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Generalized Higher Derivations on ΓM-Modules

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Abstract
The concepts of generalized higher derivations, Jordan generalized higher derivations, and Jordan generalized triple higher derivations on Γ-ring M into ΓM-modules X are presented. We prove that every Jordan generalized higher derivation of Γ-ring M into 2-torsion free ΓM-module X, such that aαbβc=aβbαc, for all a, b, c ∈M and α,β∈Γ, is Jordan generalized triple higher derivation of M into X.

Keywords: generalized higher derivations, Jordan generalized higher derivations, ΓM-module.

تعميمات المشتقات العليا على المقاسات من النمط –ΓM

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الخلاصة
قدمنا المفاهيم التالية تعميمات المشتقات العليا , تعميمات جوردان للمشتقات العليا وتعميمات جوردان الثلاثية للمشتقات العليا من الحلقات M من النمط – Γ الى المقاسات X من النمط –ΓM وايضا برهنا كل تعميم جوردان للمشتقات العليا من حلقات M من النمط – Γ الى المقاسات X طليقة الالتواء –2 من النمط –ΓM تحقق aαbβc=aβbαc لكل a,b,c∈M , α,β∈Γ هو تعميم جوردان الثلاثي للمشتقات العليا من M الى X.

1. Introduction

Let M and Γ be two additive abelian groups. Suppose that there is a mapping from M×Γ×M→M (the image of (a,α,b) being denoted by aαb, a,b∈M and α ∈Γ) satisfying the following for all a,b,c∈M, α,β∈Γ:

i) (a+b)αc= aαc+ bαc
a(α +β)c = aαc + aβc
aα(b +c) = aα b + aα c
ii) (aα b)βc = aα(bβc)

Then M is called a Γ-ring. This definition is due to Barnes [1], where every ring is a Γ-ring. M is said to be 2-torsion free if 2a = 0 implies a=0 for all a∈M. Besides, M is called a prime Γ-ring if for all a,b∈M, aMΓMb = (0) implies either a=0 or b=0. M is called a semiprime if aMΓMa = (0) with a∈M

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implies a=0. Note that every prime Γ-ring is obviously a semiprime [2].

Let M be a Γ-ring and X be an additive abelian group. X is a left ΓM- module if there exists a mapping M×Γ×X →X (sending (m,α,x) into mαx where m∈M , α,β∈Γand x∈X) satisfying the following, for all m,m_1,m_2∈M, α,β∈Γ and x,x_1,x_2 ∈X [3]:

i) $(m_1+m_2)\alpha x= m_1\alpha x+m_2\alpha x$
ii) $m(\alpha+\beta)x=m\alpha x+m\beta x$
iii) $m\alpha(x_1+x_2)=m\alpha x_1+m\alpha x_2$
iv) $(m_1\alpha m_2)\beta x= m_1\alpha(m_2\beta x)$

X is called a right ΓM- module if there exists a mapping X×Γ×M →X. X is called a ΓM-module if X is both a left and right ΓM- module. X is called a left prime (right prime) if aΓMΓb=(0) then a=0 or b=0, a∈M, b∈X (a∈X,b∈M respectively) and X is a prime if it is both a left and right prime. X is called a semipeime if aΓMΓa =(0) where a∈X implies a=0. X is called 2-torsion free if 2x=0 implies x=0 for all x∈X [3].

Paul and Halder [3] defined a left derivation and a Jordan left derivation of Γ-ring M onto ΓM-module X as follows: d:M→X is a left derivation if $d(a\alpha b) = a\alpha d(b) + b\alpha\, d(a)$, and a Jordan left derivation d(aαa)=2aαd(a). Also Paul and Halder proved that every Jordan left derivation of Γ-ring M into ΓM-module is a left derivation. Salih [4] defined derivation and Jordan derivation on a ΓM-module as follows:

d:M→X is a derivation if $d(a\alpha b) = d(a)\alpha b + a\alpha d(b)$, and a Jordan derivation
$d(a\alpha\, a) = d(a)\alpha\, a\, +\, a\alpha d(a)$

They also proved that every Jordan derivation of a Γ-ring M into ΓM-module X is a derivation . In addition, Salih [5] defined the generalized derivation and the Jordan generalized derivation of a Γ-ring M into ΓM-module X as follows :

f: M → X is an additive mapping of M into a ΓM-module X , then f is called a
generalized derivation of M into X if there exists a derivation d:M→X such that for every a,b∈M, α∈Γ.
f(aαb)= f (a)αb + aαd(b), then f is called a Jordan generalized derivation of M into X if there exists a Jordan derivation d:M→X such that for every a∈M, α∈Γ.
f (aαa)= f(a)αa + aαd(a) . Salih [5] also proved that every Jordan generalized derivation of a Γ-ring M into a 2-tortion free prime ΓM-module X is a generalized derivation of M into X.

In this paper we present the concepts of higher derivations and Jordan higher derivations of a Γ-ring M into ΓM-module X. We also prove that every Jordan higher derivation of a Γ-ring M into a 2-torsion free prime ΓM-module X is a higher derivation of M into X.

