Symmetry Group for Solving Elliptic Euler-Poisson-Darboux Equation

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Abstract:
The aim of this article is to study the solution of Elliptic Euler-Poisson-Darboux equation, by using the symmetry of Lie Algebra of orders two and three, as a contribution in partial differential equations and their solutions.

Keywords: symmetry, Elliptic Euler-poisson-Darboux, Lie Algebra

1. Introduction
The Euler-Poisson-Darboux equation is very important in physics and mathematics. It is one of the most extensively studied singular linear hyperbolic equations. The general formula of Euler-Poisson-Darboux Equation ([1, 2, 3]) is:

$$\frac{\partial^2 u}{\partial r^2} + \frac{a}{r} \frac{\partial u}{\partial r} = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = (r, z; \alpha), r \in R^n, z > 0 , -\infty < \alpha < \infty$$

(1)

where $\alpha$ is a real parameter. The classical Euler-Poisson-Darboux equation is defined as [4]

$$\frac{\partial^2 u}{\partial r^2} + \frac{a}{r} \frac{\partial u}{\partial r} = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$$

(2)

For equation (1), if $n=1$, then this equation can be defined as

$$\frac{\partial^2 u}{\partial r^2} + \frac{a}{r} \frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial z^2}$$

(3)

where $\alpha$ is a real parameter. Equation (3) is an Elliptic Euler-Poisson-Darboux equation. It is also referred to as a generalized axisymmetric Laplace equation [5, 6]. For $\alpha=1$, it is the axisymmetric Laplace equation, which was studied in [7]. Equation (4) is a hyperbolic Elliptic Euler-Poisson-Darboux equation. The Euler-Poisson-Darboux equations was considered for the first time by Euler [8] and later by Poisn [9], Riemann [10], and Darboux [11]. In the recent time, it was studied by a number of authors [1, 4, 12] because of its importance in mathematics and mathematical physics.

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In this article, we use the method of Lie group theory. Symmetry group methods are amongst powerful universal tools for the study of differential equations. There has been a rapid progress on these methods over the last few decades. Methods and algorithms were introduced for classifying subalgebras of Lie algebra. New results on the structure and classification of abstract finite and infinite dimensional Lie algebra were published [13]. Also, methods for solving group classification problems for differential equations greatly facilitated to systematically obtain exact analytical solutions [13, 14, 15]. During the solution, we used some methods of solving partial differential equations and ordinary differential equations [16, 17, 18].

### Symmetry of Elliptic Euler-Poisson-Darboux Equation

Similarity is an extremely powerful method of solving the linear and nonlinear differential equations [13, 14]. For solving the Elliptic Euler-Poisson-Darboux equation (3), we need to introduce the following Lie algebra approach as a transformation method.

We apply Lie group method to solve (3), as follows:

\[
X^2 \left( \frac{\partial^2 u}{\partial r^2} + \frac{a}{r} \frac{\partial u}{\partial r} + \frac{\partial^3 u}{\partial r^3} \right) \bigg|_{u_{rr} = -\frac{a}{r} u_r - u_r} = 0
\]

Its 2\(^{nd}\) extension, \(X^2\), is defined by

\[
X^2 = X + \eta_r \frac{\partial}{\partial u_r} + \eta_z \frac{\partial}{\partial u_z} + \eta_{zz} \frac{\partial}{\partial u_{zz}}
\]

The generator \(X\) is defined by

\[
X = \xi_r \frac{\partial}{\partial r} + \xi_z \frac{\partial}{\partial z} + \eta \frac{\partial}{\partial \eta}
\]

where \(\eta_r \), \(\eta_z \), \(\eta_{zz} \) are

\[
\eta_r = \eta + \left[ \eta_u - \xi_{1r} \right] u_r - \xi_{2r} u_z - \xi_{1u} u_2 - \xi_{2u} u_z
\]

\[
\eta_z = \eta + \left[ \eta_u - \xi_{2z} \right] u_z - \xi_{1z} u_1 - \xi_{2u} u_z
\]

\[
\eta_{zz} = \eta + \left[ 2\eta_u - \xi_{2zz} \right] u_z - \xi_{1zz} u_1 - \xi_{2zz} u_z + \left[ \eta_u - 2\xi_{2zz} \right] u_z + [\eta_u - 2\xi_{2zz} u_z - 2\xi_{1zz} u_z] + [\eta_u - 2\xi_{2zz}] u_z - \xi_{1zz} u_1 - \xi_{2zz} u_z - \xi_{1zu} u_z - 2\xi_{1uu} u_z, \]

\[
[\eta_u - 2\xi_{2uu}] u_z - \xi_{1uu} u_z - 3\xi_{2uu} u_z - 3\xi_{2zu} u_z \]

