Salih and Sulaiman Iraqi Journal of Science, 2020, Vol. 61, No. 10, pp: 2671-2680
DOI: 10.24996/ijs.2020.61.10.24

ISSN: 0067-2904

Jordan Triple Higher (σ,τ)-Homomorphisms on Prime Rings

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Received: 13/11/2019 Accepted: 21/12/2019

Abstract
In this paper, the concept of Jordan triple higher (σ,τ)-homomorphisms on prime rings is introduced. A result of Herstein is extended on this concept from the ring R into the prime ring R'. We prove that every Jordan triple higher (σ,τ)-homomorphism of ring R into prime ring R' is either triple higher (σ,τ)-homomorphism or triple higher (σ,τ)-anti-homomorphism of R into R'.

Keywords: Jordan homomorphisms, triple homomorphism, Jordan triple higher (σ,τ)- homomorphism.

الثلاثية العالية(σ,τ)تشاكلات جوردان على الحلقات الاولية

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الخلاصة
في هذا البحث ,قدمت مفهوم تشاكلات جوردان (σ,τ) الثلاثية العالية على الحلقات الاولية. لقد عممت نتائج Herstein على هذا المفهوم من الحلقة R الى الحلقة الاولية R'. حيث برهنا كل تشاكلات جوردان (σ,τ) الثلاثية العالية من الحلقة R الى الحلقة الاولية R' اما تشاكلات (σ,τ) الثلاثية العالية او تشاكلات المضادة (σ,τ) الثلاثية العالية من R الى R'.

Introduction

The idea of Jordan homomorphism of rings initially appeared in Ancochea's [1] study of semi-automorphisms, the later investigated by Kaplansky, Jacobson and Rickart [2, 3]. Herstein [4] studied Jordan homomorphisms in prime rings. He proved that a Jordan homomorphism onto prime ring of characteristic different from 2 and 3 is either a homomorphism or an anti-homomorphism. Bresar [5] generalized Herstein's work on semiprime rings.

Throughout this paper, R is a ring with the center $Z(R)$ prime if $aRb=(0)$ implies $a=0$ or $b=0$ with $a, b \in R$, and is semiprime if $aRa=(0)$ implies $a=0$. R is n-torsion free if $na=0$; $a\in R$, then $a=0$.

In this paper, we extend the result of Herstein to triple higher (σ,τ)-homomorphism and Jordan triple higher (σ,τ)-homomorphism. We show that every Jordan triple higher (σ,τ)-homomorphism, from prime ring R into prime ring R', is triple higher (σ,τ)-homomorphism or triple higher (σ,τ)-anti-homomorphism.

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2. Preliminaries

We begin by the following definition.

Definition 2.1. [3, 4, 5]

An additive mapping θ of a ring R into a ring R' is called,

(a) a homomorphism if $\theta(ab) = \theta(a)\theta(b)$ for all $a, b \in R$,

(b) anti-homomorphism if $\theta(ab) = \theta(b)\theta(a)$for all $a, b \in R$,

(c) a Jordan homomorphism if $\theta(ab + ba) = \theta(a)\,\theta(b) + \theta(b)(a)$ for all $a, b \in R$ and

(d) a Jordan triple homomorphism if $\theta(aba) = \theta(a)\theta(b)\theta(a)$ for all $a, b \in R$.

Obviously, every homomorphism or anti-homomorphism is a Jordan homomorphism and every Jordan homomorphism is Jordan triple homomorphism but the converse needs not to be true in general.

Definition 2.2. [6]

Let $\mathbb{N}$ be the set of natural numbers. A family of additive mappings $\theta = (\phi_{\mathrm{i}})_{\mathrm{i}\in\mathbb{N}}$ of R into R' is called

(a) a higher homomorphism if for all $n \in \mathbb{N}, a, b \in R$,

$$\phi_{\mathrm{n}}(ab) = \sum_{i=1}^{n} \phi_{\mathrm{i}}(a)\,\phi_{\mathrm{i}}(b),$$

(b) a higher anti-homomorphism if for all $n \in \mathbb{N}, a, b \in R$,

$$\phi_{\mathrm{n}}(ab) = \sum_{i=1}^{n} \phi_{\mathrm{i}}(b)\,\phi_{\mathrm{i}}(a),$$

(c) a Jordan higher homomorphism if for all $n \in \mathbb{N}, a, b \in R$,

$$\phi_{\mathrm{n}}(ab + ba) = \sum_{i=1}^{n} \phi_{\mathrm{i}}(a)\,\phi_{\mathrm{i}}(b) + \phi_{\mathrm{i}}(b)\phi_{\mathrm{i}}(a)$$

(d) a triple higher homomorphism if for all $n \in \mathbb{N}, a, b \in R$,

$\phi_{\mathrm{n}}(abc) = \sum_{i=1}^{n} \phi_{\mathrm{i}}(a)\,\phi_{\mathrm{i}}(b)\phi_{\mathrm{i}}(c)_{\mathrm{i}}$,

(e) a Jordan triple higher homomorphism if for all $n \in \mathbb{N}, a, b \in R$,

$$\phi_{\mathrm{n}}(aba) = \sum_{i=1}^{n} \phi_{\mathrm{i}}(a)\,\phi_{\mathrm{i}}(b)\phi_{\mathrm{i}}(a)_{\mathrm{i}}$$

Definition 2.3. [7]

Let $\mathbb{N}$ be the set of natural numbers. A family of additive mappings $\theta = (\phi_{\mathrm{i}})_{\mathrm{i}\in\mathbb{N}}$ of R into R' and σ, τ as two homomorphisms of R is said to be

(a) a (σ, τ) −higher homomorphism if for each $n \in \mathbb{N}$ and for all $a, b \in R$,

$$\phi_n(ab) = \sum_{i=1}^{n} \phi_i\big(\sigma^i(a)\phi_i\left(\tau^i(b)\right)$$