We need the following lemma

Lemma 1.1: [6]

Let M be a 2-torsion free semiprime Γ-ring and suppose that a,b∈M, if aΓmΓb + bΓmΓa = 0 for all m∈M, then aΓmΓb = bΓmΓa = 0.

2. Generalized Higher derivations on Γ-ring into ΓM-module

The generalized higher derivations, Jordan generalized higher derivations and Jordan generalized triple higher derivations on a Γ-ring into a ΓM-module are introduced. We begin with the following definition:

Definitions 2.1

Let M be a Γ-ring and $F=(f_i)_{i\in N}$ be a family of additive mappings of M into a ΓM-module X, such that f_0=id, then F is called a generalized higher derivation of M into X if there exists a higher derivations $D=(d_i)_{i\in N}$ of M into X, if for every a,b∈M, α∈Γ and n∈N.

$$f_n(a\alpha b)= \sum_{i+j=n} f_i(a)\alpha d_j(b) \qquad \text{... (i)}$$

F is said to be a Jordan generalized higher derivation of M into X if there exist Jordan higher derivations $D=(d_i)_{i\in N}$ of M into X if for every a∈M, α∈Γ and n∈N.

$$f_n(a\alpha a)=\sum_{i+j=n} f_i(a)\alpha d_j(a) \quad \text{... (ii)}$$

F is called a Jordan generalized triple higher derivation of M into X if there exist Jordan higher triple derivations $D=(d_i)_{i\in N}$ of M into X if for every a,b∈M, α,□∈Γ and n∈N.

$$f_n(a\alpha b\beta a)=\sum_{i+j+l=n} f_i(a)\alpha d_j(b)\beta d_l(a) \quad \text{... (iii)}$$

The following is an example of the generalized higher derivation of M into X:

Example 2.2

Let R be a ring, $f=(f_i)_{i\in N}$ be a generalized higher derivations of R into an R-module Y associated with $d=(d_i)_{i\in N}$, which is a higher derivation of R into R-module Y. Let $M=M_{1\times 2}(R)$, $\Gamma=\left\{\begin{pmatrix} m \\ 0 \end{pmatrix} : \text{m is an integer number}\right\}$, then M is a Γ-ring and $X = M_{1\times 2}(Y)$. We use the usual addition and multiplication on matrices. We define

$F=(F_i)_{i\in N}$ be a family of additive mapping of M into a ΓM-module X such that

$F_n(a\ \ b)= (f_n(a)\ \ f_n(b))$ associated with $D=(D_i)_{i□N}$ be a family of additive mappings of M into a ΓM-module X such that $D_n(a\ \ b) = (d_n(a)\ \ d_n(b))$. Then F is a generalized higher derivation of M into X.

It is clear that every higher derivation of a Γ-ring M into a ΓM-module X is a Jordan higher derivation of M into X, but the converse is not true in general, as shown by the following example:

Example 2. 3

Let M be a Γ-ring , X be a ΓM-module and let a∈X such that aΓa=(0) and xαaβx=0, for all x∈M, α, β∈Γ, but xαaβy ≠ 0, for some x,y∈M, x ≠ y. Also, let $D=(d_i)_{i□N}$ be a family of mappings on M into a ΓM-module X defined by the following relation, for each n∈N:

$d_n(x)=nx\alpha a + a\alpha x$, for all x∈M, α∈Γ, a∈X .

Let $F=(f_i)_{i\in N}$ be a family of mappings on M into a ΓM-module X, defined by the following, for each n∈N:

$f_n(x)= nx\alpha a$, for all x∈M, α∈Γ, a∈X.

It is clear that F is a Jordan generalized higher derivation of M into X but not a higher derivation of M into X.

Now, we give some properties of the generalized higher derivation on a □-ring into a ΓM-module.

Lemma 2.4

Let M be a Γ-ring and $F=(f_i)_{i□N}$ be a Jordan generalized higher derivation of M into a 2-torsion free ΓM-module X. Then for all a,b,c∈M ,α,β∈ Γ, and n∈N, the following statements hold:

i) $f_n(a\alpha b+b\alpha a) = \sum_{i+j=n} f_i(a)\alpha d_j(b)+d_i(b)\alpha d_j(a)$

ii) $f_n(a\alpha b\beta a+a\beta b\alpha a) = \sum_{i+j+l=n} f_i(a)\alpha d_j(b)\beta d_l(a)+ f_i(a)\beta d_j(b)\alpha d_l(a)$

iii) $f_n(a\alpha b\alpha a) = \sum_{i+j+l=n} f_i(a)\alpha d_j(b)\alpha d_l(a)$

iv) $f_n(a\alpha b\alpha c+c\alpha b\alpha a) = \sum_{i+j+l=n} f_i(a)\alpha d_j(b)\alpha d_l(c) + f_i(c)\alpha d_l(b)\alpha d_q(a)$

v) $f_n(a\alpha b\beta c+c\alpha b\beta a) = \sum_{i+j+l=n} f_i(a)\alpha d_j(b)\beta d_l(c)+ f_i(c)\alpha d_j(b)\beta d_l(a)$

i) $f_n((a+b)\,\alpha(a+b)) = \sum_{i+j=n} f_i(a+b)\,\alpha d_j(a+b)$

$$= \sum_{i+j=n} f_i(a)\,\alpha d_j(a)+ d_i(a)\,\alpha d_j(b) + f_i(b)\,\alpha d_j(a) + f_i(b)\,\alpha d_j(b) \quad \ldots (1)$$

