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Dual of Extending Acts

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Abstract

Since 1980s, the study of the extending module in the module theory has been a major area of research interest in the ring theory and it has been studied recently by several authors, among them N.V. Dung, D.V. Huyn, P.F. Smith and R. Wisbauer. Because the act theory signifies a generalization of the module theory, the author studied in 2017 the class of extending acts which are referred to as a generalization of quasi-injective acts. The importance of the extending acts motivated us to study a dual of this concept, named the coextending act. An S-act M_S is referred to as coextending act if every coclosed subact of Ms is a retract of M_S where a subact A_S of M_S is said to be coclosed in M_S if whenever the Rees factor ${A_S}/{B_S}$ is small in the Rees factor ${M_S}/{B_S}$ then $A_S=B_S$ for each subact B_S of A_S. Various properties of this class of acts have been examined. Characterization of this concept is intended to show the behavior of a coextending property. In addition, based on the results obtained by us, the conditions under which subacts inherit a coextending property were demonstrated. Ultimately, a part of this paper focused on studying the relationships between these acts and other related acts.

Keywords: Coextending acts, Extending acts, Essential subacts, Coessential subacts, Closed subacts, Coclosed subacts AMS Subject Classification: 20M30, 20M99, 08B30.

الثنائية لأنظمة التوسع

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الخلاصه

منذ الثمانينات من القرن الماضي، كانت دراسة المقاسات الموسعة في نظرية المقاس مجالًا رئيسيًا من مجالات الاهتمام البحثي في نظرية الحلقات حيث تمت دراستها مؤخرًا من قبل العديد من المؤلفين، من بينهم N.V. Dung، D.V. هوين ، وسميث و R. Wisbauer. بسبب أن نظرية النظام هي تعميم لنظرية المقاس ، درس المؤلف في عام 2017 فئة من الأنظمة الموسعة التي يشار إليها باسم اعمام الأنظمة شبه الاغمارية. دفعتنا أهمية الانظمة الموسعة إلى دراسة المفهوم الثنائي او المقابل لهذا المفهوم ، أطلق عليه اسم الانظمة الموسعة المشاركة (coextending). يشار إلى الانظمة بانها موسعة مشاركة (coextending) إذا كان كل نظام جزئي مغلق مشارك coclosed من النظام M_S هو تراجع من M_S حيث يقال أن النظام الجزئي A_S من M_S يكون مغلق مشارك coclosed في M_S متى ما كان العامل ${A_S}/{B_S}$ هو صغيرًا في العامل

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فان $^{M_S}/_{B_S}$ لكل نظام جزئي B_S من A_S. تم فحص خصائص مختلفة لهذه الفئة من الانظمة. $A_S = B_S$
يهدف توصيف هذا المفهوم إلى إظهار سلوك الانظمة الموسعة المشاركة (coextending). بالإضافة إلى ذلك ، بناءً على النتائج التي حصلنا عليها ، فقد تم توضيح الشروط التي بموجبها ترث الانظمة الجزئية خاصية التوسع المشارك (coextending property). في نهاية المطاف، ركز جزء من هذه الورقة على دراسة العلاقات بين هذه الأنظمة وغيرها من الأنظمة ذات الصلة.

1. Introduction

It is well-known that the extending modules have been extensively studied in a monograph by Dung *et al.* [1], as well as in an earlier book by Mohammed and Müller [2]. As for the act theory which is referred to as a generalization of the module theory, and for the importance of this subject, the extending act was studied by the author who then submitted generalizations for it [3, 4].

Note that we will use terminologies and notations from previous works [5,6,7,8,9]. In addition, for more information about generalization of injective acts, we refer the reader to other references [7,10,11,12,13].

