DOI: 10.24996/ijs.2020. SI.1.8

ISSN: 0067-2904

Lotka-Volterra Model with Prey-Predators Food Chain

A.G. Farhan
Department of Mathematics, College of Basic Education, Mustansiriyah University, Baghdad, Iraq

Received: 15/11/ 2019 Accepted: 15/ 3/2020

Abstract

In this work, we consider a modification of the Lotka-Volterra food chain model of three species, each of them is growing logistically. We found that the model has eight equilibrium points, four of them always exist, while the rest exist under certain conditions. In terms of stability, we found that the system has five unstable equilibrium points, while the rest points are locally asymptotically stable under certain satisfying conditions. Finally, we provide an example to support the theoretical results.

Keywords: Prey, Predator, Equilibrium points, Local stability.

نموذج Lotka–Volterra مع السلسلة الغذائية فريسة–مفترس

عبدالخضر غالي فرحان

قسم الرياضيات، كلية التربيه الاساسيه، الجامعة المستنصرية، بغداد، العراق

الخلاصه

في هذا العمل ، نعتبر تعديلا لنموذج Lotka–Volterra لسلسلة الغذاء لثلاثة أنواع ، كل نوع منها ينمو لوجستياً. وجدنا أن للنموذج ثمان نقاط توازن ، أربعة منها موجودة دائمًا ، بينما الباقي موجودة تحت شروط معينة. من حيث الاستقرار ، وجدنا أن النظام لديه خمس نقاط توازن غير مستقرة ، في حين أن بقية نقاط التوازن مستقرة محليًا بأضطراد تحت شروط محددة. أخيرًا ، قدمنا مثالًا لدعم النتائج النظرية.

1. INTRODUCTION

In 1920, Vito Volterra introduced his famous mathematical model of the equation that described the relationship between two reactive species. Volterra was interested in studying and analyzing the phenomenon of the increase of fish as a predator, and thus the decrease of fish as prey, in the Adriatic Sea, during the First World War. The Lotka-Volterra equation is a pair of first-order nonlinear differential equations that can be written as stated below [1, 2, 3]:

$\dot{x} = ax - bxy, \quad \dot{y} = cxy - dy.$

In the first article [1], an exogenous control in a model of population dynamics of two species was considered on the level of predators only, as follows:

$\dot{x} = ax - bxy, \quad \dot{y} = cxy - dy - u.$

The equation of Lotka-Volerra has been modified and studied by many researchers. Modifications are presented as a system of a pair of nonlinear differential equations of the first order, describing the prey and predator. It also involves aspects of logistic growth and carrying capacity of prey, carrying predator capability, and predator factor, as follows [3]:

$\dot{x} = ax - bx^2 - bxy, \qquad \dot{y} = -dy + exy - fy^2,$

or where the two species are grown logistically and compete with each other , as follows [4,5] :

Email: a.g.farhan.edbs@uomustansiriyah.edu.iq

$\dot{x} = ax(1 - cx - bxy), \quad \dot{y} = dy(1 - ex - fy).$

which involves logistic growth and carrying capacity of prey, carrying predator capability, and predator factor. Other modifications were presented as a system of three nonlinear differential equations of the first-order. It describes the relationship between one prey and two predators, in which the prey grows logistically, as in below [3, 6]:

$\dot{x} = ax - xy - xz, \dot{y} = -by + xy, \dot{z} = -cz + xz.$

According to this model, in the absence of the prey, the predators will be dying. In [7], a modification is presented as the following system of three nonlinear differential equations of the first-order, that describes the relationship between two prey and one predator, in which the prey grow logistically:

$\dot{x} = ax - xz, \quad \dot{y} = by - yz, \quad \dot{z} = -cz + xz + yz.$

In [8], a prey-predator model was studied when there was a disease in the both species, and in [9] a mathematical model for two prey and two predators was studied, whereby all the species grow logistically.

In this paper, we propose a modified model of the Lotka-Volterra equation, which is a system of three dimensional first-order nonlinear differential equations. It describes the population dynamics of a food chain in which all species grow logistically, which means that there is no death of any species in the absence of other species.