The invariance condition for equation (3) is

\[
\eta_{rr} = \frac{a}{r} \xi + \frac{a}{r} \eta_r + \eta_{zz} \bigg|_{u_{rr} = \cdots} = 0
\]

By substituting Eqs. (8-11) and write equation (3) as

\[
\frac{\partial^2 u}{\partial r^2} = -\frac{a}{r} \frac{\partial u}{\partial r} - \frac{\partial^3 u}{\partial r^3}
\]

into (12), we get

\[
\eta_{rr} + [2\eta_r - \xi_{2r}] u_r - \xi_{2r} u_z + \left[ \eta_u - 2\xi_{1r} \right] u_r - 2\xi_{2r} u_z + [\eta_u - 2\xi_{2u}] u_z - 2\xi_{2u} u_z + \left[ \eta_u - 2\xi_{2uu} \right] u_z - 2\xi_{2uu} u_z + \left[ \eta_u - 2\xi_{2u} \right] u_z - 2\xi_{2u} u_z - 3\xi_{2uu} u_z - 3\xi_{2uu} u_z
\]

From this equation, we obtain a polynomial equation in \(u_r, u_z, u_{rr}, u_{zz}, u_{rrr}, u_{zzz}\),

From the coefficient of \(u_2 u_{rr} u_z \) we have got

\[
\xi_{1u} = \xi_{2u} = 0
\]

From the coefficient of \(u_{r} \), we have got

\[
\xi_{2r} = -\xi_{1z}.
\]

From the coefficient of \(u_{zz} \), we have got

\[
\xi_{1z} = \xi_{2z}.
\]

