



ISSN: 0067-2904

Fuzzy Maximal Sub-Modules

Maysoun A.Hamel*, Hatam Y. Khalaf

Department of Mathematics, Ibn-Al-Haitham College of Education, University of Baghdad, Baghdad, Iraq

Received: 24/11/ 2019

Accepted: 21/1/2020

Abstract

In this paper, we introduce and study the notions of fuzzy quotient module, fuzzy (simple, semisimple) module and fuzzy maximal submodule.

Also, we give many basic properties about these notions.

Keywords: (Simple, Semisimple) Fuzzy Modules, Quotient Fuzzy Modules And Maximal Fuzzy Submodule.

المقاسات الجزئية العظمى الضبابية

ميسون علي هامل*، حاتم يحيى خلف

قسم الرياضيات، كلية التربية (ابن الهيثم)، جامعة بغداد، بغداد، العراق

الخلاصة

في هذا البحث قدمنا ودرسنا المفاهيم مقاس القسمة الضبابي ، المقاس البسيط (وشبه البسيط) الضبابي والمقاس الجزئي الاعظم الضبابي . وكذلك اعطينا العديد من الخواص الاساسية حول المفاهيم

Introduction

Recall that an R -module M is called simple if and only if it has no proper non trivial submodules [1]. M is called semisimple if and only if it is a sum of simple submodules of M . While, a submodule N of an R -module M is called maximal if and only if there is no proper submodule M that is different from N containing N properly [1]. Also, a semi $-T$ - maximal submodule introduced as follows [2]: A submodule K of R -module M is called semi $-T$ - maximal if $T + K/K$ is a semisimple R -module M (where T is a submodule of M).

Hamel [3] introduced the definition of fuzzy simple and fuzzy semisimple modules. Some properties of these concepts which are useful in the next sections are given, while we add many other results.

A fuzzy module X is called simple if X has no nontrivial fuzzy submodules .

In other words , X is simple if whenever $A \leq X$, either $A = X$ or $A = 0$. Moreover , let $A \leq X$, A is a fuzzy simple submodule of X if A is a fuzzy simple module .

Also, we define and introduce quotient fuzzy modules and fuzzy maximal submodules as a generalization of maximal submodules and give some properties of these concepts.

This paper consists of three sections . In section 1, we recall many definitions and properties which are needed in our work. In section 2, various basic properties about fuzzy simple (semisimple) modules are discussed. In section 3, we study the behavior of fuzzy maximal submodules .

1. Preliminaries

This section contains some definitions and properties of fuzzy sets , fuzzy modules , and quotient fuzzy modules.

*Email: mathmaysoon@gmail.com

Definition 1.1 [4]

Let S be a non-empty set and I be the closed interval $[0,1]$ of the real line (real numbers). A fuzzy set A in S (a fuzzy subset of S) is a function from S into I .

Definition 1.2 [4]

A fuzzy set A of a set S is called a fuzzy constant if $A(x) = t$, for all $x \in S$, where $t \in [0,1]$.

Definition 1.3[5]

Let $x_t : S \rightarrow [0,1]$ be a fuzzy set in S , where $x \in S, t \in [0, 1]$ defined by:

$$x_t(y) = \begin{cases} t & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

for all $y \in S$. x_t is called a fuzzy singleton or fuzzy point in S .

Definition 1.4[5]

Let A and B be two fuzzy sets in S , then:

- 1- $A = B$ if and only if $A(x) = B(x)$, for all $x \in S$.
- 2- $A \leq B$ if and only if $A(x) \leq B(x)$, for all $x \in S$.

If $A < B$ and there exists $x \in S$ such that $A(x) < B(x)$, then A is a proper fuzzy subset of B and written as $A < B$.

By part (2), we can deduce that $x_t \leq A$ if and only if $A(x) \geq t$.