2. THE MATHEMATICAL MODEL

Consider the mathematical model:

$$\begin{aligned} \dot{x} &= x(a - \gamma x - by - cz), \\ \dot{y} &= y(d - \beta y - ez + fx), \\ \dot{z} &= z(g - \delta z + hy + mx), \end{aligned} \tag{2.1}$$

of three species x, y and z. All Species grow logistically, as shown by the terms $x(a - \gamma x)$,$y(d - \beta y)$ and $z(g - \delta z)$, where a , d, *and* g are the natural growth rates of *x*, *y* and *z* respectively;γ, β and δ are the natural deaths of x, y and z, respectively; b and c are the rates of change of x due to the presence of y and z, respectively*;* e and f are the rates of change of y due to the presence of x and z; respectively *;* h and m are the rates of change of z due to the presence of y and x, respectively.

3. BOUNDEDNESS OF THE POSITIVE SOLUTIONS

In this section, some sufficient conditions are provided, in order for the positive solutions of the system (2.1) to be bounded.

THEOREM 1: Suppose that $f < b, m < c, h < e$. Then all the positive solutions of the system (2.1) are bounded, if the initial point belongs to $\mathcal{B}$, where:

$$\mathcal{B} = \{(x, y, z): x \le H_1, y \le H_2, z \le H_3, H_i \text{are real posive numbers}, i = 1,2,3\}$$

Proof:

Consider the function:

$u(t) = x(t) + y(t) + z(t).$

$$\dot{u} = x(a - \gamma x - by - cz) + y(d - \beta y - ez + fx) + z(g - \delta z + hy + mx)$$
$$\le x(a - \gamma x) + y(d - \beta y) + z(g - \delta z).$$

Now

$\dot{u} + pu \le x(a - \gamma x + p) + y(d - \beta y + p) + z(g - \delta z + p) \le x(a + p) + y(d + p) + z(g + p) \le (a + p)^2 + (d + p)^2 + (g + p)^2.$

So we have

$\dot{u} \le -pu + (a + p)H_1 + (d + p)H_2 + (g + p)H_3.$

So that $0 \le u(t) \le \frac{(a+p)H_1 + (d+p)H_2 + (g+p)H_3}{p} + u(0)e^{-\rho t}$, and for t→∞,

$$0 \le u(t) \le \frac{(a + p)^2 + (d + p)^2 + (g + p)^2}{\rho}.$$

Thus, we have completed the proof of the theorem.

4. EXISTENCE AND LOCAL STABILITY OF THE EQUILIBRIA POINTS

In this section, we will give the necessary and sufficient conditions for the existence of 8 equilibrium points for the system (2.1) and study their local stability. The system has in origin an equilibrium point, three axial equilibrium points, three boundary equilibrium points, and one interior equilibrium point.

1) The equilibrium point $P_0 = (0,0,0)$ always exists.
Easy calculations show that the characteristic equation of the matrix of Jacobi at the equilibrium point $P_0 = (0,0,0)$ takes the following form:

$$\begin{vmatrix} a-\lambda & 0 & 0 \\ 0 & d-\lambda & 0 \\ 0 & 0 & g-\lambda \end{vmatrix} = 0, \tag{4.1}$$

It is clear that the solutions of the equation (4.1) are:
$\lambda_1 = a > 0, \lambda_2 = d > 0$, and $\lambda_3 = g > 0$.
Since λ_1 is positive and λ_2, λ_3 are negative, so $P_0 = (0,0,0)$ is an unstable node point (source).
2) The equilibrium point $P_1 = (\frac{a}{\gamma}, 0, 0)$ always exists.
The characteristic equation of the matrix of Jacobi of the system (2.1) at the equilibrium points $P_1 = (\frac{a}{\gamma}, 0, 0)$ can be written as follows:

$$\begin{vmatrix} -a-\lambda & -\frac{ba}{\gamma} & -\frac{ca}{\gamma} \\ 0 & d-\lambda & 0 \\ 0 & 0 & g-\lambda \end{vmatrix} = 0, \tag{4.2}$$

The solutions of the equation (4.2) are:
$\lambda_1 = -a < 0, \lambda_2 = d > 0$ and $\lambda_3 = g > 0$.
Since λ_1 is negative and λ_2, λ_3 are positive, so $P_1 = (\frac{a}{\gamma}, 0, 0)$ is a saddle point (unstable).

