - (hollow - lifting).

Example (3-4)

Z as Z - module is indecomposable and Z - a - (hollow - lifting), thus by the previous proposition, Z as Z - module is strongly Z - a - (hollow - lifting).

The abovementioned study [5, Examples and Remarks (3.1.2)(7), P. 50] proved that if W is indecomposable, then W is strongly hollow - lifting if and only if W is hollow - lifting. Thus, we provide the following remark.

Remark (3-5)

Strongly R - a - (hollow - lifting) needs not to be strongly hollow - lifting module. For example, consider that W = Z as Z - module is indecomposable. By [6, Examples and Remarks (2.2.2)(3), P. 34], Z as Z - module is not hollow - lifting and by [5, Examples and Remarks (3.1.2)(7), P. 50] Z as Z - module is not strongly hollow - lifting module.

But Z as Z - module is Z - a - (hollow - lifting), thus by proposition (3-5) Z as Z - module is strongly Z - a - (hollow - lifting) module.

We think that strongly hollow - lifting is not strongly R - a - (hollow - lifting).

Remark (3-6)

Every R - a - hollow module is strongly R - a - (hollow - lifting) module. But the converse is not true. For example Z_4 as Z - module is strongly Z - a - (hollow - lifting) but it is not Z - a - hollow.

Proposition (3-7)

If W is strongly R - a - (hollow - lifting) module, then every R - a - coclosed submodule L of W with $\frac{W}{L}$ R - a - hollow is a fully invariant direct summand.

Proof

Let W be strongly R - a - (hollow - lifting) module and let L be R - a - coclosed submodule of W such that $\frac{W}{L}$ R - a - hollow. Thus, there is a fully invariant direct summand B of W such that B ≤ L and $\frac{L}{B} \ll^{a} \frac{W}{B}$. But L is R - a - coclosed of W, so L = B and hence L is a fully invariant direct summand of W.

The following proposition shows the case when the quotient module is strongly R - a - (hollow - lifting).

Proposition (3-8)

Let W be an R - module. If W is strongly R - a - (hollow - lifting), then $\frac{W}{F}$ is strongly R - a - (hollow - lifting) for every fully invariant submodule F of W.

Proof: Let $\frac{A}{F}$ be a submodule of $\frac{W}{F}$ such that $\frac{\frac{W}{F}}{\frac{A}{F}}$ is R - a - hollow. So by the third isomorphism theorem, $\frac{\frac{W}{F}}{\frac{A}{F}} \cong \frac{W}{A}$, then $\frac{W}{A}$ is R - a - hollow.

Now, since W is strongly R - a - (hollow - lifting), then there exists a fully invariant direct summand L of W such that $\frac{A}{L} \ll^{a} \frac{W}{L}$.

We complete the proof in the same way of proposition (2-10).

Proposition (3-9)

Let W be an R - module. If W is SS - module, then W is strongly R - a - (hollow - lifting) if and only if W is R - a - (hollow - lifting).

Proof: Clear.

Corollary (3-10)

Let W be a duo (fully invariant) module. Then W is strongly R - a - (hollow - lifting) if and only if W is R - a - (hollow - lifting).

Proposition (3-11)

If W is strongly R - a - (hollow - lifting) module, then for every submodule A of W with $\frac{W}{A}$ is R - a - hollow, there exists a fully invariant direct summand B of W with B ≤ A such that W = B ⊕ L and A ∩ L <<^{a} L.

Proof

Suppose that W is strongly R - a - (hollow - lifting), and let A be a submodule of W such that $\frac{W}{A}$ is R - a - hollow. Then by our assumption, there exists a fully invariant direct summand B of W such that B ≤ A and $\frac{A}{B}$ <<^{a} $\frac{W}{B}$. We complete the proof in the same way of proposition (2-13).

Proposition (3-12)

If W is strongly R - a - (hollow - lifting), then every submodule A of W with $\frac{W}{A}$ is R - a - hollow can be written as A = B ⊕ S, with B is a fully invariant direct summand of W and S <<^{a} W.

Proof

Suppose that W is strongly R - a - (hollow - lifting), and let A ≤ W with $\frac{W}{A}$ is R - a - hollow. Then by our assumption, there exists a fully invariant direct summand B of W such that B ≤ A and $\frac{A}{B}$ <<^{a} $\frac{W}{B}$.

We complete the proof in the same way of proposition (2-16).

Proposition (3-13)

If W is strongly R - a - (hollow - lifting), then every submodule A of W with $\frac{W}{A}$ is R - a - hollow can be written as A = B + S, with B is a fully invariant direct summand of W and S <<^{a} W.

Proof: Clear.

Corollary (3-14)

Let W be a projective SS - module. If W is a strongly R - a - (hollow - lifting), then for every submodule A of W such that $\frac{W}{A}$ is R - a - hollow, $\frac{W}{A}$ has a projective R - a - cover.

Proof

Suppose that W s a projective SS - module and W is a strongly R - a - (hollow - lifting) module, and let A be a submodule of W with $\frac{W}{A}$ is R - a - hollow. By proposition (3-9), we have that W is R - a - (hollow - lifting), and by proposition(3-7), we have that $\frac{W}{A}$ has a projective R - a - cover.

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