(b) a (σ, τ) −higher anti-homomorphism if for each $n \in \mathbb{N}$ and for all $a, b \in R$,

$$\phi_n(ab) = \sum_{i=1}^{n} \phi_i\big(\sigma^i(b)\phi_i\left(\tau^i(a)\right)$$

(c) a Jordan (σ, τ) −higher homomorphism if for each $n \in \mathbb{N}$ and for all $a, b \in R$,

$$\phi_n(ab + ba) = \sum_{i=1}^{n} \phi_i\big(\sigma^i(a)\phi_i\left(\tau^i(b)\right) + \phi_i\big(\sigma^i(b)\phi_i(\tau^i(a))$$

(d) a Jordan triple (σ, τ) −higher homomorphism if for all $n \in \mathbb{N}, a, b \in R$,

$$\phi_n(aba) = \sum_{i=1}^{n} \phi_i\left(\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(b)\right)\phi_i(\tau^i(a))$$

Definition 2.4.

Let $\mathbb{N}$ be the set of natural numbers. A family of additive mappings $\theta = (\phi_i)_{i\in\mathbb{N}}$ of R into R' and σ,τ as two homomorphisms of R is said to be

(a) a triple (σ,τ) −higher homomorphism if for all $n \in \mathbb{N}, a, b \in R$,

$$\phi_n(abc) = \sum_{i=1}^{n} \phi_i\left(\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(b)\right)\phi_i(\tau^i(c))$$

(b) a triple (σ,τ) −higher anti-homomorphism if for all $n \in \mathbb{N}, a, b \in R$,

$$\phi_n(abc) = \sum_{i=1}^{n} \phi_i\left(\sigma^i(c)\right)\phi_i\left(\sigma^i\tau^{n-i}(b)\right)\phi_i\left(\tau^i(a)\right),$$

Now, we give an example of triple (σ,τ)-higher homomorphism and Jordan triple (σ,τ)-higher homomorphism.

Example 2.5:

Let $\theta = (\phi_i)_{i\in\mathbb{N}}$ be a triple (σ,τ)-higher homomorphism from R into R'. Then for each $n \in \mathbb{N}$ and for all $a, b, c \in R$, we have:

$$\phi_n(abc) = \sum_{i=1}^{n} \phi_i\left(\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(b)\right)\phi_i(\tau^i(c))$$

Let $T = R \times R \times R$ and $T' = R' \times R' \times R'$. Then T and T' are rings. We define $\theta' = (\phi'_i)_{i\in\mathbb{N}}$ to be a family of mappings from T to T' by:

$$\phi'_n((a,b,c)) = (\phi_n(a), \phi_n(b), \phi_n(c))$$

for all $(a,b,c) \in T$.

Then ϕ is a triple (σ,τ)-higher homomorphism.

Let S be the subset $\{(a,a,a): a \in R\}$ of T and S' be the subset $\{(b,b,b): b \in R'\}$ of T'. Then S and S' are rings and the family of mappings $\theta' = (\phi'_i)_{i\in\mathbb{N}}$ from S to S' is defined in terms of the Jordan (σ,τ)-higher homomorphism by

$$\phi'_n((a,b,a)) = (\phi_n(a), \phi_n(b), \phi_n(a))$$

for all $(a,a,a) \in S$.

Then ϕ is a Jordan triple (σ,τ)- higher homomorphism from S to S'.

Obviously, every triple (σ,τ) −higher homomorphism or triple (σ,τ) −higher anti-homomorphism is a Jordan triple (σ,τ) −higher homomorphism but the converse needs not to be true in general. In an earlier work[6], the author provided an example of Jordan higher homomorphism but not higher homomorphism on a ring. We extend it to triple (σ,τ) −higher homomorphism on ring as follows.

Example 2.6.

Suppose that S is a ring with non-trivial involution $*$, $R = S\oplus S\oplus S$, $a \in S$ such that $a \in Z(S)$ and $s_1 a s_2 = 0$, for all $s_1, s_2 \in R$. Let $\theta = (\phi_i)_{i\in\mathbb{N}}$ be a family of mappings of R into itself defined, for each $n \in N$ and $(s,t,s) \in R$, by:

$$\phi_n(s,t,s) = \begin{cases} \left((2-n)a\sigma^i(s), (n-1)\sigma^i\tau^{n-i}(t^*), (2-n)a\sigma^i(s)\right), & n = 1,2 \\ 0 & n \geq 3 \end{cases}$$

Therefore, it is clear that ϕ is a Jordan triple (σ,τ)-higher homomorphism but not a triple (σ,τ)-higher homomorphism.

Now, we will give the following lemmas which are used in the proofs of the main results.

Lemma 2.7: [5]

Let R be a 2-torsion free semiprime ring. If $x, y \in R$ such that $xry + yrx = 0$, for all $r \in R$, then $xry = yrx = 0$.

Lemma 2.8:

Let $\theta = (\phi_i)_{i\in\mathbb{N}}$. be a Jordan triple (σ,τ)-higher homomorphism of R into R'. Then for each $n \in \mathbb{N}$ and for all $a, b, c \in R$,

$$\phi_n(abc + cba) = \sum_{i=1}^{n} \phi_i\left(\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(b)\right)\phi_i\left(\tau^i(c)\right) + \phi_i\left(\sigma^i(c)\right)\phi_i\left(\sigma^i\tau^{n-i}(b)\right)\phi_i\left(\tau^i(a)\right)$$

Proof: Since ϕ is a Jordan triple (σ,τ)-higher homomorphism, hence

$$\phi_n(aba)=\sum_{i=1}^{n}\phi_i\left(\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(b)\right)\phi_i\left(\tau^i(a)\right)$$