3) The equilibrium point $P_2 = \left(0, \frac{d}{\beta}, 0\right)$ always exist*s*.
The characteristic equation of the matrix of Jacobi of the system (2.1) at the equilibrium point $P_2 = \left(0, \frac{d}{\beta}, 0\right)$ is:

$$\begin{vmatrix} a-\lambda & 0 & 0 \\ \frac{fd}{\beta} & -d-\lambda & -\frac{ed}{\beta} \\ 0 & 0 & g-\lambda \end{vmatrix} = 0, \tag{4.3}$$

Thus the solutions of the equation (4.3) are:
$\lambda_1 = a > 0, \lambda_2 = -d < 0$, and $\lambda_3 = g > 0$.
Since λ_2 is negative and λ_1, λ_3 are positive, so $P_2 = (0, \frac{d}{\beta}, 0)$ is a saddle point (unstable).

4) The equilibrium point $P_3 = \left(0,0,\frac{g}{\delta}\right)$ always exists.
The characteristic equation of the matrix of Jacobi of the system (2.1) at the equilibrium points $P_3 = (0,0,\frac{g}{\delta})$ is written as follows:

$$\begin{vmatrix} a-\lambda & 0 & 0 \\ 0 & d-\lambda & 0 \\ \frac{mg}{\delta} & \frac{g}{\delta} & -g-\lambda \end{vmatrix} = 0, \tag{4.4}$$

Thus the solutions of the last equation are as follows:
$\lambda_1 = a > 0, \lambda_2 = d > 0$, and $\lambda_3 = -g < 0$.
Since λ_3 is negative and λ_1, λ_2 are positive, so $P_3 = (0,0,\frac{g}{\delta})$ is a saddle point (unstable).
5) The equilibrium point

$$P_4 = (\tilde{x}, \tilde{y}, 0) = \left(\frac{a\beta - bd}{\gamma\beta + bf}, \frac{\gamma d + af}{\gamma\beta + bf}, 0\right)$$

exists if and only if:

$$a\beta > bd. \tag{4.5}$$

The characteristic equation of the matrix of Jacobi of the system (2.1) at the equilibrium point $P_4 = (\tilde{x}, \tilde{y}, 0)$ is written as follows:

$$\begin{vmatrix} -\gamma\tilde{x}-\lambda & -b\tilde{x} & -c\tilde{x} \\ f\tilde{y} & -\beta\tilde{y}-\lambda & -e\tilde{y} \\ 0 & 0 & g+h\tilde{y}+m\tilde{x}-\lambda \end{vmatrix} = 0, \quad (4.6)$$

and the solutions of the equation (4.6) are written as follows:

$$\lambda_i = \frac{-(\gamma\tilde{x}+\beta\tilde{y}) \mp \sqrt{(\gamma\tilde{x}+\beta\tilde{y})^2 - 4\tilde{x}\tilde{y}(\gamma\beta+bf)}}{2}, \text{i} = 1,2.$$

$\lambda_3 = g + h\tilde{y} + m\tilde{x} > 0.$

Now, if $(\gamma\tilde{x}+\beta\tilde{y})^2 \geq 4\tilde{x}\tilde{y}(\gamma\beta+bf)$, then λ_1 and λ_2 are negatives, different or repeated roots. Otherwise $\lambda_1$and λ_2 are complex conjugate with real part negatives, while $\lambda_3 = g + h\tilde{y} + m\tilde{x} > 0$, so that $P_4 = (\tilde{x},\tilde{y},0)$ is a saddle point (unstable).