On the other hand:

$f_n((a+b)\,\alpha(a+b)) = f_n(a\alpha a+a\alpha b+b\alpha a+b\alpha b)$

$= f_n(a\alpha a+ b\alpha b) + f_n(a\alpha b+b\alpha a)$

$$= \sum_{i+j=n} f_i(a)\,\alpha d_j(a)+ f_i(b)\,\alpha d_j(b)+ f_n(a\alpha b+b\alpha a) \quad \ldots(2)$$

By comparing (1) and (2) we get:

$$f_n(a\alpha b+b\alpha a)= \sum_{i+j=n} f_i(a)\,\alpha d_j(b)+ f_i(b)\alpha d_j(a)$$

ii) By replacing aβb+bβa for b in (i) we have:

$f_n(a\alpha(a\beta b+b\beta a)\mathbf{+} (a\beta b+b\beta a\mathbf{)}\,\alpha a)$

$= \sum_{i+j=n} f_i(a)\alpha d_j(a\beta b+b\beta a)+ f_i(a\beta b+b\beta a)\,\alpha d_j(a)$

$= \sum_{i+j=n} f_i(a)\,\alpha\left(\sum_{p+h=l} d_p(a)\,\beta d_h(b)+d_p(b)\,\beta d_h(a)\right)+ \sum_{i+j=n}\left(\sum_{r+t=i} f_r(a)\,\beta d_t(b) +d_r(b)\,\beta d_t(a)\right)\alpha d_j(a)$

$= \sum_{i+j+h=n} f_i(a)\,\alpha d_j(a)\,\beta d_h(b)+ d_i(a)\alpha d_j(b)\,\beta d_h(a) + \sum_{r+t+l=n} f_r(a)\,\beta d_t(b)\,\alpha d_l(a)+ d_r(b)\,\beta d_t(a)\,\alpha d_l(a) \quad \ldots (1)$

On the other hand:

$f_n(a\alpha(a\beta b+b\beta a)+ (a\beta b+b\beta a\mathbf{)}\,\alpha a)$

$= f_n(a\alpha a\beta b+a\alpha b\beta a+a\beta b\alpha a+b\beta a\alpha a)$

$= \sum_{i+r+t=n} f_i(a)\alpha d_r(a)\beta d_t(b)+f_i(b)\beta d_r(a)\alpha d_t(a)+f_n(a\alpha b\beta a+a\beta b\alpha a) \quad \ldots (2)$

By comparing (1) and (2) we get:

$$f_n(a\alpha b\beta a+a\beta b\alpha a) = \sum_{i+r+t=n} f_i(a)\alpha d_r(b)\,\beta d_t(a)+ f_i(a)\,\beta d_r(b)\,\alpha d_t(a)$$

iii) By replacing α for β in (ii) we have:

$$f_n(a\alpha b\alpha a+a\alpha b\alpha a) = 2\, f_n(a\alpha b\alpha a) = 2 \sum_{i+r+t=n} f_i(a)\,\alpha d_r(b)\,\alpha d_t(a)$$

Since X is a 2-torsion free, then:

$$f_n(a\alpha b\alpha a) = \sum_{i+r+t=n} f_i(a)\,\alpha d_r(b)\,\alpha d_t(a)$$

iv) By replacing a+c for a in (iii) we get:

$f_n((a+c)\alpha b\alpha(a+c)) = \sum_{i+r+t=n} f_i(a+c)\,\alpha d_r(b)\alpha d_t(a+c) = \sum_{i+r+t=n} f_i(a)\alpha d_r(b)\alpha d_t(a)+ f_i(a)\alpha d_r(b)\alpha d_t(c)$

$+ f_i(c)\alpha d_r(b)\alpha d_t(a) +f_i(c)\,\alpha d_r(b)\alpha d_t(c) \quad \ldots.(1)$

On the other hand:

$f_n((a+c)\alpha b\alpha(a+c))= f_n(a\alpha b\alpha a+a\alpha b\alpha c+c\alpha b\alpha a+c\alpha b\alpha c)$

$$= \sum_{i+r+t=n} f_i(a)\alpha d_r(b)\alpha d_t(a)+ f_i(c)\alpha d_r(b)\alpha d_t(c) + f_n(a\alpha b\alpha c+c\alpha b\alpha a) \quad ... (2)$$