By linearizing a, we get

$$\phi_n((a+c)b(a+c))=\sum_{i=1}^{n}\phi_i\left(\sigma^i(a+c)\right)\phi_i\left(\sigma^i\tau^{n-i}(b)\right)\phi_i\left(\tau^i(a+c)\right)$$

$$=\sum_{i=1}^{n}\phi_i\left(\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(b)\right)\phi_i\left(\tau^i(a)\right)+\phi_i\left(\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(b)\right)\phi_i\left(\tau^i(c)\right)$$
$$+\phi_i\left(\sigma^i(c)\right)\phi_i\left(\sigma^i\tau^{n-i}(b)\right)\phi_i\left(\tau^i(a)\right)+\phi_i\left(\sigma^i(c)\right)\phi_i\left(\sigma^i\tau^{n-i}(b)\right)\phi_i\left(\tau^i(c)\right) \quad (1)$$

On the other hand:

$$\phi_n((a+c)b(a+c))=\phi_n(aba+abc+cba+cbc)=\phi_n(aba)+\phi_n(abc+cba)+\phi_n(cbc)$$
$$=\sum_{i=1}^{n}\phi_i\left(\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(b)\right)\phi_i\left(\tau^i(a)\right)+\phi_i\left(\sigma^i(c)\right)\phi_i\left(\sigma^i\tau^{n-i}(b)\right)\phi_i\left(\tau^i(c)\right)$$
$$+\phi_n(abc+cba) \quad \dots (2)$$

By comparing (1) and (2), we achieve the result.

Remark 2.9:

Let $\theta=(\phi_{\mathrm{i}})_{\mathrm{i}\in\mathbb{N}}$ be a Jordan triple (σ,τ)-higher homomorphism from R into R'. Then for each $n\in\mathbb{N}$ and for all $a,b\in R$, we will write

$$A_n(a,b,c)=\phi_n(abc)-\sum_{i=1}^{n}\phi_i\left(\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(b)\right)\phi_i\left(\tau^i(c)\right)$$
$$B_n(a,b,c)=\phi_n(abc)-\sum_{i=1}^{n}\phi_i\left(\sigma^i(c)\right)\phi_i\left(\sigma^i\tau^{n-i}(b)\right)\phi_i\left(\tau^i(a)\right)$$

Note that $A_n(a,b,c)=0$, if and only if ϕ is a triple (σ,τ)-higher homomorphism, and $B_n(a,b,c)=0$, if and only if ϕ is a triple (σ,τ)- higher anti-homomorphism.

For the purpose of this paper, we can list the following elementary properties about the above:

1- $A_n(a,b,c)+A_n(c,b,a)=0$,

2- $B_n(a,b,c)+B_n(c,b,a)=0$,

Lemma 2.10:

If $\theta=(\phi_{\mathrm{i}})_{\mathrm{i}\in\mathbb{N}}$ is a Jordan triple (σ,τ)-higher homomorphism from a ring R into a ring R', then for all $a,b\in R$ and $n\in\mathbb{N}$,

i) $A_n(a+b,c,d)=A_n(a,c,d)+A_n(b,c,d)$

ii) $A_n(a,b+c,d)=A_n(a,b,d)+A_n(a,c,d)$

iii) $A_n(a,b,c+d)=A_n(a,b,c)+A_n(a,b,d)$

Proof:

i) $A_n(a+b,c,d)=\phi_n((a+b)cd)-\sum_{i=1}^{n}\phi_i\left(\sigma^i(a+b)\right)\phi_i\left(\sigma^i\tau^{n-i}(c)\right)\phi_i\left(\tau^i(d)\right)$

$$=\phi_n(acd+bcd)-\sum_{i=1}^{n}\phi_i\left(\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(c)\right)\phi_i\left(\tau^i(d)\right)$$
$$-\sum_{i=1}^{n}\phi_i\left(\sigma^i(b)\right)\phi_i\left(\sigma^i\tau^{n-i}(c)\right)\phi_i\left(\tau^i(d)\right)$$

Since ϕ_n is an additive mapping for each n, then

$=\phi_n(acd)-\sum_{i=1}^{n}\phi_i\left(\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(c)\right)\phi_i\left(\tau^i(d)\right)+\phi_n(bcd)-$
$\sum_{i=1}^{n}\phi_i\left(\sigma^i(b)\right)\phi_i\left(\sigma^i\tau^{n-i}(c)\right)\phi_i\left(\tau^i(d)\right)=A_n(a,c,d)+A_n(b,c,d)$

In a similarly way, we can prove (ii) and (iii).

3. Main Results

Lemma 3.1:

If $\theta = (\phi_i)_{i\in\mathbb{N}}$ is a Jordan triple higher (σ,τ)-homomorphism of R into R', then for each $n \in \mathbb{N}$ and for all $a,b,c,r \in R$,

$A_n(\sigma^n(a\,b,c))\phi_n(\sigma^n(r))B_n(\tau^n(a\,b,c)) + B_n(\sigma^n(abc))\phi_n(\sigma^n(r))A_n(\tau^n(a,b,c)) = 0.$

Proof:

We proceed by the induction on $n \in \mathbb{N}$. Assume that θ is a Jordan triple higher (σ,τ)-homomorphism and take $a,b,c,r \in R$.

If $n = 1$: Define $w = abcrcba + cbarabc$, then we get the required result.

We can assume that the following equation is true for all $a,b,c,r \in R, n \in \mathbb{N}$ and $m < n$:

$A_m(\sigma^m(a,b\,,c))\phi_m(\sigma^m(r))B_m(\tau^m(a,b\,,c))+B_m(\sigma^m(a,b,c))\phi_m(\sigma^m(r))A_m(\tau^m(a,b,c)) = 0$