6) The equilibrium point

$$P_5 = (\bar{x},0,\bar{z}) = \left(\frac{\delta a - cg}{\delta\gamma + cm}, 0, \frac{(\gamma g + am)}{\delta\gamma + cm}\right)$$

exists if and only if:

$$\delta a > cg. \quad (4.7)$$

The characteristic equation of the matrix of Jacobi of the system (2.1) at the equilibrium point $P_5 = (\bar{x},0,\bar{z})$ can be written as follows:

$$\begin{vmatrix} -\gamma\bar{x}-\lambda & -b\bar{x} & -c\bar{x} \\ 0 & d-e\bar{z}+f\bar{x}-\lambda & 0 \\ m\bar{z} & h\bar{z} & -\delta\bar{z}-\lambda \end{vmatrix} = 0, \quad (4.8)$$

and the solutions of the equation (4.8) are written as follows:

$\lambda_2 = d - e\bar{z} + f\bar{x},$

$$\lambda_i = \frac{-(\gamma\bar{x}+\delta\bar{z}) \mp \sqrt{(\gamma\bar{x}+\delta\bar{z})^2 - 4\bar{x}\bar{z}(\gamma\delta+mc)}}{2}, \text{i} = 1,3.$$

Now, if $(\gamma\bar{x}+\delta\bar{z})^2 \geq 4\bar{x}\bar{z}(\gamma\delta+me)$, then λ_1 and λ_3 are negatives, different or repeated roots. Otherwise $\lambda_1$and λ_3 are complex conjugates with negative real parts. While $\lambda_2 < 0$ if and only if $d < e\bar{z} - f\bar{x}$.

So that $P_5 = (\bar{x},0,\bar{z})$ is locally asymptotically stable if and only if $d > e\bar{z} - f\bar{x}$. So we have the following result:

Theorem2: Suppose that $\delta a > cg$.If$d^* = e\bar{z} - f\bar{x}$ is a positive number, then the equilibrium point $P_5 = (\bar{x},0,\bar{z})$ is:

1) locally asymptotically stable if $d < d^*$,
2) unstable if $d > d^*$.

Proof:

It is easy to show that $\frac{d\bar{x}}{dd} = 0$, $\frac{d\bar{z}}{dd} = 0$ and $\frac{d\lambda_i}{dd} = 0, i = 1,3.$

So that $\bar{x},\bar{y},\bar{z}$ and $\lambda_i, i = 1,3$ are constants with respect to the parameter d and the real parts of the eigenvalues λ_1 and λ_3 are always negative, while the eigenvalueλ_2 is negative if $d > e\bar{z} - f\bar{x} = d^*$ and positive if $d < e\bar{z} - f\bar{x} = d^*$. So the theorem is proved.

7) The equilibrium point

$$P_6 = (0,\hat{y},\hat{z}) = \left(0, \frac{(\delta d - eg)}{\delta\beta + he}, \frac{g\beta + hd}{\delta\beta + he}\right)$$

exists if and only if:

$$\delta d > eg. \quad (4.9)$$

The characteristic equation of the matrix of Jacobi of the system (2.1) at the equilibrium point $P_6 = (0,\hat{y},\hat{z})$ can be written as follows:

$$\begin{vmatrix} a-b\hat{y}-c\hat{z}-\lambda & 0 & 0 \\ f\hat{y} & -\beta\hat{y}-\lambda & -e\hat{y} \\ m\hat{z} & h\hat{z} & -\delta\hat{z}-\lambda \end{vmatrix} = 0, \quad (4.10)$$

and the solutions of the equation (4.10) are:

$\lambda_1 = a - b\hat{y} - c\hat{z},$

$$\lambda_i = \frac{-(\beta\hat{y}+\delta\hat{z}) \mp \sqrt{(\beta\hat{y}+\delta\hat{z})^2 - 4\hat{y}\hat{z}(\beta\delta+he)}}{2}, i = 2,3.$$

Now, if $(\beta\hat{y}+\delta\hat{z})^2 \geq 4\hat{y}\hat{z}(\beta\delta+he)$, then λ_2 and λ_3 are negatives, different or repeated roots. Otherwise λ_2 and λ_3 are complex conjugates with negative real parts. While $\lambda_1 < 0$ if and only if $a < b\hat{y}+c\hat{z}$.
So that $P_5 = (\bar{x}, 0, \bar{z})$ is locally asymptotically stable if and only if: $a < b\hat{y}+c\hat{z}$.
Similarly, as the proof of theorem 2, we get the following results:
Theorem 3: Suppose that $\delta d > eg$. If $a^* = b\hat{y}+c\hat{z}$ is a positive number, then the equilibrium point $P_6 = (0, \hat{y}, \hat{z})$ is:
1) locally asymptotically stable if $d < d^*$,
2) unstable if $d > d^*$.
Proof: See the proof of Theorem 2.
8) The equilibrium point $P_7 = (\breve{x}, \breve{y}, \breve{z})$
where $\breve{x} = \frac{|A_1|}{|A|}, \breve{y} = \frac{|A_2|}{|A|}, \breve{z} = \frac{|A_3|}{|A|}$,