By comparing (1) and (2) we get:

$$f_n(a\alpha b\alpha c+c\alpha b\alpha a) = \sum_{i+r+t=n} f_i(a)\alpha d_r(b)\ \alpha d_t(c)+ f_i(c)\alpha d_r(b)\alpha d_t(a)$$

v) By replacing a+c for a in Definition 1.1 (iii) we get:

$$f_n((a+c)\alpha b\beta(a+c)) = \sum_{i+r+t=n} f_i(a+c)\alpha d_r(b)\beta d_t(a+c)$$

$$= \sum_{i+r+t=n} f_i(a)\alpha d_r(b)\beta d_t(a)+f_i(a)\alpha)d_r(b)\beta d_t(c) + f_i(c)\alpha d_r(b)\beta d_t(a)+f_i(c)\alpha d_r(b)\beta d_t(c) \quad ... (1)$$

On the other hand:

$f_n((a+c)\alpha b\beta(a+c)) = f_n(a\alpha b\beta a+a\alpha b\beta c+c\alpha b\beta a+c\alpha b\beta c)$

$$= \sum_{i+r+t=n} f_i(a)\alpha d_r(b)\beta d_t(a)+f_i(c)\alpha d_r(b)\beta d_t(c)+f_n(a\alpha b\beta c+c\alpha b\beta a) \quad ...(2)$$

By comparing (1) and (2) we get:

$$f_n(a\alpha b\beta c+c\alpha b\beta a) = \sum_{i+r+t=n} f_i(a)\alpha d_r(b)\beta d_t(c)+f_i(c)\alpha d_r(b)\ \beta d_t(a)$$

Definition 2.5

Let $F=(f_i)_{i\in N}$ be a Jordan generalized higher derivation of a Γ-ring M into a ΓM-module X associated with $D=(d_i)_{i\in N}$ of M into X . For every n∈N, for each a,b∈M and for each α∈Γ, we define $\psi_n(a,b)_\alpha$ by:

$$\delta_n(a,b)_\alpha = f_n(a\alpha b)- \sum_{i+r=n} f_i(a)\alpha d_r(b)$$

In the following lemma, we give the properties of $\delta_n(a,b)_\alpha$.

Lemma 2.6

Let $F=(f_i)_{i\in N}$ be a Jordan generalized higher derivation of a Γ-ring M into a ΓM-module X. Then for all a,b∈M, α,β∈Γ and n∈N:

i) $\delta_n(a,b)_\alpha= - \delta_n(b,a)_\alpha$

ii) $\delta_n(a+b,c)_\alpha= \delta_n(a,c)_\alpha+ \delta_n(b,c)_\alpha$

iii) $\delta_n(a,b+c)_\alpha= \delta_n(a,b)_\alpha+ \delta_n(a,c)_\alpha$

iv) $\delta_n(a,b)_{\alpha+\beta}= \delta_n(a,b)_\alpha+ \delta_n(a,b)_\beta$

proof

i) By Lemma 3.4(i) and since f_n is additive mapping for each n∈N then:

$$f_n(a\alpha b+b\alpha a) = \sum_{i+r=n} f_i(a)\alpha d_r(b)+f_i(b)\alpha d_r(a)$$

$$f_n(a\alpha b) + f_n(b\alpha a) = \sum_{i+r=n} f_i(a)\ \alpha d_r(b)+ \sum_{i+r=n} f_i(b)\alpha d_r(a)$$

$$f_n(a\alpha b) - \sum_{i+r=n} f_i(a)\alpha d_r(b) =- f_n(b\alpha a) + \sum_{i+r=n} f_i(b)\alpha d_r(a) \quad \delta_n(a,b)_\alpha = - \delta_n(b,a)_\alpha$$

ii) $\delta_n(a+b,c)_\alpha = f_n((a+b)\alpha c) - \sum_{i+r=n} f_i(a+b)\alpha d_r(c)$

$$= f_n(a\alpha c+b\alpha c) - \sum_{i+r=n} f_i(a)\alpha d_r(c)+f_i(b)\alpha d_r(c)$$

$= f_n(a\alpha c) - \sum_{i+r=n} f_i(a)\alpha d_r(c) + f_n(b\alpha c) - \sum_{i+r=n} f_i(b)\alpha d_r(c) = \delta_n(a,c)_\alpha + \delta_n(b,c)_\alpha$

iii) $\delta_n(a,b+c)_\alpha = f_n(a\alpha(b+c)) - \sum_{i+r=n} f_i(a)\alpha d_r(b+c) = f_n(a\alpha b+a\alpha c) - \sum_{i+r=n} f_i(a)\alpha d_r(b) + f_i(a)\alpha d_r(c)$

$= f_n(a\alpha b) - \sum_{i+r=n} f_i(a)\alpha d_r(b) + f_n(a\alpha c) - \sum_{i+r=n} f_i(a)\alpha d_r(c)$

$= \delta_n(a,b)_\alpha + \delta_n(a,c)_\alpha$

iv) $\delta_n(a,b)_{\alpha+\beta} = f_n(a(\alpha+\beta)b) - \sum_{i+r=n} f_i(a)(\alpha+\beta)d_r(b)$

$= f_n(a\alpha b+a\beta b) - \sum_{i+r=n} f_i(a)\alpha d_r(b) + f_i(a)\beta d_r(b) = f_n(a\alpha b) - \sum_{i+r=n} f_i(a)\alpha d_r(b) + f_n(a\beta b) - \sum_{i+j+r=n} f_i(a)\beta d_r(b)$

$= \delta_n(a,b)_\alpha + \delta_n(a,b)_\beta$

We present the following remark.