Now, we have

$$\phi_n(w) = \phi_n(a(bcrcb)a + c(barab)c)$$

$$= \sum_{i=1}^{n} \phi_i\left(\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(bcrcb)\right)\phi_i\left(\tau^i(a)\right) + \phi_i\left(\sigma^i(c)\right)\phi_i\left(\sigma^i\tau^{n-i}(barab)\right)\phi_i\left(\tau^i(c)\right)$$

$$= \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i(b))(\sum_{j=1}^{i} \phi_j\left(\sigma^i\tau^{n-i}(crc)\right))\,\phi_i\left(\tau^i(b)\right)\phi_i\left(\tau^i(a)\right)$$

$$+ \phi_i\left(\sigma^i(c)\right)\phi_i(\sigma^i(b))(\sum_{j=1}^{i} \phi_j\left(\sigma^i\tau^{n-i}(ara)\right))\,\phi_i\left(\tau^i(b)\right)\phi_i\left(\tau^i(c)\right)$$

$$= \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i(b))\,(\sum_{j=1}^{i} \phi_j\left(\sigma^j\sigma^i\tau^{n-i}(c)\right)\phi_j\left(\sigma^j\tau^{i-j}\sigma^i\tau^{n-i}(r)\right)\phi_j(\tau^j\sigma^i\tau^{n-i}(c)))\phi_i\left(\tau^i(b)\right)\phi_i\left(\tau^i(a)\right)$$

$$+ \sum_{i=1}^{n} \phi_i(\sigma^i(c))\phi_i(\sigma^i(b))\,(\sum_{j=1}^{i} \phi_j\left(\sigma^j\sigma^i\tau^{n-i}(a)\right)\phi_j\left(\sigma^j\tau^{i-j}\sigma^i\tau^{n-i}(r)\right)\phi_j(\tau^j\sigma^i\tau^{n-i}(a)))\phi_i\left(\tau^i(b)\right)\phi_i\left(\tau^i(c)\right)$$

$$= \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i(b))\phi_i\left(\sigma^i\sigma^i\tau^{n-i}(c)\right)\phi_i\left(\sigma^i\tau^{n-i}\sigma^i\tau^{n-i}(r)\right)\phi_i(\tau^i\sigma^i\tau^{n-i}(c)))\phi_i\left(\tau^i(b)\right)\phi_i\left(\tau^i(a)\right)$$

$$+ \sum_{i=1}^{n} \phi_i(\sigma^i(c))\phi_i(\sigma^i(b))\phi_i\left(\sigma^i\sigma^i\tau^{n-i}(a)\right)\phi_i\left(\sigma^i\tau^{n-i}\sigma^i\tau^{n-i}(r)\right)\phi_i(\tau^i\sigma^i\tau^{n-i}(a)))\phi_i\left(\tau^i(b)\right)\phi_i\left(\tau^i(c)\right)$$

$$= \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i(b))\phi_i\left(\sigma^i\sigma^i\tau^{n-i}(c)\right)\phi_i\left(\sigma^i\tau^{n-i}\sigma^i\tau^{n-i}(r)\right)\sum_{j=1}^{i} \phi_j(\tau^j\sigma^j\tau^{n-j}(c)))\,\phi_j\left(\tau^j(b)\right)\phi_j\left(\tau^j(a)\right)$$

$$+ \sum_{i=1}^{n} \phi_i(\sigma^i(c))\phi_i(\sigma^i(b))\phi_i\left(\sigma^i\sigma^i\tau^{n-i}(a)\right)\phi_i\left(\sigma^i\tau^{n-i}\sigma^i\tau^{n-i}(r)\right)\sum_{j=1}^{i} \phi_j(\tau^j\sigma^j\tau^{n-i}(a)))\,\phi_j\left(\tau^j(b)\right)\phi_j\left(\tau^j(c)\right)$$

$$= \phi_n(\sigma^n(a))\phi_n(\sigma^n(b))\phi_n(\sigma^n\sigma^n(c)\phi_i(\sigma^n\sigma^n(r))\sum_{j=1}^{i} \phi_j(\tau^j\sigma^j\tau^{n-j}(c)))\,\phi_j\left(\tau^j(b)\right)\phi_j\left(\tau^j(a)\right)$$

$$+ \sum_{i=1}^{n-1} \phi_i\left(\sigma^i(a)\right)\phi_i\left(\sigma^i(b)\right)\phi_i\left(\sigma^i\sigma^i\tau^{n-i}(c)\right)\phi_i\left(\sigma^i\tau^{n-i}\sigma^i\tau^{n-i}(r)\right)\sum_{j=1}^{i} \phi_j(\tau^j\sigma^j\tau^{n-j}(c)))\,\phi_j\left(\tau^j(b)\right)\phi_j\left(\tau^j(a)\right)$$

$$+ \phi_n(\sigma^n(c))\phi_n(\sigma^n(b))\phi_n(\sigma^n\sigma^n(a)\phi_i(\sigma^n\sigma^n(r))\sum_{j=1}^{i} \phi_j(\tau^j\sigma^j\tau^{n-j}(a)))\,\phi_j\left(\tau^j(b)\right)\phi_j\left(\tau^j(c)\right)$$

$$+ \sum_{i=1}^{n-1} \phi_i(\sigma^i(c))\phi_i(\sigma^i(b))\phi_i\left(\sigma^i\sigma^i\tau^{n-i}(a)\right)\phi_i\left(\sigma^i\tau^{n-i}\sigma^i\tau^{n-i}(r)\right)\sum_{j=1}^{i} \phi_j(\tau^j\sigma^j\tau^{n-i}(a)))\,\phi_j\left(\tau^j(b)\right)\phi_j\left(\tau^j(c)\right)$$

… (3)

On the other hand

$$\phi_n(w) = \phi_n((abc)r(cba) + (cba)r(abc))$$

$$= \sum_{i=1}^{n} \phi_i\left(\sigma^i(abc)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\phi_i\left(\tau^i(cba)\right) + f_i\left(\sigma^i(cba)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\phi_i\left(\tau^i(abc)\right)$$