Throughout this paper, S is a commutative monoid with zero element and every S-act is unitary right S-act with zero element Θ which is denoted by M_S. Besides, the symbol $^{A_S}/_{B_S}$ is referred to as Rees factor. It is familiar that an S-act can be found by some other terminologies, as follows: S-systems, S-sets, S-operands, S-polygons, transition systems, and S-automata [14]. An S-act N_S is referred to as a retract of S-act M_S if and only if there exists a subact H_S of M_S and S-epimorphism $f:M_S \longrightarrow H_S$ such that $N_S \cong H_S$ and f(h)= h for every $h \in H_S$ [14,P.84]. An S-homomorphism f which maps an S-act A_S into S-act B_S is said to be split if there exists S-homomorphism g which maps B_S into A_S such that $fg=1_B$[3]. A subact A_S of B_S is called large (or essential) in B_S if and only if any homomorphism $f:B_S \longrightarrow H_S$, where H_S is any S-act with restriction to A_S is one to one, then f is itself one to one [4]. In this case, we say that B_S is essential extension of A_S . In a previous article [4], Berthiaume showed that every S-act has a maximal essential extension which is injective and it is unique up to S-isomorphism over M_S. A non-zero subact B_S of A_S is intersection large if for all non-zero subact C_S of A_S , $C_S \cap B_S \neq \Theta$, and will be denoted by B_S is ∩-large in A_S [15]. In another study [16], Feller and Gantos proved that every large subact of A_S is ∩-large, but the converse is not true in general. An equivalence relation ρ on a right S-act N_S is a congruence relation if aρb implies that as ρbs for all $a,b \in N_S$ and $s \in S$ [17].The congruence ψ_N is called singular on N_S and it is defined by $a\psi_N b$ if and only if ax = bx for all x in some ∩-large right ideal of S [18] . A subact B_S of S-act A_S is called closed if it has no proper ∩-large extension in A_S, that is the only solution of $B_S \hookrightarrow^{\cap l} C_S \ {}_{\neq}\!\hookrightarrow A_S$ is $B_S=C_S$ .A subact B_S of a right S-act A_S is called small (or superfluous) in A_S if for every subact C_S of A_S , $B_S \cup C_S=A_S$ implies $C_S=A_S$[8]. as Also, an S-act M_S is called extending, if every subact of M_S is ∩-large in a retract [6]. Equivalently, M_S is extending if and only if every closed subact of M_S is a retract [6].

In this paper, we introduce a new concept, namely the coextending act, as a dual of the class of extending acts, where M_S is referred to as coextending, if every coclosed subact of M_S is a retract of M_S, where a subact N_S of M_S is said to be coclosed in M_S if whenever $^{N_S}/_{H_S}$ is small in $^{M_S}/_{H_S}$ then $N_S=H_S$ for each subact H_S of N_S.

This article consists of three sections. Some essential properties and examples of coextending acts are given in section two. Like extending acts, the direct sum of coextending act may not be coextending. We show this fact by an example in section two also. For this reason, we give certain conditions under which the direct sum of coextending acts is coextending act, in theorem (3.5) and theorem (3.6)). In section three, some relationships between coextending acts and other related acts, such as lifting and semisimple acts are investigated. Conclusions and discussions are presented in section four.

2. Dual of Extending Acts

In this section, we introduce and study a dual of the class of extending acts which is coextending acts, but before that we need the following concepts:

Definition (2.1):[14]. Let M_S be S-acts and N_S any subact of M_S that defines the Rees congruence ρ_N on M, by setting $a\rho_N a^/$ if $a, a^/ \in N_S$ or $a = a^/$. The resulting factor act is referred to as **Rees factor** of M_S by subact N_S and it is denoted by M_S/N_S .

Definition (2.2):[14]. Let $f{:}M_S \longrightarrow N_S$ be S-homomorphism. Then the kernel equivalence kerf is defined by a(kerf) $a^/$ if and only if $f(a)=f(a^/)$ for $a, a^/ \in M_S$ is an **act congruence** which is referred to as kernel congruence of f.

Definition (2.3):[14] Let M_S be an S-act and $m \in M_S$. Then the homomorphism from S_S into M_S(or $_SM$) is defined by $\lambda_m(s) = ms$(or sm) for every $s \in S$. The kernel congruence $\ker\lambda_m$ on S_S is referred to as **annihilator congruence** of $m \in M_S$.

Definition(2.4): A subact N_S of M_S is said to be **coclosed** in M_S if whenever N_S/H_S is small in M_S/H_S then $H_S=N_S$ for each subact H_S of N_S.