$$A = \begin{bmatrix} \gamma & b & c \\ e & \beta & -f \\ m & h & -\delta \end{bmatrix},\quad A_1 = \begin{bmatrix} a & b & c \\ d & \beta & -f \\ -g & h & -\delta \end{bmatrix},\quad A_2 = \begin{bmatrix} \gamma & a & c \\ e & d & -f \\ m & -g & -\delta \end{bmatrix},\quad A_3 = \begin{bmatrix} \gamma & b & a \\ e & \beta & d \\ m & h & -g \end{bmatrix}$$

exists if and only if:

$$|A||A_i| > 0, i = 1,2,3. \tag{4.11}$$

The characteristic equation of the matrix of Jacobi of the system (2.1) at the equilibrium point $P_7 = (\breve{x}, \breve{y}, \breve{z})$ can be written as follows:

$$\begin{vmatrix} -\gamma\breve{x}-\lambda & -b\breve{x} & -c\breve{x} \\ f\breve{y} & -\beta\breve{y}-\lambda & -e\breve{y} \\ m\breve{z} & h\breve{z} & -\delta\breve{z}-\lambda \end{vmatrix} = 0. \tag{4.12}$$

The equation (4.12) can be written as:

$$\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3 = 0, \tag{4.13}$$

where:
$B_1 = \gamma\breve{x}+\beta\breve{y}+\delta\breve{z}$,
$B_2 = (\gamma\beta+bf)\breve{x}\breve{y} + (\gamma\delta+cm)\breve{x}\breve{z} + (\beta\delta+he)\breve{y}\breve{z}$,
$B_3 = [(\delta\beta+he)\gamma + (f\delta-em)b + (m\beta+fh)c]\breve{x}\breve{y}\breve{z}$.
The necessary and sufficient conditions to guarantee that the equilibrium point $P_7 = (\breve{x}, \breve{y}, \breve{z})$ is locally asymptotically stable are:

$$\begin{cases} B_3 > 0, \\ B_1B_2 > B_3. \end{cases} \tag{4.14}$$

5. NUMERICAL SIMULATIONS

Consider the following set of values of the parameters included in the model (2.1):

$$\boldsymbol{S} = \{a = 3, b = 1, c = 1.5, d = 0.2, e = 2, f = 2, g = 0.1, h = 0.1, m = 0.2, \gamma = 0.5, \beta = 0.1, \delta = 1.2\}.$$

The system of differential equation, that we obtain from the compensation of the parameters of the set $\boldsymbol{S}$ in the model (2.1), has the following 8 equilibrium points:
$P_0 = (0,0,0)$, $P_1 = (6,0,0)$, $P_2 = (0,2,0)$, $P_3 = (0,0,0.0833)$,
$P_4 = (0.04878, 2.97561, 0)$, $P_5 = (3.83333,0,0.72222)$ and $P_6 = (0,0.125,0.0.09375)$ which, are unstable, while equilibrium points $P_7 = (0.16995, 2.71362,0.13427)$ are locally asymptotically stable Figures-(1-4). Note that $a^* = 0.265625$, so the point is unstable if $a = 3 > a^*$ and is locally asymptotically stable if $a = 0.2 < a^*$, Figures-(3 and 5).