Since θ is a Jordan triple higher (σ,τ)-homomorphism, then

$$= \sum_{i=1}^{n} \phi_i\left(\sigma^i(abc)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\left(\sum_{j=1}^{i} \phi_j\left(\tau^j(c)\right)\phi_j\left(\tau^j(b)\right)\phi_j\left(\tau^j(a)\right)\right.$$

$$\left. + \phi_j\left(\tau^j(a)\right)\phi_j\left(\tau^j(b)\right)\phi_j\left(\tau^j(c)\right) - \phi_j(\tau^j(abc)\right)$$

$$+ \sum_{i=1}^{n}(\sum_{j=1}^{i}(\phi_j\left(\sigma^j\sigma^j(c)\right)\phi_j\left(\sigma^j\tau^{i-j}\sigma^j(b)\right)\phi_j\left(\tau^j\sigma^j(a)\right)$$

$$+\phi_j\left(\sigma^j\sigma^j(a)\right)\phi_j\left(\sigma^j\tau^{i-j}\sigma^j(b)\right)\phi_j\left(\tau^j\sigma^j(c)\right) - \phi_i\left(\sigma^i(abc)\right))\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\phi_i\left(\tau^i(abc)\right)$$

$$= \sum_{i=1}^{n} \phi_i\left(\sigma^i(abc)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\sum_{j=1}^{i} \phi_j\left(\tau^j(c)\right)\phi_j\left(\tau^j(b)\right)\phi_j\left(\tau^j(a)\right)$$

$$+\sum_{i=1}^{n} \phi_i\left(\sigma^i(abc)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\sum_{j=1}^{i} \phi_j\left(\tau^j(a)\right)\phi_j\left(\tau^j(b)\right)\phi_j\left(\tau^j(c)\right)$$

$$-\sum_{i=1}^{n} \phi_i\left(\sigma^i(abc)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\phi_i(\tau^i(abc)$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{i} \phi_j\left(\sigma^j\sigma^j(c)\right)\phi_j\left(\sigma^j\tau^{i-j}\sigma^j(b)\right)\phi_j\left(\tau^j\sigma^j(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\phi_i\left(\tau^i(abc)\right)$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{i}(\phi_j\left(\sigma^j\sigma^j(a)\right)\phi_j\left(\sigma^j\tau^{i-j}\sigma^j(b)\right)\phi_j\left(\tau^j\sigma^j(c)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\phi_i\left(\tau^i(abc)\right)$$

$$-\sum_{i=1}^{n}\sum_{j=1}^{i} \phi_i\left(\sigma^i(abc)\right))\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\phi_i\left(\tau^i(abc)\right)$$

$$= -\sum_{i=1}^{n} \phi_i\left(\sigma^i(abc)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)(\sum_{j=1}^{i} \phi_i(\tau^i(abc) - \phi_j\left(\tau^j(c)\right)\phi_j\left(\tau^j(b)\right)\phi_j\left(\tau^j(a)\right)$$

$$-\sum_{i=1}^{n} \phi_i\left(\sigma^i(abc)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)(\sum_{j=1}^{i} \phi_i(\tau^i(abc) - \phi_j\left(\tau^j(a)\right)\phi_j\left(\tau^j(b)\right)\phi_j\left(\tau^j(c)\right)$$

$$+\sum_{i=1}^{n} \phi_i\left(\sigma^i\sigma^i(c)\right)\phi_i\left(\sigma^i\tau^{n-i}\sigma^i(b)\right)\phi_i\left(\tau^i\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\phi_i\left(\tau^i(abc)\right)$$

$$+\sum_{i=1}^{n} \phi_i\left(\sigma^i\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}\sigma^i(b)\right)\phi_i\left(\tau^i\sigma^i(c)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\phi_i\left(\tau^i(abc)\right)$$

$$= -\sum_{i=1}^{n} \phi_i\left(\sigma^i(abc)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)B_i(\tau^i(a),\tau^i(b),\tau^i(c))$$

$$-\sum_{i=1}^{n} \phi_i\left(\sigma^i(abc)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)A_i(\tau^i(a),\tau^i(b),\tau^i(c)$$

$$+\sum_{i=1}^{n} \phi_i\left(\sigma^i\sigma^i(c)\right)\phi_i\left(\sigma^i\tau^{n-i}\sigma^i(b)\right)\phi_i\left(\tau^i\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\phi_i\left(\tau^i(abc)\right)$$

$$+\sum_{i=1}^{n} \phi_i\left(\sigma^i\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}\sigma^i(b)\right)\phi_i\left(\tau^i\sigma^i(c)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\phi_i\left(\tau^i(abc)\right)$$

$$= -\phi_n\left(\sigma^n(abc)\right)\phi_n\left(\sigma^n(r)\right)B_n(\tau^n(a,b,c)) - \sum_{i=1}^{n-1} \phi_i\left(\sigma^i(abc)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)A_i(\tau^i(a,b,c))$$

$$-\phi_n\left(\sigma^n(abc)\right)\phi_n\left(\sigma^n(r)\right)A_n(\tau^n(a,b,c)) - \sum_{i=1}^{n-1} \phi_i\left(\sigma^i(abc)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)B_i(\tau^i(a,b,c))$$

$$+\phi_n\left(\sigma^n(c)\right)\phi_n\left(\sigma^n(b)\right)\phi_n\left(\sigma^n(a)\right)\phi_n\left(\sigma^n(r)\right)\phi_n\left(\tau^n(abc)\right)$$

$$+\sum_{i=1}^{n-1} \phi_i\left(\sigma^i\sigma^i(c)\right)\phi_i\left(\sigma^i\tau^{n-i}\sigma^i(b)\right)\phi_i\left(\tau^i\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\phi_i\left(\tau^i(abc)\right)$$

$$+\phi_n\left(\sigma^n(a)\right)\phi_n\left(\sigma^n(b)\right)\phi_n\left(\sigma^n(c)\right)\phi_n\left(\sigma^n(r)\right)\phi_n\left(\tau^n(abc)\right)$$

$$+\sum_{i=1}^{n-1} \phi_i\left(\sigma^i\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}\sigma^i(b)\right)\phi_i\left(\tau^i\sigma^i(c)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\phi_i\left(\tau^i(abc)\right)$$

… (4)