Definition (2.5): An S-act M_S is referred to as **coextending** act if every coclosed subact of Ms is a retract of M_S.

Definition(2.6): A subact B_S of an S-act M_S is called **coessential** subact of A_S in M_S if A_S/B_S is small in M_S/B_S.

Remarks and Examples (2.7)

1. Unlike for modules, not every kernel-congruence can be described by a subact, but any subact N_S of M_S gives rise to a kernel congruence $\ker\pi$, where π: $M_S \longrightarrow M_S/N_S$ is the canonical epimorphism. It is well-known that M_S/N_S has a zero, which is the class consisting of N_S. Notice that if M_S/N_S is the Rees factor act of M_S by the subact N_S, then the class$[n]_{\rho_N}$of an element $n \in N_S$ is a zero in M_S/N_S and the class $[m]_{\rho_N}$ of $m \in M_S/N_S$ is the one-element classes {m}. Thus, the Rees factor act M_S/N_Scould be considered as $M_S/N_S = (M_S/N_S \ \dot{\cup}\{\Theta_S\})$.

2. It is well-known that every hollow act is coextending act, where an S-act M_S is called hollow if every proper subact of M_S is small [19].

Proof: Let M_S be a hollow act and let N_S be a coclosed subact of M_S, then N_S is small in M_S, and so for each subact K_S of N_S, N_S/K_S is small in M_S/K_S. But N_S is a coclosed subact of M_S, which implies that $K_S=N_S$, and then $N_S=(\Theta_S)$, which is a retract of M_S.

3. It is obvious that the converse of 2 is generally not true. This means that the coextending act is not a hollow act. For example, Z as Z-act is coextending act, but not hollow act.

4. Every semisimple act is coextending-act, but the converse is not true in general; for examples: Z as Z act is coextending act but not semisimple.

5. Every local act (i.e., an act that has only maximal subact), is a coextending act.

6. Every uniserial act is a coextending act, where an act is referred to as uniserial act if its sub-acts are linearly ordered by inclusion [18]. Also, a monoid S is called uniserial monoid if it is uniserial as an S-act.

7. It is clear that $M_S = Z_2 \oplus Z_4$ is a coextending act.

8. Isomorphic to coextending act is coextending act.

9. Recall that an S-act A_S is called co-uniform if all proper subacts of A_S are coessential [19]. In other words, we reformulate it as follows: an S-act M_S is referred to as couniform, if every proper subact K_S of M_S is either (Θ) or there exists a proper subact $K^/$ of K such that $K_S/K^/$ is small in $M_S/K^/$.

Proof: Let K_S be subact of M_S. If $K_S=\Theta$, then K_S is coclosed retract of M_S and the proof is complete. If $K_S \neq \Theta$, and as M_S is couniform act, so there exists a proper subact $K^/$of K_S where $K_S/K^/$ is small in $M_S/K^/$.Thereby K_S is not coclosed in M_S and Θ is the only proper coclosed subact of M_S, and then M_S

is coextending act.

It is obvious that every couniform act is coextending act, but the converse is generally not true; for example the Z-act Z_6 is coextending act but not couniform (since a semisimple act is coextending act but not couniform act).

Besides, every Artinian couniform act is a hollow act, hence it is a coextending act.

The following proposition gives some important properties of the coextending acts.

Proposition (2.8): A retract subact of coextending act is a coextending act.

Proof: Let M_S be S-act, and let N_S be a retract subact of M_S, Let K_S be a coclosed subact of N_S. Since N_S is a retract subact of M_S, so N_S is a coclosed subact of M_S. It implies that K_S is a coclosed subact of M_S, hence K_S is a retract of M_S , that is $M_S = K_S \oplus L_S$ for some subact L_S of M_S. $N_S = M_S \cap N_S = (K_S \oplus L_S) \cap N_S = K_S \oplus (L_S \cap N_S)$. Thus K_S is a retract subact of N_S, i.e N_S is a coextending act.

Corollary (2.9): If M_S is a coextending act and N_S is a coclosed subact of M_S, then M_S/N_S is a coextending act.