Definition 1.5 [5],[6]

Let M be an R -module. A fuzzy set X of M is called fuzzy module of an R -module M if :

- 1- $X(x - y) \geq \min\{X(x), X(y)\}$, for all $x, y \in M$
- 2- $X(rx) \geq X(x)$ for all $x \in M$ and $r \in R$.
- 3- $X(0) = 1$

Definition 1.6 [6]

Let X and A be two fuzzy modules of R -module M . A is called a fuzzy submodule of X if $A \leq X$.

Definition 1.7 [6]

If A is a fuzzy module of an R -module M , then the submodule A_t of M is called the level submodule of M where $t \in [0,1]$.

Proposition 1.8 [6]

Let A be a fuzzy set of an R -module M . Then the level subset $A_t, t \in [0,1]$ is a submodule of M if and only if A is a fuzzy submodule of X where X is a fuzzy module of an R -module M . Now, we shall give some properties of fuzzy submodules, which are used in the next section.

Proposition 1.9[7]

Let A be a fuzzy module in M , then we define $A_* = A_{A(0_M)} = \{x \in M, A(x) = 1\} = A(0_M) = 1$.

Proposition 1.10 [8]

Let A be a fuzzy module of an R -module in M , then A_* is a submodule of M .

We add the following results.

Proposition 1.11

Let X be a fuzzy module of an R -module M and $A \leq X$ such that $X(0) = 1$, then $A(0) = 1$.

Proof: It is clear by the definition of fuzzy modules.

Remark 1.12

If A and B are fuzzy modules of a fuzzy module X such that $A \leq B$, then $A_* \leq B_*$.

Proof:

Let $x \in A_*$, then $A(x) = A(0)$.

But $B(x) \geq A(x)$

, $\forall x \in M$, hence $B(x) \geq A(x) = A(0) = B(0)$. Thus $x \in B_{B(0)} = B_*$.

Remark 1.13

The converse of the above Remark is not true in general as the following example shows :

Let $X: Z \rightarrow [0,1]$ defined by : $X(x) = 1, \forall x \in Z$.

$$\text{Let } A(x) = \begin{cases} 1 & \text{if } x \in 2Z, \\ 0.9 & \text{otherwise} \end{cases}$$

$$B(x) = \begin{cases} 1 & \text{if } x \in 4Z, \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

It is clear that A and B are fuzzy submodules of X and that $A_* = 2Z = B_* = 4Z$.
Hence $A_* \leq B_*$. But $A \neq B$, since $A(3) = 0.9, B(3) = 1/2 = 0.5$.

Remark

We assume that if $A_* = B_*$, then $A = B$ is called a **Condition (*)**.

Lemma 1.14 [9]

Let A be a fuzzy submodule of fuzzy module X, $(A: X)_t \leq (A_t: X_t)$, For all $t \in (0,1]$. Also, we can prove that by Lemma 2.3.3.[8].

It follows that if, $X = A \oplus B$, for $A, B \leq X$, then $X_* = (A \oplus B)_* = A_* \oplus B_*$.

Remark 1.15[9]

It is not necessarily that $(A: X)_t = (A_t: X_t)$ for any fuzzy submodule A of a fuzzy module X.

Proposition 1.16[9]

If A is a prime fuzzy submodule of fuzzy X, then $(A: X)_t = (A_t: X_t)$, for all $t \in (0,1]$ where $A_t \neq X_t$.

Recall that a submodule N of R-module M is called an essential if $N \cap H = 0$.

For nontrivial submodule H of M.[10],[11], [12].

Rabi [9] introduced the following definition : Let X be a fuzzy module of an R-module M. A fuzzy submodule A of X is called an essential if $A \cap B = 0_1$, for nontrivial fuzzy submodule B of X .

Now we add the following definition.

Definition 1.17

Let X be a fuzzy module of an R – module M and let A be fuzzy submodule of X . Define $X/A: M/A_* \rightarrow [0,1]$ by :

$$X/A(a + A_*) = \begin{cases} 1 & \text{if } a \in A_* \\ \sup\{X(a + b) & \text{if } b \in A_*, a \notin A_* \end{cases}$$

for all coset $a + A_* \in M/A_*$.