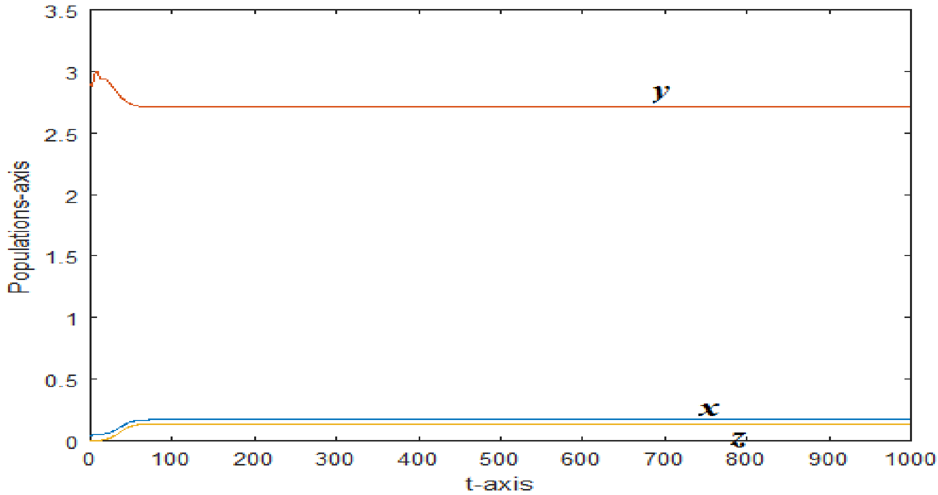

Figure 1-The trajectory of the system (2.1) with the parameters of the set $\boldsymbol{S}$ and the initial points $(0.04,\ 2.9, 0.001\ )$ is located close to $\boldsymbol{P_4}$ and it is moving away from $\boldsymbol{P_4}$ and approaching $\boldsymbol{P_7}$.

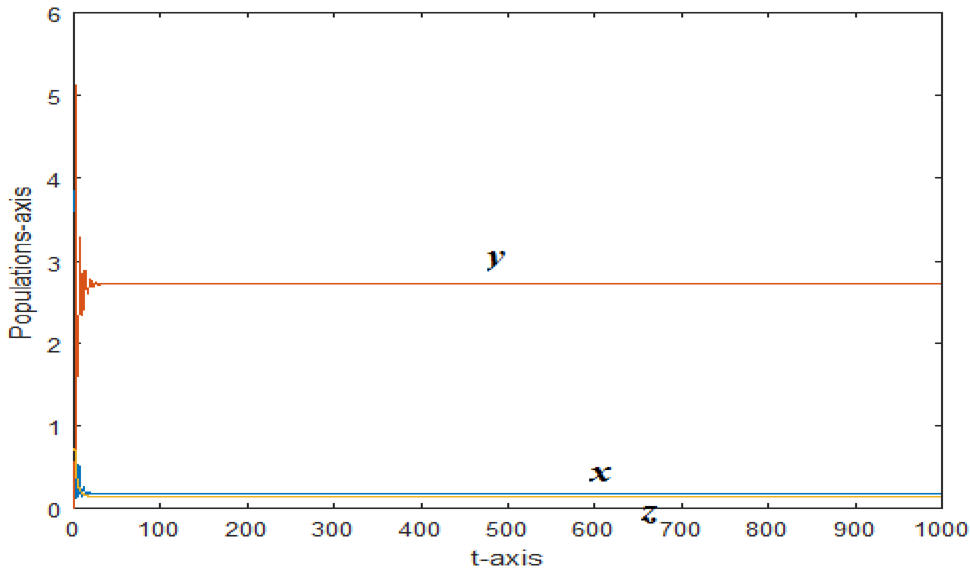

Figure 2-The trajectory of the system (2.1) with the parameters of the set $\boldsymbol{S}$ and the initial points $(3.8,\ 0.001, 0.7\ )$ is located close to $\boldsymbol{P_5}$ and it is moving away from $\boldsymbol{P_5}$ and approaching $\boldsymbol{P_7}$.