Proof: Since M_S is a coextending act and N_S is a coclosed subact of M_S, then N_S is a retract of M_S, so $M_S = N_S \oplus W_S$ for some subact W_S of M_S. Hence $M_S/N_S \cong W_S$. But W_S is a retract of M_S, so by proposition (2.7), W_S is a coextending act. For this reason and by remarks and examples (2.7) and (8), M_S/N_S is coextending act.

Definition (2.10): An S-act M_S is referred to as **hereditary** if every subact of M_S is projective. Especially, a monoid S is called hereditary if all subacts of projective acts over S are again projective. If this is required only for finitely generated subacts, it is referred to as semihereditary.

The following theorem gives the hereditary property for the coextending act. Before that, we need the following concepts: recall that an S-act M_S is called **multiplication** if for each subact N_S of M_S there exists an ideal I of S, such that N=MI [20]. Recall that an S-act M_S is called faithful if J=(Θ), this means that the annihilator of M_S is the zero ideal where the ideal $J_{(\Theta)} = J = \{s \in S \mid M_S = (\Theta)\}$ of S is referred to as annihilator of M_S in S [21]. It is obvious that the field of rational number Q as Z-act is faithful, but Z_n as Z-act is not faithful.

Theorem (2.11): Let M_S be a finitely generated faithful multiplication S-act. Then S is a coextending monoid if and only if M_S is a coextending act.

Proof: ⟹) Let N_S be a coclosed subact of M_S. Since M_S is a multiplication S-act, then N=MI for some ideal I of S. It is easy to see that I is a coclosed in S. Hence, I is a retract of S, and so $S=I\oplus J$ for some ideal J of S. It follows that $M_S=MI\oplus MJ=N\oplus MJ$. This means that N_S is a retract subact of M_S.

⟸) Let I be a coclosed ideal of S. By putting N=MI, then N_S is a coclosed subact of M_S. But M_S is a coextending act, so N_S is a retract subact of M_S; that is, there exists a subact W_S of M_S such that $N_S \oplus W_S = M_S$. But W_S=MJ for some ideal J of R. Now $MI\oplus MJ=M$ implies that $M(I\cup J)=MS$. Since M_S is a finitely generated faithful multiplication act, then $I\cup J=S$, which means that I is a retract of S.

Lemma (2.12): Let $f:M_1 \longrightarrow M_2$ be an epimorphism from an S-act M_1 to a projective S-act M_2. If M_1 is coextending act, then M_2 is coextending.

Proof: Let f be epimorphism, and since M_2 is projective, so every epimorphism is split. This means that there exists an S-homomorphism g from M_2 into M_1 such that $fg = 1_{M_2}$, and since every act is epimorphic image to free act, so M_1 is free. This implies that M_2 is isomorphic to a retract of M_1, and by proposition (2.8), every retract of M_1 is coextending act. Thereby, by remarks and examples (2.7) and (8), M_2 is coextending act.

The following proposition gives a necessary and sufficient condition on a free act to be a coextending act

Proposition (2.13): Let S is a monoid, and then every free S-act is a coextending act if and only if every free projective S-act is a coextending act.

Proof: ⟹) Let M_S be a projective S-act. M_S is an epimorphic image of a free S-act say F. This means that there exists epimorphism $h:F \longrightarrow M_S$. By the hypothesis, F is a coextending act, and since M_S is projective and h is epimorphism, so by lemma (2.12), M_S is a coextending act.

⟸) It is obvious.

Corollary (2.14): Let S be a monoid, and then every finitely generated free S- act is a coextending act if and only if every finitely generated projective S-act is a coextending act.

In the following, we study when the direct sum of the coextending act is coextending. In fact, this is not true in general. Now, we study some cases in which the direct sum of coextending act is a coextending act. Before that we need the following lemmas.

Lemma (2.15): Let $M_S=M_1\oplus M_2$ where M_1 and M_2 be two S-acts, and let $A=A_1\oplus A_2$, where $A_1 \subseteq M_1$ and $A_2 \subseteq M_2$. If A is a coclosed subact of M_S, then A_1 is a coclosed subact of M_1 and A_2 is a coclosed subact of M_2.