Proposition 1.18

If X is a fuzzy module of an Rmodule M and A is a fuzzy submodule of X, then X/A is a fuzzy module of M/A_* .

Proof:

By definition 1.17, X/A is a nonempty fuzzy subset of M/A_* .

Firstly, we must prove that X/A is well defined.

Let $a_1 + A_*, a_2 + A_* \in X_*/A_*$, if $a_1 + A_* = a_2 + A_*$ implies that $a_1 - a_2 \in A_*$.

Therefore $a_1 - a_2 = b_1, b_1 \in A_*$, implies that $a_1 = a_2 + b_1 \in A_*, b' = b_1 + b' = X/A(a_2 + A_*)$.

Hence X/A is well defined .

Now, we must prove that X/A is a fuzzy module of M/A_* .

1) If $a_1, a_2 \notin A_*$, then either $a_1 - a_2 \in A_*$ or $a_1 - a_2 \notin A_*$ and either $a_1, a_2 \in A$ or $a_1, a_2 \notin A$.

Thus, if $a_1 - a_2 \in A_*$, then, $X/A(a_1 - a_2 + A_*) = 1 \geq \min\{X/A(a_1 + A_*), X/A(a_2 + A_*)\}$

If $a_1 - a_2 \notin A_*$, then

$$\begin{aligned} X/A(a_1 - a_2 + A_*) &= \sup\{X(m), m \in a_1 - a_2 + A_*\} = \sup\{X(m_1 - m_2), \\ & \quad m_1 \in a_1 + A_*, m_2 \in a_2 + A_*\} \\ &\geq \sup\{\min\{X(m_1), X(m_2)\} \mid m_1 \in a_1 + A_*, m_2 \in a_2 + A_*\} \\ &\geq \min\{\sup\{X(m_1) \mid m_1 \in a_1 + A_*, \sup\{X(m_2) \mid m_2 \in a_2 + A_*\}\} \\ & \quad = \min\{\sup\{X(a_1 + n_1) \mid n_1 \in A_*\} \sup\{X(a_2 + n_2), n_2 \in A_*\}\} \\ &= \min\{X/A(a_1 + A_*), X/A(a_2 + A_*)\} \end{aligned}$$

Hence, $X/A(a_1 - a_2 + A_*) \geq \min\{X/A(a_1 + A_*), X/A(a_2 + A_*)\}$.

2– To prove that $X/A(rx + A_*) \geq X/A(x + A_*)$, for $r \in R$ and $x \in M$.

if $rx \in A_*$, then $X/A(rx + A_*) = 1$, which implies that $X/A(rx + A_*) \geq X/A(x + A_*)$.

If $rx \notin A_*$ then $x \notin A_*$. Hence, $X/A(rx + A_*) = \sup\{X/A(rx + b) : b \in A_*\}$ and

$X/A(x + A_*) = \sup\{X/A(x + b) : b \in A_*\}$.

$$\begin{aligned} & \text{But, } \sup\{X(rx + b) : b \in A_*\} \geq \sup\{X(rx + rb) : b \in A_*\} \\ & = \sup\{X[r(x + b)] : b \in A_*\} \\ & \geq \sup\{X[(x + b)] : b \in A_*\} \\ & = X|A(x + A_*) \end{aligned}$$

Thus, $\{\sup\{X(rx + b) : b \in A_*\} \geq X|A(x + A_*)\}$

That is $X/A(rx + A_*) \geq X/A(x + A_*)$

3) To prove that $(0 + A_*) = 1, [0 + A_*] = A_*$,

$$\begin{aligned} X/A(a + A_*) &= \sup\{X(a + b) \mid \begin{matrix} 1 & \text{if } a \in A_* \\ \text{if } b \in A_*, a \notin A_* \end{matrix}\} \\ &= \begin{cases} 1 & \text{if } a \in A_* \\ \sup\{X(a + b) \mid \text{if } b \in A_*, a \notin A_* \end{cases} \end{aligned}$$

If $a \in A_*$, we get

$$X/A(a + A_*) = 1, \text{ but } 0 \in A_*, \text{ then } X/A(0 + A_*) = 1.$$

Thus X/A is called a quotient fuzzy module.