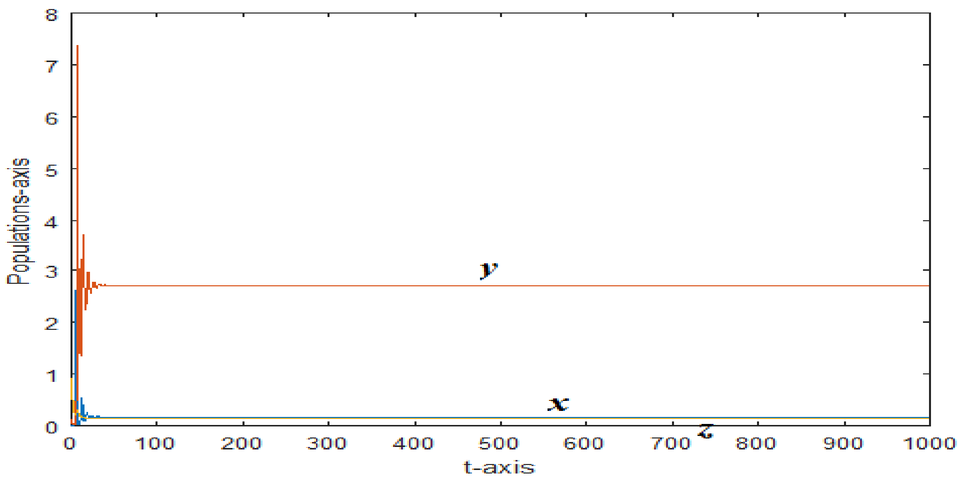

Figure 3-The trajectory of the system (2.1) with the parameters of the set $\boldsymbol{S}$ and the initial points $(0.00001, 0.12, 0.09)$ is located close to $\boldsymbol{P_6}$ and it is moving away from $\boldsymbol{P_6}$ and approaching $\boldsymbol{P_7}$.

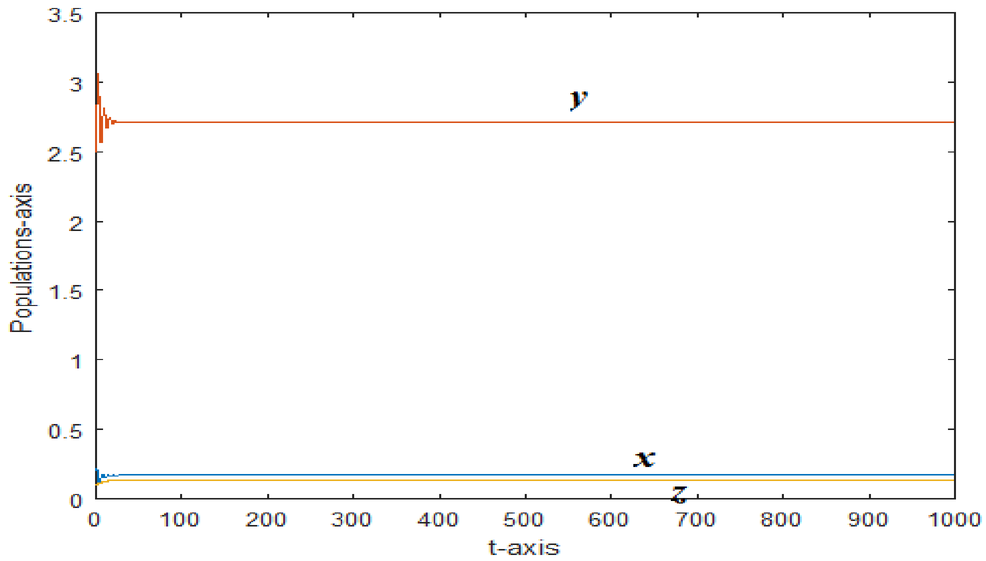

Figure 4-The trajectory of the system (2.1) with the parameters of the set $\boldsymbol{S}$ and the initial points $(0.2, 2.7, 0.1)$ is located close to $\boldsymbol{P_7}$ and it is approaching $\boldsymbol{P_7}$.