Proof: Assume that the Rees factor A_1/B_1 is small in the Rees factor M_1/B_1 and the Rees factor A_2/B_2 is small in the Rees factor M_2/B_2, where $B_1\subseteq A_1$ and $B_2\subseteq A_2$. Therefore, Rees factor $\frac{A_1\oplus A_2}{B_1\oplus B_2}$ is small in the Rees factor $\frac{M_1\oplus M_2}{B_1\oplus B_2}$. Because of that, A is coclosed subact of M_S, then we obtain that $A_1=B_1$ and $A_2=B_2$ .Hence, $A_1$and A_2 are coclosed subacts in M_1 and M_2, respectively.

Lemma (2.16): Let $M_S=M_1\oplus M_2$, where M_1 and M_2 are S-acts. If $ann_SM_1\cup ann_SM_2=S$, then any subact of M_S can be written in the form $N=N_1\oplus N_2$, where N_1 is a subact of M_1 and N_2 is a subact of M_1.

Proof: Let N be any subact of M_S. We claim that $N=N_1\oplus N_2$, for some subacts N_1 of M_1 and N_2 of M_2. In fact, if $n\in N$, then $n=(x,y)$, for some $x\in N_1\subseteq M_1$ and $y\in N_2\subseteq M_2$. So, $x\in M_1$ and $y\in M_2$. Furthermore, there exist elements $(a_1,a_2)\in ann_S(M_1)$ and $(b_1,b_2)\in ann_S(M_2)$, such that (a_1,b_1) or $(a_2,b_2)=(1,1)$. Let $N_1=ann_S(M_2)x$ and $N_2=ann_S(M_1)y$, then N_1 is subact of M_1 and N_2 is subact of M_2. Now, $x=(x_1,x_2)=(1,1)(x_1,x_2)=(a_1,b_1)(x_1,x_2)=(a_1x_1,b_1x_2)=b_1x_2\in M_1$and $y=(y_1,y_2)=(1,1)(y_1,y_2)=(a_1,b_1)(y_1,y_2)=(a_1y_1,b_1y_2)=a_1y_1\in M_2$. Then $n=(x,y)=(b_1x_2,a_1y_1)\in N_1\oplus N_2$, therefore $N\subseteq N_1\oplus N_2$. For the other direction, let $h=(cx,dy)$ for some $c=(c_1,c_2)\in ann_SM_2$ and $d=(d_1,d_2)\in ann_SM_1$. Thus, $h=((c_1,c_2)x,(d_1,d_2)y)=(c,d)(x,y)=(c,d)n\in N$. For this reason, we have $N_1\oplus N_2\subseteq N$. Therefore, the proof is complete.

Lemma(2.17): Let $M_S=M_1\oplus M_2$, where M_1 and M_2 be S-acts and let $ann_SM_1\cup ann_SM_2=S$. Then N_S is a coclosed subact of M_S if and only if there exist coclosed subacts N_1 of M_1 and N_2 of M_2 such that $N_S= N_1\oplus N_2$.

Proof:⟹) Since N_S is subact of M_S and $M_S=M_1\oplus M_2$, $ann_SM_1\cup ann_SM_2=S$, so by lemma(2.16), there exists subacts N_1 and N_2 of M_1 and M_2, respectively, such that $N_S=N_1\oplus N_2$, and by lemma(2.15) both of N_1 and N_2 are coclosed subacts in M_1 and M_2, respectively.

⟸) In order to prove that N_S is a coclosed subact of M_S, assume that N/B is small in M/B where B is a subact of M_S. Since $ann_SM_1\cup ann_SM_2=S$, so $B_S=B_1\oplus B_2$ for some subacts B_1 and B_2 of M_1 and M_2, respectively. Thus:

$N/B = \frac{N_1\oplus N_2}{B_1\oplus B_2}$ which is small in $\frac{M_1\oplus M_2}{B_1\oplus B_2}$. Then, we have $N_1/B_1 \oplus N_2/B_2$ is small in $M_1/B_1 \oplus M_2/B_2$which implies that $N_1/B_1$is small in M_1/B_1 and $N_2/B_2$is small in M_2/B_2. Since N_1 and N_2 are coclosed subact of M_1 and M_2, respectively, thus $N_1=B_1$ and $N_2=B_2$, and hence $N_S= B_1\oplus B_2=B_S$.