From equation (3) and (4), we get

$$0 = -\phi_n\left(\sigma^n(abc)\right)\phi_n\left(\sigma^n(r)\right)B_n(\tau^n(a,b,c) - \phi_n\left(\sigma^n(abc)\right)\phi_n\left(\sigma^n(r)\right)A_n(\tau^n(a,b,c)$$

$$+\sum_{i=1}^{n-1} \phi_i\left(\sigma^i\sigma^i(c)\right)\phi_i\left(\sigma^i\tau^{n-i}\sigma^i(b)\right)\phi_i\left(\tau^i\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\Bigg(\phi_i\left(\tau^i(abc)\right)$$

$$-\phi_i\left(\tau^i(a))\phi_i\left(\tau^i(b)\right)\phi_i\left(\tau^i(c)\right)\right)\Bigg)$$

$$+\sum_{i=1}^{n-1} \phi_i\left(\sigma^i\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}\sigma^i(b)\right)\phi_i\left(\tau^i\sigma^i(c)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\Bigg(\phi_i\left(\tau^i(abc)\right)$$

$$-\phi_i\left(\tau^i(c))\phi_i\left(\tau^i(b)\right)\phi_i\left(\tau^i(a)\right)\right)\Bigg)$$

$$+\phi_n\left(\sigma^n(c)\right)\phi_n\left(\sigma^n(b)\right)\phi_n\left(\sigma^n(a)\right)\phi_n\left(\sigma^n(r)\right)\left(\phi_n\left(\tau^n(abc)\right)\right)$$

$$-\sum_{i=1}^{n} \phi_i\left(\tau^i(a)\right)\phi_i\left(\tau^i(b)\right)\phi_i\left(\tau^i(c)\right)$$

$$+\phi_n\left(\sigma^n(a)\right)\phi_n\left(\sigma^n(b)\right)\phi_n\left(\sigma^n(c)\right)\phi_n\left(\sigma^n(r)\right)\left(\phi_n\left(\tau^n(abc)\right)\right)$$

$$-\sum_{i=1}^{n} \phi_i\left(\tau^i(c)\right)\phi_i\left(\tau^i(b)\right)\phi_i\left(\tau^i(a)\right)$$

$$-\sum_{i=1}^{n-1}\phi_i\left(\sigma^i(abc)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)A_i(\tau^i(a,b,c))$$
$$-\sum_{i=1}^{n-1}\phi_i\left(\sigma^i(abc)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)B_i(\tau^i(a,b,c))$$

$$= -\phi_n\big(\sigma^n(abc)\big)\phi_n\big(\sigma^n(r)\big)B_n(\tau^n(a,b,c) - \phi_n\big(\sigma^n(abc)\big)\phi_n\big(\sigma^n(r)\big)A_n(\tau^n(a,b,c)$$
$$+\phi_n\big(\sigma^n(c)\big)\phi_n\big(\sigma^n(b)\big)\phi_n\big(\sigma^n(a)\big)\phi_n\big(\sigma^n(r)\big)A_n(\tau^n(a,b,c)+\phi_n\big(\sigma^n(a)\big)\phi_n\big(\sigma^n(b)\big)\phi_n\big(\sigma^n(c)\big)\phi_n\big(\sigma^n(r)\big)B_n(\tau^n(a,b,c)$$
$$+\sum_{i=1}^{n-1}\phi_i\left(\sigma^i\sigma^i(c)\right)\phi_i\left(\sigma^i\tau^{n-i}\sigma^i(b)\right)\phi_i\left(\tau^i\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)A_n(\tau^n(a,b,c)$$
$$+\sum_{i=1}^{n-1}\phi_i\left(\sigma^i\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}\sigma^i(b)\right)\phi_i\left(\tau^i\sigma^i(c)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)B_n(\tau^n(a,b,c)$$
$$-\sum_{i=1}^{n-1}\phi_i\left(\sigma^i(abc)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)A_i(\tau^i(a,b,c))$$
$$-\sum_{i=1}^{n-1}\phi_i\left(\sigma^i(abc)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)B_i(\tau^i(a,b,c))$$

$$= -\phi_n\left(\sigma^n(abc)-\phi_n\big(\sigma^n(a)\big)\phi_n\big(\sigma^n(b)\big)\phi_n\big(\sigma^n(c)\big)\right)\phi_n\big(\sigma^n(r)\big)B_n(\tau^n(a,b,c)$$
$$-\phi_n\left(\sigma^n(abc)-\phi_n\big(\sigma^n(c)\big)\phi_n\big(\sigma^n(b)\big)\phi_n\big(\sigma^n(a)\big)\right)\phi_n\big(\sigma^n(r)\big)A_n(\tau^n(a,b,c)$$
$$-\left(\sum_{i=1}^{n-1}\phi_i\left(\sigma^i(abc)\right)-\phi_i\left(\sigma^i\sigma^i(c)\right)\phi_i\left(\sigma^i\tau^{n-i}\sigma^i(b)\right)\phi_i\left(\tau^i\sigma^i(a)\right)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)A_n(\tau^n(a,b,c)$$
$$-\left(\sum_{i=1}^{n-1}\phi_i\left(\sigma^i(abc)\right)-\phi_i\left(\sigma^i\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}\sigma^i(b)\right)\phi_i\left(\tau^i\sigma^i(c)\right)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)B_n(\tau^n(a,b,c)$$

$$= -A_n(\sigma^n(a,b,c)\phi_n\big(\sigma^n(r)\big)B_n(\tau^n(a,b,c) - B_n(\sigma^n(a,b,c)\phi_n\big(\sigma^n(r)\big)A_n(\tau^n(a,b,c)$$
$$-\left(\sum_{i=1}^{n-1}B_n(\sigma^n(a,b,c)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)A_n(\tau^n(a,b,c)$$
$$-\left(\sum_{i=1}^{n-1}A_n(\sigma^n(a,b,c)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)B_n(\tau^n(a,b,c)$$

Hence, we have

$$A_n(\sigma^n(a,b,c)\phi_n\big(\sigma^n(r)\big)B_n(\tau^n(a,b,c) + B_n(\sigma^n(a,b,c)\phi_n\big(\sigma^n(r)\big)A_n(\tau^n(a,b,c) = 0.$$

Lemma 3.2:

Let $\theta = (\phi_{\mathrm{i}})_{\mathrm{i}\in\mathbb{N}}$ be a Jordan triple higher (σ,τ)-homomorphism of R into R', then for each $n \in \mathbb{N}$ and for all $a, b, c, r \in R$,

$$A_n(\sigma^n(a,b,c)\phi_n\big(\sigma^n(r)\big)B_n(\tau^n(a,b,c) = B_n(\sigma^n(a,b,c)\phi_n\big(\sigma^n(r)\big)A_n(\tau^n(a,b,c) = 0.$$

Proof.