From the coefficient of \(u_2 \) we have got

\[
-\xi_{2rr} - \frac{a}{r} \xi_{2r} + (2\eta_u - \xi_{2zz}) = 0
\]
From the coefficient of $u_r$, we have got
\[-\frac{\alpha}{r^2} \xi_1 - \frac{\alpha}{r} \xi_1 r + (2\eta u_r - \xi_{1rr}) = 0 \tag{18}\]
From the free Coefficient we have got
\[-\eta_{rr} + \frac{\alpha}{r^2} \eta_r + \eta_{zz} = 0 . \tag{19}\]
We have noticed that, due to the complexity of finding Lie group of transformation from Eq. (13) through Eqs. (14-19), so we derive eq(3) with respect to $r$ to obtain the equation of the form:
\[
\frac{\partial^3 u}{\partial r^3} + \frac{\alpha}{r^2} \frac{\partial u}{\partial r} + \frac{\alpha}{r} \frac{\partial^2 u}{\partial r^2} + \frac{\partial^4 u}{\partial r^4} \eta_{rrr} \frac{\partial}{\partial u_{zz}} = 0 . \tag{20}\]
The third extension of generator is:
\[X[3] = X + \eta_{1r} \frac{\partial}{\partial r} + \eta_{zz} \frac{\partial}{\partial u_{zz}} + \eta_{rrr} \frac{\partial}{\partial u_{zz}} \tag{21}\]
Now, we find the extended infinitesimal coefficient from the formula:
\[
\eta_{iiz_2...i_k} = D_{ik} \eta^{(k+1)} - D_{ik} \eta_r \tag{22}\]
Where $D_i = \frac{\partial}{\partial r} + u_i \frac{\partial}{\partial u_z} + \ldots + u_{i_1...i_m} \frac{\partial}{\partial u_{zz}}$, $i=1,2,...n \tag{23}$
Then, from (21-23), we found the extended infinitesimal coefficient.
\[
\eta_{zzr} = \eta_{zzz} + \eta_{izzz} \xi_1 + \frac{\alpha}{r^2} \eta_r + \frac{\alpha}{r} \frac{\partial^2 u}{\partial r^2} + \frac{\partial^4 u}{\partial r^4} V_{1r} \tag{24}\]
The invariance condition for equation (20) is
\[
\left( \eta_{rrr} + \frac{2a}{r^2} \frac{\partial u}{\partial r} - \eta_{zz} \right) \frac{\partial \xi_1}{\partial r} + \frac{\partial u}{\partial r} \frac{\partial \xi_1}{\partial r} + \frac{\partial^4 u}{\partial r^4} \xi_1 = 0 \tag{25}\]
By the substitution of (8-11 and 24-25) into (27), we get
\[
\eta_{rrr} + \left[ 3\eta_{zz} - \xi_{1rr} \right] \eta_r + \left[ \eta_{zzz} - 2\xi_{1zz} u_z + 3\theta_{1zz} u_z^2 \right] u_r + \left[ \eta_{zzz} - \xi_{1zz} u_z - 2\xi_{1zz} u_z u_r \right] u_r + 3\xi_{1zz} u_r + \xi_{2rrr} u_z + (2\eta u_r - \xi_{1rr}) u_r + 3\xi_{1zz} u_z + \xi_{2zzz} u_z + [\eta_{zzzz} - \xi_{1zz} u_z + 2\xi_{1zz} u_z u_r] u_r + 3\xi_{1zz} u_z u_r + \xi_{2zzz} u_z - 3\xi_{1zzz} u_z u_r + [\eta_{zzzz} - \xi_{1zz} u_z + 2\xi_{1zz} u_z u_r] u_r + 3\xi_{1zz} u_z u_r + \xi_{2zzz} u_z - 3\xi_{1zz} u_z u_r + \xi_{2zzz} u_z - 3\xi_{1zz} u_z u_r + \xi_{2zzz} u_z = 0 \tag{26}\]
From this equation, we obtain a polynomial equation in $u_z$, $u_{zz}$, $u_{zzz}$, $u_{zzzz}$, $u_{rrr}$, $u_{zzzzz}$, $u_{zzzzzz}$.
The coefficient of $u_{zzzzzz}$, $u_{rrr}$ is
\[ \xi_{1u} = \xi_{2u} = 0 \]  
\[ \text{The coefficient of } u_{rzz} \text{ is } \eta_{u} - \xi_{1r} + \xi_{2z} - 2\xi_{2r} = 0 \]  
\[ \text{The coefficient of } u_{zzz} \text{ is } \xi_{1r} = 0 \]  
\[ \text{The coefficient of } u_{rzz} \text{ is } \xi_{2r} = 0 \]  
\[ \text{The coefficient of } u_{zzz} \text{ is } \eta_{uur} = 0 \]  
\[ \text{The coefficient of } u_{xx} \text{ is } \eta_{uuu} = 0 \]  
\[ \text{The coefficient of } u_{x} \text{ is } 2\eta_{xx} - \xi_{1zz} = 0 \]  
\[ \text{The coefficient of } u_{zz} \text{ is } 2\xi_{xx} - \xi_{1z} = 0 \rightarrow \xi_{2z} = \xi_{1z} \]  
\[ \text{The free coefficient is } \eta_{rrr} - \frac{a}{r^2} \eta_{r} + \frac{a}{r} \eta_{rr} + \eta_{zrr} = 0 \]  
From 12, 13, 14, 27, and 29, we have \[ \xi_{2z} = \xi_{1z} = \xi_{1u} = \xi_{2u} = \xi_{2r} = \xi_{1r} = 0 \]  
This yields that \( \xi_{1z} = \xi_{2z} = \) constants. \[ \text{Let } \xi_{1} = c_{1} \text{ and } \xi_{2} = c_{2} \]  of the coefficient of \( u_{rr} \) is
\[ \eta_{u} = 0 \]  
From (33 and 35), we get
\[ \eta_{uz} = \eta_{wr} = 0 \]  
Then by solving this equation, we have
\[ \eta(r, z) = F(r, z)r + c_{3} \]  
or
\[ \eta(r, z) = G(r, z)z + c_{4} \]  
Since 42 = 43, then we get
\[ F(r, z) = \frac{1}{r}, \text{ or } G(r, z) = \frac{1}{z} \text{ and } c_{3} = c_{4} = c \]  
Hence \( \eta(r, z) = 1 + c \)  
Using lagrange method, we have
\[ \frac{dr}{c_{1}} = \frac{dx}{c_{2}} = \frac{du}{\eta} \]  
\[ \frac{dx}{c_{1}} = \frac{du}{
\frac{1}{1+c} \text{ or } u = \frac{1}{c_{2}} z + f(S) \]  
\[ u = (1+c) \frac{1}{c_{2}} z + (1+c)f(S) \]  
To find \( f(S) \), we substitute (49) in (3), as follows:
\[ u_{r} = \frac{1}{c_{1}} (1 + c)f'(S) \]  
\[ u_{rr} = \frac{1}{c_{1}^{2}} (1 + c)f''(S) \]  
\[ u_{zz} = \frac{1}{c_{2}^{2}} (1 + c)f''(S) \]  
Substituting (50-52) in (3) yields the ordinary differential equation
\[ \frac{1}{c_{1}^{2}} (1 + c)f''(S) + \frac{a}{r} \frac{1}{c_{1}} (1 + c)f'(S) + \frac{1}{c_{2}^{2}} (1 + c)f''(S) = 0 \]
\[ f''(S) + \frac{a}{r} c_1 f'(S) + \frac{c_2}{c_2^2} f''(S) = 0 \]  

(53)

We have got

\[ f(S) = A + Be^{-\frac{ac_2^2}{r(c_1+c_2)}} \]

(54)

By substituting (54) in (49), we get

\[ u = \frac{(1+c)}{c_2} z + (1 + c)(A + Be^{-\frac{ac_2^2}{r(c_1+c_2)}}) \]

(55)

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References