Remark 2.7

Note that $F=(f_i)_{i\in N}$ is a higher generalized derivation of a Γ-ring M into a ΓM-module X if and only if $\delta_n(a,b)_\alpha = 0$ for all a,b∈M, α,β∈Γ and n∈N.

3. Main Results

We prove some lemmas which make us able to give the next results.

Lemma 3.1.

Let $F=(f_i)_{i\in N}$ be a Jordan generalized higher derivation of a Γ-ring M into a ΓM-module X. Assume that n∈N, a,b,m∈M, and α,β∈Γ, if $\delta_t(a,b)_\alpha=0$ for every t<n, then:

i) $\delta_n(a,b)_\alpha \beta m\beta [a,b]_\alpha + [a,b]_\alpha \beta m\beta \psi_n(a,b)_\alpha = 0$

ii) $\delta_n(a,b)_\alpha \alpha m\alpha [a,b]_\alpha + [a,b]_\alpha \alpha m\alpha \psi(a,b)_\alpha = 0$

iii) $\delta_n(a,b)_\beta \alpha m\alpha [a,b]_\beta + [a,b]_\beta \alpha m\alpha \psi_n(a,b)_\beta = 0$

Where $\psi_n(a,b)_\alpha = d_n(a\alpha b) - \sum_{i+r=n} d_i(a)\alpha d_r(b)$

Proof

i) Since mαa, aαb, bαa∈M and aαmβa, bαmβb∈M, it follows that $a\alpha b\beta m\beta b\alpha a + b\alpha a\beta m\beta a\alpha b = a\alpha(b\beta m\beta b)\alpha a + b\alpha(a\beta m\beta a)\alpha b$

Since f_n is additive mappings for each n∈N and by lemma 3.4 (v), we obtain:

$f_n(a\alpha b\beta m\beta b\alpha a + b\alpha a\beta m\beta a\alpha b)$

$= \sum_{s+t=n} f_s(a\alpha b\beta m\beta b\alpha a) + f_t(b\alpha a\beta m\beta a\alpha b)$

$= \sum_{i+r+t+q+l=n} f_i(a)\alpha d_r(b)\beta d_t(m)\beta d_q(b)\alpha d_l(a) + f_i(b)\alpha d_r(a)\beta d_t(m)\beta d_q(a)\alpha d_l(b)$

and since: $f_n(a\alpha b\beta m\beta b\alpha a + b\alpha a\beta m\beta a\alpha b)$

$= f_n((a\alpha b)\beta m\beta(b\alpha a) + (b\alpha a)\beta m\beta(a\alpha b))$

$= \sum_{i+r+s=n} f_i(a\alpha b)\beta d_r(m)\beta d_t(b\alpha a) + f_i(b\alpha a)\beta d_r(m)\beta d_t(a\alpha b)$

By the inductive assumption, we can substitute $d_r(u\alpha v)$ for:

$\sum_{i+l=r} f_i(u)\alpha d_l(v)$

when r<n, for u=a,b and v=b,a, thus an easy computation gives:

$$\sum_{i+l+s+t+p+h=n} f_i(u)\alpha d_l(v)\beta d_s(m)\beta d_p(v)\alpha d_h(u) - \sum_{r+t+q=n} f_r(u\alpha v)\beta d_t(m)\beta d_q(v\alpha u)$$

$=-(\delta_n(u,v)_\alpha \beta m\beta v\alpha u + u\alpha v\beta m\beta\, \psi_n(v,u)_\alpha)$

Thus, by comparing both expressions of:

$f_n(a\alpha b\beta m\beta b\alpha a + b\alpha a\beta m\beta a\alpha b)$

we obtain:

$-\left(\delta_n(a,b)_\alpha \beta m\beta b\alpha a + a\alpha b\beta m\beta\psi_n(b,a)_\alpha + \delta_n(b,a)_\alpha\beta m\beta a\alpha b + b\alpha a\beta m\beta\psi_n(a,b)_\alpha\right)=0$

By Lemma 2.6(i), we get:

$-\left(\delta_n(a,b)_\alpha\beta m\beta b\alpha a - \delta_n(a,b)_\alpha\beta m\beta a\alpha b + b\alpha a\beta m\beta\psi_n(a,b)_\alpha - a\alpha b\beta m\beta\psi_n(a,b)_\alpha\right)=0$

$-\left(\delta_n(a,b)_\alpha\, \beta m\beta(b\alpha a - a\alpha b) + (b\alpha a - a\alpha b)\, \beta m\beta\psi_n(a,b)_\alpha\right)=0$

$\delta_n(a,b)_\alpha\beta m\beta[a,b]_\alpha + [a,b]_\alpha\beta m\beta\psi_n(a,b) = 0$

ii) Since f_n is an additive mapping and by lemma 2.4 (iv), we get:

$f_n(a\alpha b\alpha m\alpha b\alpha a + b\alpha a\alpha m\alpha a\alpha b)$

$$= \sum_{i+l+s+p+h=n} f_i(a)\alpha d_l(b)\alpha d_s(m)\alpha d_p(b)\alpha) d_h(a)$$

$$+ f_i(b)\alpha d_l(a)\alpha d_s(m)\alpha d_p(a)\alpha d_h(b) \quad \ldots (1)$$

and since:

$f_n((a\alpha b)\alpha m\alpha(b\alpha a) + (b\alpha a)\alpha m\alpha(a\alpha b))$

$$= \sum_{i+r+t=n} f_i(a\alpha b)\alpha d_r(m)\alpha d_t(b\alpha a) + f_i(b\alpha a)\alpha d_r(m)\alpha d_r(m) d_t(a\alpha b) \quad \ldots (2)$$

Similarly, as in the proof of (i) and by comparing (1) and (2), we get:

$-(\delta_n(a,b)_\alpha\alpha m\alpha b\alpha a - a\alpha b\alpha m\alpha\psi_n(b,a)_\alpha + \delta_n(b,a)_\alpha\alpha m\alpha a\alpha b$

$+ b\alpha a\alpha m\alpha\, \psi_n(a,b)_\alpha) = 0$

By Lemma 2.6 (i), we have:

$-(\delta_n(a,b)_\alpha\alpha m\alpha b\alpha a - \delta_n(a,b)_\alpha\alpha m\alpha a\alpha b + b\alpha a\alpha m\alpha\psi_n(a,b)_\alpha - a\alpha b\alpha m\alpha\psi_n(a,b)_\alpha)=0$

$-(\delta_n(a,b)_\alpha\ \alpha m\alpha(b\alpha a - a\alpha b) + (b\alpha a - a\alpha b)\ \alpha m\alpha\ \psi_n(a,b)_\alpha) = 0$

$\delta_n(a,b)_\alpha\ \alpha m\alpha\ [a,b]_\alpha + [a,b]_\alpha\ \alpha m\alpha\ \psi_n(a,b)_\alpha = 0$

iii) By interchanging α and β in (i), we obtain (iii).

Lemma 3.2

Let $F=(f_i)_{i\in N}$ be a Jordan generalized higher derivation of a Γ-ring M into a 2-torsion free prime ΓM-module X. Then for all a,b,m∈M, α,β∈Γ and n∈N, we have:

i) $\delta_n(a,b)_\alpha\ \beta m\beta[a,b]_\alpha = [a,b]_\alpha\ \beta m\beta\ \psi_n(a,b)_\alpha = 0$

ii) $\delta_n(a,b)_\alpha\ \alpha m\alpha[a,b]_\alpha = [a,b]_\alpha\ \alpha m\alpha\ \psi_n(a,b)_\alpha = 0$

iii) $\delta_n(a,b)_\beta\ \alpha m\alpha[a,b]_\beta = [a,b]_\beta\ \alpha m\alpha\ \psi_n(a,b)_\beta = 0$

Proof

i) By Lemma 3.1 (i), we have:

$\delta_n(a,b)_\alpha\ \beta m\beta[a,b]_\alpha + [a,b]_\alpha\beta m\beta\psi_n(a,b)_\alpha = 0$

by Lemma 1.1, we get:

$\delta_n(a,b)_\alpha\ \beta m\beta[a,b]_\alpha = [a,b]_\alpha\beta m\beta\psi_n(a,b)_\alpha = 0$

ii) By Lemma 3.1 (ii), we have:

$\delta_n(a,b)_\alpha\ \alpha m\alpha[a,b]_\alpha + [a,b]_\alpha\alpha m\alpha\ \psi_n(a,b)_\alpha = 0$

by Lemma 1.1, we get:

$\delta_n(a,b)_\alpha\ \alpha m\alpha[a,b]_\alpha = [a,b]_\alpha\alpha m\alpha\ \psi_n(a,b)_\alpha = 0$

iii) By Lemma 3.1 (iii), we have:

$\delta_n(a,b)_\beta\alpha m\alpha[a,b]_\beta + [a,b]_\beta\ \alpha m\alpha\ \psi_n(a,b)_\beta = 0$

by Lemma 1.1, we have:
$\delta_n(a,b)_\beta \alpha m\alpha[a,b]_\beta = [a,b]_\beta\ \alpha m\alpha\ \psi_n(a,b)_\beta=0$.

Theorem 3.3

Let $D=(d_i)_{i\in N}$ be a Jordan higher derivation of a Γ-ring M into a 2-torsion free prime ΓM-module X . Then for all a,b,m∈M, α,β∈Γ and n∈N, we have:

i) $\delta_n(a,b)_\alpha\ \beta m\beta\ [c,d]_\alpha\ =0$

ii) $\delta_n(a,b)_\alpha\ \alpha m\alpha\ [c,d]_\alpha =0$

iii) $\delta_n(a,b)_\alpha\ \alpha m\alpha\ [c,d]_\beta =0$

Proof

i) By replacing a+c for a in Lemma 3.2(i), we get:

$\delta_n(a+c,b)_\alpha\ \beta m\beta[a+c,b]_\alpha = 0$

$\delta_n(a,b)_\alpha\ \beta m\beta[a,b]_\alpha + \delta_n(a,b)_\alpha\ \beta m\beta[c,b]_\alpha + \delta_n(c,b)_\alpha\ \beta m\beta[a,b]_\alpha$
$+ \delta_n(c,b)_\alpha\ \beta m\beta[c,b]_\alpha =0$

By Lemma 3.2 (i), we get:

$\delta_n(a,b)_\alpha\ \beta m\beta[a,b]_\alpha = \delta_n(c,b)_\alpha\ \beta m\beta[c,b]_\alpha=0$

Hence, $\delta_n(a,b)_\alpha\ \beta m\beta[c,b]_\alpha + \delta_n(c,b)_\alpha\ \beta m\beta[a,b]_\alpha=0$

Therefore, we get:

$\delta_n(a,b)_\alpha\ \beta m\beta[c,b]_\alpha\beta m\beta\psi_n(a,b)_\alpha\beta m\beta[c,b]_\alpha$
$= -\delta_n(a,b)_\alpha\ \beta m\beta[c,b]_\alpha\beta m\beta\psi_n(c,b)_\alpha\beta m\beta[a,b]_\alpha= 0$

Hence, by the primness of X, we have:

$\delta_n(a,b)_\alpha\ \beta m\beta[c,b]_\alpha=0$ ... (1)

Similarly, by replacing b+d for b in this equality, we get:

$\delta_n(a,b)_\alpha\ \beta m\beta[a,d]_\alpha=0$ ... (2)

Thus: $\delta_n(a,b)_\alpha\ \beta m\beta[a+c, b+d]_\alpha=0$

$\delta_n(a,b)_\alpha\beta m\beta[a,b]_\alpha+\psi_n(a,b)_\alpha\beta m\beta[a,d]_\alpha+\delta_n(a,b)_\alpha\beta m\beta[c,b]_\alpha$
$+\psi_n(a,b)_\alpha\beta m\beta[c,d]_\alpha=0$

By (1), (2) and Lemma 3.2 (i), we get:

$\delta_n(a,b)_\alpha\ \beta m\beta[c,d]_\alpha=0$

ii) By replacing a+c for a in Lemma 3.2 (ii), we get:

$\delta_n(a+c,b)_\alpha\ \alpha m\alpha[a+c,b]_\alpha = 0$

$\delta_n(a,b)_\alpha\ \alpha m\alpha\ [a,b]_\alpha + \delta_n(a,b)_\alpha\ \alpha m\alpha\ [c,b]_\alpha + \delta_n(c,b)_\alpha\ \alpha m\alpha[a,b]_\alpha$
$+ \delta_n(c,b)_\alpha\ \alpha m\alpha\ [c,b]_\alpha =0$

By Lemma 3.2 (ii), we get: $\delta_n(a,b)_\alpha\ \alpha m\alpha[a,b]_\alpha = \delta_n(c,b)_\alpha\ \alpha m\alpha[c,b]_\alpha=0$

hence $\psi_n(a,b)_\alpha\ \alpha m\alpha[c,b]_\alpha+ \psi_n(c,b)_\alpha\ \alpha m\alpha[a,b]_\alpha=0$.

Therefore, we get:

$\delta_n(a,b)_\alpha\ \alpha m\alpha[c,b]_\alpha\alpha m\alpha\psi_n(a,b)_\alpha\alpha m\alpha[c,b]_\alpha$
$= - \delta_n(a,b)_\alpha\ \alpha m\alpha[c,b]_\alpha\alpha m\alpha\psi_n(c,b)_\alpha\alpha m\alpha[a,b]_\alpha=0$

By primness of X, we have:

$\delta_n(a,b)_\alpha\ \alpha m\alpha[c,b]_\alpha=0$ ...(1)

Similarly, by replacing b+d for b in this equality, we get:

$\delta_n(a,b)_\alpha\ \alpha m\alpha[a,d]_\alpha=0$ ...(2)

Thus:

$\delta_n(a,b)_\alpha\ \alpha m\alpha\ [a+c, b+d]_\alpha=0$

$\delta_n(a,b)_\alpha\ \alpha m\alpha[a,b]_\alpha+ \delta_n(a,b)_\alpha\ \alpha m\alpha[a,d]_\alpha + \delta_n(a,b)_\alpha\ \alpha m\alpha[c,b]_\alpha + \delta_n(a,b)_\alpha\ \alpha m\alpha[c,d]_\alpha= 0$

By (1), (2) and lemma 3.2(ii), we get: $\delta_n(a,b)_\alpha\ \alpha m\alpha[c,d]_\alpha=0$

iii) Finally, by replacing α+β for α in (ii), we get:

$\delta_n(a,b)_{\alpha+\beta}\ \alpha m\alpha[c,d]_{\alpha+\beta} =0$

$\delta_n(a,b)_\alpha\ \alpha m\alpha[c,d]_\alpha + \delta_n(a,b)_\alpha\ \alpha m\alpha[c,d]_\beta + \delta_n(a,b)_\beta\ \alpha m\alpha[c,d]_\alpha$