By Lemma 3.1 and Lemma 2.7, we achieve the result.

Theorem 3.3:

Let $\theta = (\phi_{\mathrm{i}})_{\mathrm{i}\in\mathbb{N}}$ be a Jordan triple higher (σ,τ)-homomorphism of ring R into prime ring R'. Then for each $n \in \mathbb{N}$ and for all $a, b, c, r, x, y, z \in R$,

$A_n\big(\sigma^n(a,b,c)\big)\,\phi_n\big(\sigma^n(r)\big)B_n\big(\tau^n(x,y,z)\big) = 0.$

Proof.

By replacing $a + x$ by a in Lemma 3.2, we get

$$A_n\big(\sigma^n(a+x,b,c)\big)\phi_n\big(\sigma^n(r)\big)B_n\big(\tau^n(a+x,b,c)\big)=0$$

Hence

$$A_n\big(\sigma^n(a,b,c)\big)\,\phi_n\big(\sigma^n(r)\big)B_n\big(\tau^n(a,b,c)\big)+A_n\big(\sigma^n(a,b,c)\big)\,\phi_n\big(\sigma^n(r)\big)B_n\big(\tau^n(x,b,c)\big)+$$
$$A_n\big(\sigma^n(x,b,c)\big)\phi_n\big(\sigma^n(r)\big)B_n\big(\tau^n(a,b,c)\big)+A_n\big(\sigma^n(x,b,c)\big)\,\phi_n\big(\sigma^n(r)\big)B_n\big(\tau^n(x,b,c)\big)=0$$

By Lemma 3.2, we obtain

$$A_n\big(\sigma^n(a,b,c)\big)\,\phi_n\big(\sigma^n(r)\big)B_n\big(\tau^n(x,b,c)\big)+A_n\big(\sigma^n(x,b,c)\big)\phi_n\big(\sigma^n(r)\big)B_n\big(\tau^n(a,b,c)\big)=0$$

Therefore, we get

$$0=A_n\big(\sigma^n(a,b,c)\big)\,\phi_n\big(\sigma^n(r)\big)B_n\big(\tau^n(x,b,c)\big)\,\phi_n\big(\sigma^n(r)\big)A_n\big(\sigma^n(a,b,c)\big)\phi_n\big(\sigma^n(r)\big)B_n\big(\tau^n(x,b,c)\big)$$
$$=-A_n\big(\sigma^n(a,b,c)\big)\,\phi_n\big(\sigma^n(r)\big)B_n\big(\tau^n(x,b,c)\big)\,\phi_n\big(\sigma^n(r)\big)B_n\big(\sigma^n(x,b,c)\big)\phi_n\big(\sigma^n(r)\big)A_n\big(\tau^n(a,b,c)\big)$$

Since R' is prime, we obtain

$$A_n\big(\sigma^n(a,b,c)\big)\,\phi_n\big(\sigma^n(r)\big)\,B_n\big(\tau^n(x,b,c)\big)=0. \qquad \dots(5)$$

By replacing b+y for b in equation (5), we get

$$A_n\big(\sigma^n(a,b+y,c)\big)\phi_n\big(\sigma^n(r)\big)B_n\big(\tau^n(x,b+y,c)\big)=0$$

Hence

$$A_n\big(\sigma^n(a,b,c)\big)\,\phi_n\big(\sigma^n(r)\big)B_n\big(\tau^n(x,b,c)\big)+A_n\big(\sigma^n(a,b,c)\big)\,\phi_n\big(\sigma^n(r)\big)B_n\big(\tau^n(x,y,c)\big)+$$
$$A_n\big(\sigma^n(a,y,c)\big)\phi_n\big(\sigma^n(r)\big)B_n\big(\tau^n(x,b,c)\big)+A_n\big(\sigma^n(a,y,c)\big)\,\phi_n\big(\sigma^n(r)\big)B_n\big(\tau^n(x,y,c)\big)=0$$

We can use equation (5), then we get

$$A_n\big(\sigma^n(a,b,c)\big)\,\phi_n\big(\sigma^n(r)\big)B_n\big(\tau^n(x,y,c)\big)+A_n\big(\sigma^n(a,y,c)\big)\phi_n\big(\sigma^n(r)\big)B_n\big(\tau^n(x,b,c)\big)=0$$

Therefore, we get

$$0=A_n\big(\sigma^n(a,b,c)\big)\,\phi_n\big(\sigma^n(r)\big)B_n\big(\tau^n(x,y,c)\big)\,\phi_n\big(\sigma^n(r)\big)A_n\big(\sigma^n(a,b,c)\big)\phi_n\big(\sigma^n(r)\big)B_n\big(\tau^n(x,y,c)\big)$$
$$=-A_n\big(\sigma^n(a,b,c)\big)\,\phi_n\big(\sigma^n(r)\big)B_n\big(\tau^n(x,y,c)\big)\,\phi_n\big(\sigma^n(r)\big)B_n\big(\sigma^n(x,y,c)\big)\phi_n\big(\sigma^n(r)\big)A_n\big(\tau^n(a,b,c)\big)$$