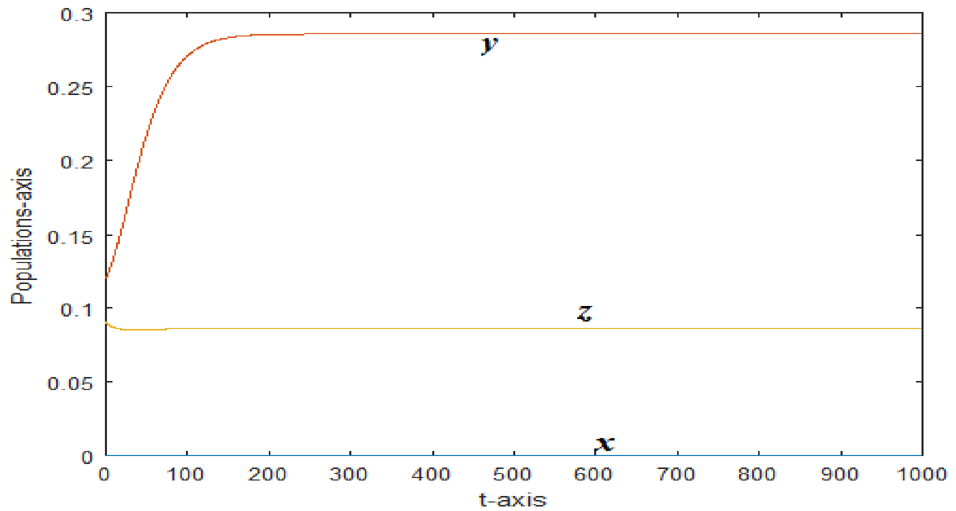

Figure 5-The trajectory of the system (2.1) with the parameters of the set $\boldsymbol{S}$, $0.2 = a < a^* = 0.265625$, where the initial point $(0.0001, 0.12, 0.09)$ is located close to $\boldsymbol{P_6}$ and it is approaching □□.

7. CONCLOSIONS

In this work, we consider a modification of the chain food of Lotka-Volterra model for three species, each of them grows logistically, which means that there is no death of any species in the absence of the others. The model is a system of differential equations that has eight equilibrium points, four of which always exist while the existence of the rest points depends on the fulfillment of the conditions mentioned in section 4 of this work. The stability analysis of the equilibrium points shows that five of the eight equilibrium points are unstable while the rest are locally asymptotically stable under specific conditions mentioned in section 4 of this paper. Finally, an example is given in this work, where the number of the equilibrium points was eight, with only one being locally asymptotically stable.

Acknowledgment

The author would like to thank the Mustansiriyah University (www.uomstansiriyah.edu.iq), Baghdad-Iraq, for its support in the present work.

REFERENCES

1. Mendoza Meza M. E., Bhaya A. and Kaszkurewicz E. **2005**. Controller design techniques for the Lotka–Volterra nonlinear system. *RevistaControle&Automação*, **16** (2): 124-135.
2. Adamu H. A. **2018.** Mathematical analysis of predator-prey model with two preys and one predator, *International Journal of Engineering and Applied Sciences*, **5**(11): 17-23.
3. Hadžiabdić V., Mehuljić M. and Bektešević J. **2017**. Lotka-Volterra model with Two Predators and Their Prey, *TEM Journal*, **6**: 132-136.
4. Bena¨ım M. and Lobry C. **2016**. Lotka-Volterra with randomly fluctuating environments or "How switching between beneficial environments can make survival harder". *The Annals of Applied and Probability*. **26**(6): 3754–3785
5. Florent Malrieu and Pierre-Andr´eZitt, **2017**. On the persistence regime for Lotka-Volterra in randomly fluctuating environments ALEA, Lat. Am. JProbab. On the persistence regime for Lotka-Volterra. *Math. Stat*. **14**: 733–749.
6. JaumeLlibre and Dongmei Xiao, **2014**. global dynamics of a Lotka–Volteera model with two predators competing for one prey, *Siam Journal of Appl. Math*., **74**(2): 434–453.
7. Korobeinikov A. and Wake G. **1999**. global properties of the three-dimensional predator-prey Lotka-Volterra systems, *Journal of Applied Mathematics & Decision Sciences,* **3**(2): 155-162.
8. Naji R.K. and Ali F.A. **2014**. Modeling and Stability of Lotka-Volterra Prey-Predator System Involving Infectious Disease in Each Population, *Iraqi Journal of Science,* **55**(2): 491-505.
9. Farhan A.G. **2020**. On the mathematical model of two-prey and two-predator species, *Iraqi Journal of Science*, **61**(3): 608-619.