In the following theorems, we put certain conditions under which the direct sum of two coextending acts is coextending act.

Theorem (2.18): Let $M_S=M_1\oplus M_2$ where M_1 and M_2 be S-acts. If $ann_SM_1\cup ann_SM_2=S$, then M_S is a coextending act if and only if both of M_1 and M_2 are coextending acts.

Proof: ⟹) It follows from proposition (2.8).

⟸)Let N_S be a coclosed subact of M_S. By lemma(2.17), for some coclosed subacts N_1 and N_2 of M_1 and M_2, respectively, we have $N=N_1\oplus N_2$. But M_1 and M_2 are coextending acts, so N_1 is retract of M_1 and N_2 is a retract of M_2, that is $N_1\oplus W_1=M_1$ and $N_2\oplus W_2=M_2$, for some subacts W_1 of M_1 and W_2 of M_2. Hence:

$N\oplus(W_1\oplus W_2)=(N_1\oplus N_2)\oplus(W_1\oplus W_2)=(N_1\oplus W_1)\oplus(N_2\oplus W_2) =M_1\oplus M_2=M_S$. Therefore N is a retract of M_S, and hence M_S is a coextending act.

Theorem(2.19): Let $M_S=\oplus_{i=1}^{n} M_i$, where each of M_i is an S-act for each i = 1, ..., n. If every subact of M_S is a fully invariant, then M_S is a coextending act if and only if each M_i is a coextending act for each i = 1, ..., n.

Proof:⟹) It follows from proposition (2.8).

⟸) Let N_S be a coclosed sub act of M_S. By assumption, N_S is a fully invariant subact of M_S, so $N_S =\oplus_{i=1}^{n} (N\cap M_i)$. On the other hand, N_S is coclosed of M_S, so by lemma (2.15), for each i=1,...,n,

$N \cap M_i$ is a coclosed subact of M_i. Since M_i is a coextending act for each i=1, ...,n, hence $N \cap M_i$ is a retract of M_i for each i=1,...,n, then $(N \cap M_i) \oplus B_i = M_i$ for some subact B_i of M_i. Therefore, $\oplus_{i=1}^{n} M_i = \oplus_{i=1}^{n} \{(N \cap M_i) \oplus B_i = \oplus_{i=1}^{n} \{(N \cap M_i)\} \oplus \{\oplus_{i=1}^{n} B_i\}$. For this reason, we have $M_S = N_S \oplus B_S$, where $B = \oplus_{i=1}^{n} B_i$. This means that M_S is coextending act.

3. THE RELATIONSHIPS OF COEXTENDING ACTS WITH OTHER RELATED CONCEPTS

In this section, we give some relationships between coextending acts and some other acts, such as the lifting and semisimple acts, but before that, we need the following concept:

Definition(3.1): An S-act M_S is referred to as **lifting**, if for every subact N_S of M_S contains a retract H_S of M_S such that N_S/H_S is small in M_S/H_S.

From the above definition, we have the following:

Proposition (3.2): If M_S is a lifting S-act, then M_S is a coextending act.

Proof: Let N_S be a coclosed subact of M_S. Since M_S is a lifting act, so N_S contains a retract W_S of M_S such that N_S/W_S is small in M_S/W_S by definition (3.2). But N_S is a coclosed subact of M_S, then $W_S=N_S$. That is M_S is a coextending act.

The converse of proposition(3.2) is generally not true; for example, Z as Z-act is a coextending act, but it is not lifting. However, as we get in the following theorem, the condition to be the converse is true, but first we need the following concept:

Definition (3.3): an S-act M_S is referred to as **amply supplemented** act, if every supplement subact of M_S is a retract of M_S. Equivalently, if for any two subact A_S and B_S of M_S with $A_S \cup B_S = M_S$, then B_S contains a supplement of A_S in M_S (where a subact A_S is a supplement of B_S if and only if $A_S \cup B_S = M_S$ and $A_S \cap B_S$ is small in A_S. Equivalently, a subact A_S of M_S is called supplement of B_S in M_S if $A_S \cup B_S = M_S$ and A_S is a minimal element in the set of subacts L_S of M_S with $B_S \cup L_S = M_S$)

Theorem (3.4): Let M_S be an S-act, then M_S is a lifting act if and only if M_S is a coextending act and amply supplemented.