Since R' is prime, we obtain

$$A_n\big(\sigma^n(a,b,c)\big)\,\phi_n\big(\sigma^n(r)\big)B_n\big(\tau^n(x,y,c)\big)=0 \qquad \dots(6)$$

By replacing $c+z$ for c in equation (6), we get

$$A_n\big(\sigma^n(a,b,c+z)\big)\phi_n\big(\sigma^n(r)\big)B_n\big(\tau^n(x,y,c+z)\big)=0$$

Hence

$$A_n\big(\sigma^n(a,b,c)\big)\,\phi_n\big(\sigma^n(r)\big)B_n\big(\tau^n(x,y,c)\big)+A_n\big(\sigma^n(a,b,c)\big)\,\phi_n\big(\sigma^n(r)\big)B_n\big(\tau^n(x,y,z)\big)+$$
$$A_n\big(\sigma^n(a,b,z)\big)\phi_n\big(\sigma^n(r)\big)B_n\big(\tau^n(x,y,c)\big)+A_n\big(\sigma^n(a,b,z)\big)\,\phi_n\big(\sigma^n(r)\big)B_n\big(\tau^n(x,y,z)\big)=0$$

We can use equation (5), then we get

$$A_n\big(\sigma^n(a,b,c)\big)\,\phi_n\big(\sigma^n(r)\big)B_n\big(\tau^n(x,y,z)\big)+A_n\big(\sigma^n(a,b,z)\big)\phi_n\big(\sigma^n(r)\big)B_n\big(\tau^n(x,y,c)\big)=0$$

Therefore, we get

$$0=A_n\big(\sigma^n(a,b,c)\big)\,\phi_n\big(\sigma^n(r)\big)B_n\big(\tau^n(x,y,z)\big)\,\phi_n\big(\sigma^n(r)\big)A_n\big(\sigma^n(a,b,c)\big)\phi_n\big(\sigma^n(r)\big)B_n\big(\tau^n(x,y,z)\big)$$
$$=-A_n\big(\sigma^n(a,b,c)\big)\,\phi_n\big(\sigma^n(r)\big)B_n\big(\tau^n(x,y,z)\big)\,\phi_n\big(\sigma^n(r)\big)B_n\big(\sigma^n(x,y,z)\big)\phi_n\big(\sigma^n(r)\big)A_n\big(\tau^n(a,b,c)\big)$$

Since R' is prime, we obtain

$$A_n\big(\sigma^n(a,b,c)\big)\,\phi_n\big(\sigma^n(r)\big)B_n\big(\tau^n(x,y,z)\big)=0$$

In the following theorem we give the conditions which make the Jordan triple higher (σ,τ)-homomorphism is either triple higher (σ,τ)-homomorphism or triple higher (σ,τ)-anti-homomorphism.

Theorem 3.4:

Every Jordan triple higher (σ,τ)-homomorphism of ring R into prime ring R' is either triple higher (σ,τ)-homomorphism or triple higher (σ,τ)-anti-homomorphism.

Proof.

Let θ be a Jordan triple higher (σ,τ)-homomorphism. Then by Theorem 3.3, we have

$$A_n\big(\sigma^n(a,b,c)\big)\,\phi_n\big(\sigma^n(r)\big)B_n\big(\tau^n(x,y,z)\big)=0$$

Since R' is prime, therefore either $A_n\big(\sigma^n(a,b,c)\big)=0$ or $B_n\big(\tau^n(x,y,z)\big)=0$, for each $n\in\mathbb{N}$ and for all $a,b,c,x,y,z\in R$.

If $B_n\big(\tau^n(x,y,z)\big)=0$, then by Remark 2.9, we obtain θ is triple higher (σ,τ)-anti-homomorphism.

But if $A_n\big(\sigma^n(a,b,c)\big)=0$, then by Remark 2.9, we obtain θ is triple higher (σ,τ)-homomorphism.

Proposition 3.5:

Let $\theta = (\phi_{\rm i})_{{\rm i}\in\mathbb{N}}$ be a Jordan triple higher (σ,τ)-homomorphism from prime ring R into prime ring R', then θ is higher (σ,τ)-homomorphism.

Proof:

Since θ is a Jordan triple higher (σ,τ)-homomorphism, then for all $a, r \in R$ and $n \in \mathbb{N}$, we have

$$\phi_n(ara) = \sum_{i=1}^{n} \phi_i\left(\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\phi_i(\tau^i(a))$$

By replacing a by ab, we get

$$\phi_n\big((ab)r(ab)\big) = \sum_{i=1}^{n} \phi_i\left(\sigma^i(ab)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\phi_i(\tau^i(ab))$$

$$= \phi_n\big(\sigma^n(ab)\big)rab + ab\sum_{i=1}^{n} \phi_i\left(\sigma^i\tau^{n-i}(r)\right)\phi_i(\tau^i(ab)) \quad \dots (7)$$

On the other hand, we get

$$\phi_n\big((ab)r(ab)\big) = \sum_{i=1}^{n} \phi_i\big(\sigma^i(ab)\big)\phi_{\rm i}\big(\sigma^i\tau^{n-i}(r)\big)\phi_{\rm i}(\tau^i(ab))$$

$$= \sum_{i=1}^{n} \phi_i\left(\sigma^i(a)\right)\phi_i\left(\sigma^i(b)\right)rab + ab\sum_{i=1}^{n} \phi_i\left(\sigma^i\tau^{n-i}(r)\right)\phi_i(\tau^i(ab)) \quad \dots (8)$$

By comparing (7) and (8), we get

$$\left(\phi_n\big(\sigma^n(ab)\big) - \sum_{i=1}^{n} \phi_i\left(\sigma^i(a)\right)\phi_i\left(\sigma^i(b)\right)\right)rab = 0.$$

Since R is prime and $ab \neq 0$, we get

$$\phi_n(ab) = \sum_{i=1}^{n} \phi_i\left(\sigma^i(a)\right)\phi_{\rm i}(\tau^i(b))$$

Hence θ is a higher (σ,τ)-homomorphism.

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