$+ \delta_n(a,b)_\beta \alpha m\alpha[c,d]_\beta = 0$

By (i) and (ii), we get:

$\delta_n(a,b)_\alpha \alpha m\alpha[c,d]_\beta + \delta_n(a,b)_\beta \alpha m\alpha[c,d]_\alpha = 0$

Therefore, we have:

$\delta_n(a,b)_\alpha \alpha m\alpha[c,d]_\beta \alpha m\alpha\ \psi_n(a,b)_\alpha \alpha m\alpha[c,d]_\beta$

$= - \delta_n(a,b)_\alpha \alpha m\alpha[c,d]_\beta \alpha m\alpha\ \psi_n(a,b)_\beta \alpha m\alpha[c,d]_\alpha = 0$

Hence, by the primness of X, we have:

$\delta_n(a,b)_\alpha \alpha m\alpha[c,d]_\beta = 0$

Theorem 3.4

Every Jordan generalized higher derivation of a Γ-ring M into a 2-torsion free prime ΓM-module X is a higher derivation of M into X.

Proof

Let $F=(f_i)_{i\in N}$ be a Jordan higher derivation of a Γ-ring M into a 2-torsion free prime ΓM-module X. Since X is a prime, we get from Theorem 3.3 (i) that either $\delta_n(a,b)_\alpha=0$ or $[c, d]_\alpha=0$ for all $a,b,c,d\in M$, $\alpha\in\Gamma$, and $n\in N$.

If $[c,d]_\alpha\neq 0$ for all $c,d\in M$ and $\alpha\in\Gamma$. Then $\delta_n(a,b)_\alpha=0$ for all $a,b\in M$, $\alpha\in\Gamma$ and $n\in N$. Hence, by Remark 2.7 we get that F is a generalized higher derivation of M into X.

But, if $[c,d]_\alpha=0$ for all $c,d\in M$ and $\alpha\in\Gamma$, then M is commutative and, therefore, we have from lemma 2.4(i):

$$f_n(2a\alpha b) = 2\sum_{i+j+l=n} f_i(a)k_j(\alpha)d_l(b)$$

Since X is a 2-torsion free, we obtain that F is a generalized higher derivation of M into X.

Proposition 3.5

Every Jordan generalized higher derivation of a Γ-ring M into a 2-torsion free ΓM-module X, such that $a\alpha b\beta c=a\beta b\alpha c$, for all $a,b,c\in M$ and $\alpha,\beta\in\Gamma$, is a Jordan generalized triple higher derivation of M into X.

Proof

Let $F=(f_i)_{i\in N}$ be a Jordan generalized higher derivation of M into X.

By replacing b by $a\beta b+b\beta a$ in Lemma 2.4 (i), we get:

$$f_n(a\alpha(a\beta b+b\beta a)+(a\beta b+b\beta a)\alpha a)$$

$$=\sum_{i+l=n} f_i(a)\alpha d_l(a\beta b+b\beta a)+ f_i(a\beta b+b\beta a)\alpha d_l(a)$$

$$=\sum_{i+l=n} f_i(a)\alpha\left(\sum_{r+s+t=l} d_r(a)\beta d_t(b)+ d_r(b)\beta d_t(a)\right)$$

$$+\left(\sum_{p+h=i} f_p(a)\beta d_h(b)+ d_p(b)\beta d_h(a)\right)\alpha d_l(a)$$

$$=\sum_{i+l=n}\ \sum_{r+t=l} f_i(a)\alpha d_r(a)\beta d_t(b)+ d_i(a)\alpha d_r(b)\beta d_t(a)$$

$$+\sum_{i+l=n}\ \sum_{p+h=i} f_p(a)\beta d_h(b)\alpha d_l(a)+d_p(b)\beta d_h(a)\alpha d_l(a)$$

$$=\sum_{i+r+t=n} f_i(a)\alpha d_r(a)\beta d_t(b)+d_i(a)\alpha d_r(b)\beta d_t(a) + f_i(a)\beta d_r(b)\alpha d_t(a)+d_i(b)\beta d_r(a)\alpha d_t(a) \quad ...(1)$$

On the other hand:

$f_n(a\alpha(a\beta b+b\beta a)+(a\beta b+b\beta a)\alpha a)=f_n(a\alpha a\beta b+a\alpha b\beta a+a\beta b\alpha a+b\beta a\alpha a)$

$$= \sum_{i+r+t=n} f_i(a)\alpha d_r(a)\beta d_t(b)+f_i(b)\beta d_r(a)\alpha d_t(a)+f_n(a\alpha b\beta a+a\beta b\alpha a) \quad \dots (2)$$

By comparing (1) and (2), and since $a\alpha b\beta c=a\beta b\alpha c$ for all $a,b,c\in M$ and $\alpha,\beta\in\Gamma$, we get:

$2f_n(a\alpha b\beta a)$

$$= 2\sum_{i+r+t=n} f_i(a)\alpha d_r(b)\beta d_t(a)$$

Since X is a 2-torsion free, we have: $f_n(a\alpha b\beta a)$

$$= \sum_{i+r+t=n} f_i(a)\alpha d_r(b)\beta d_t(a)$$

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