In the following result we give a condition to obtain the coincide among the concepts of the coextending act, lifting act and semisimple act:

Proposition (3.5): Let M_S be an S-act. If every subact of M_S is a coclosed, then the following statements are equivalent:

1. M_S is a lifting act.
2. M_S is a coextending act.
3. M_S is a semisimple act.

Proof: (1)⟹(2): It follows from proposition (3.2).
(2) ⟹ (3) It is obvious.
(3) ⟹ (1) It is clear.

The next proposition gives another condition so that the converse of proposition (3.2) is true:

Definition (3.6): A subact B_S of an S-act M_S is called **coclosure** of A_S in M_S, if A_S/B_S is small in M_S/B_S and B_S is coclosed subact of M_S .

Proposition (3.7): If an S-act M_S is a coextending act, such that every subact N_S of M_S has a coclosure, then M_S is a lifting act.

Proof: Let N_S be a subact of M_S. By assumption, N_S has coclosure subact. For this reason, there exists a coclosed subact B_S of M_S such that N_S/B_S is small in M_S/B_S . But M_S is a coextending act, therefore B_S is a retract of M_S. Thereby M_S will be a lifting act by definition(3.1).

4. Conclusions and Discussions

From the previous theorems, examples, remarks, and propositions, we can present some major points, as follows:

1. Proposition (2.8) and corollary (2.9) answered the earlier submitted question; what are the conditions on subacts to inherit the property of coextending? Accordingly, they gave the following two results:

a. When subacts are retracted.

b. If a subact N_S is coclosed, then the quotient subact ${M_S}/{N_S}$ is coextending.

2. Theorem(2.11) demonstrated the hereditary property for the coextending act, where it was stated that a monoid S is coextending if and only if M_S is coextending under the following conditions on S-act M_S:

a. Finitely generated.

b. Faithful.

c. Multiplication.

3. Lemma (2.12) explained that the epimorphic image of a coextending act is coextending when the epimorphic act is projective.

4. Based on lemma (2.12), proposition (2.13) gave another result on the epimorphic image; that is, every free S-act is coextending if and only if every free projective is coextending.

5. Lemma (2.15) revealed that subacts are coclosed if they have a decomposition form, as follows: if $A_S=A_1\oplus A_2$ is coclosed subact of $M_S=M_1\oplus M_2$, then A_1 is coclosed subact of M_1 and A_2 is coclosed subact of M_2.

6. Lemma(2.17) gave another condition on any subact to be coclosed when $ann_S M_1 \cup ann_S M_2=S$, as follows: A subact A_S of $M_S= M_1\oplus M_2$ is coclosed if and only if there exists A_1 and A_2 are coclosed subacts of M_1 and M_2, respectively, such that $A_S=A_1\oplus A_2$.

7. We gave certain conditions in theorem (2.18) to the direct sum of coextending acts to be coextending depending on lemma (2.17).

8. Theorem (2.19) explained an important result; that is, a finite direct sum of a coextending act is coextending under the condition that all the subacts are fully invariant.

9. Proposition (3.2) showed the relation between the lifting act and coextending, as in the following: every lifting act is coextending.

10. Theorem (3.4) gave a condition to the converse of proposition (3.2) to be true, as follows: an S-act M_S is lifting if and only if M_S is amply supplemented and coextending.

11. It was given a condition in proposition (3.5) to coincide the following concepts: lifting acts, coextending acts, and semisimple acts. This condition was that every subact of S-act M_S must be coclosed.

12. In proposition (3.7), we obtained the equivalence between the lifting act and the coextending. Thereby, this proposition suggested another condition to the converse of proposition (3.2) to be true. This condition stated that every subact of S-act has a coclosure.

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