However, in a previous work [10] there exists a definition of quotient fuzzy module as follows, which is an equivalent to Definition 3.1 where $X(0) = 1$.

Proposition 1.20[10]

Let X be a fuzzy module of an R -module M and A be a fuzzy submodule of X . Define $X/A: M/A_* \rightarrow [0,1]$, such that

$$X/A(a + A_*) = \sup\{X(a + b), a \in M, b \in A_*\}.$$

Lemma 1.21

Let A be fuzzy submodule of fuzzy module X , then $X_*/A_* \leq (X/A)_*$.

Proof :

Let $y' \in X_*/A_*$, hence $y' = x + A_*$, for $x \in X_*$, which implies that $X(x) = 1$.

Now

$$\begin{aligned} X/A(y') &= X/A(x + A_*) = \sup\{X(x + a), a \in A_*\} \\ &\geq \sup\{\min\{X(x), A(a)\}\} \\ &= \sup\{\min\{1, 1\}\} = 1 = \sup\{1\} = 1 \end{aligned}$$

Hence $X/A(y') = 1$ so, $y' \in (X/A)_*$. Hence $X_*/A_* \leq (X/A)_*$.

Proposition 1.22

Let X be a fuzzy module of an R -module M , such that $X(x) = 1, \forall x \in M$. Then, $(X/A)_* = X_*/A_*$, for each $A \leq X$

Proof:

$$X_*/A_* \leq (X/A)_*, \text{ by Lemma 1.21.}$$

Now $X_* = M$ and $X/A: M/A_* \rightarrow [0,1]$, so for each $a + A_* \in M + A_*$.

$$\frac{X}{A}(a + A_*) = \sup\{B(a + b) : a \notin A_*, b \in A_*\}$$

Since $X(a + b) \geq \min\{X(x), X(b)\} = 1$, hence $X/A(a + A_*) = 1$.

Proposition 1.23

Let A, B be fuzzy submodules of fuzzy module X . Then $X/A \cap B \approx$ Fuzzy submodules of $(X/A \oplus X/B)$.

Proof:

Define $f: X \rightarrow X|A \oplus X|B$ by $f(x_t + A, x_t + B)$, for each $x_t \in X$. Then f is a homomorphism.

$$\begin{aligned} \ker f &= \{x_t : x_t \in X \text{ and } f(x_t) = 0_1\} \\ &= \{x_t : x_t \in X \text{ and } f(x_t + A, x_t + B) = (0_1, 0_1)\} \\ &= \{x_t : x_t \in X \text{ and } x_t \in A \cap B\} = A \cap B. \end{aligned}$$

Thus by First Fundamental theorem

$$X/\ker f \approx \text{Im } f \approx \text{fuzzy submodules of } (X|A \oplus X|B)$$

S.2 Fuzzy simple (semisimple) modules

Recall that an R -module M is called simple if and only if it has no proper non trivial submodules and that M is called semisimple if and only if M is a sum of simple submodules of M [1].

Hamel [3] introduced the definition of fuzzy simple

modules and fuzzy semisimple modules . Some properties of these concepts which are useful in next the sections are given, while we add many other results .

Definition 2.1

A fuzzy module X is called simple if X has no nontrivial fuzzy submodules .

In other words , X is simple if whenever $A \leq X$,either $A = X$ or $A = 0_1$

Moreover, let $A \leq X$, A is a fuzzy simple submodule of X if A is a fuzzy simply module .

Remarks 2.2 [3]

Let X be a fuzzy module , then the followings hold:

1) Every simple fuzzy module is F_- regular fuzzy module ,where F_- regular is every fuzzy submodule of X is pure .

2) If X is a simple fuzzy module, then X_t is a simple module , $\forall t(0,1]$.

3) If X_t is a simple module , $\forall t \in (0,1]$, then is not necessarily that X is a simple fuzzy module.

Proposition 2.3

Let X be a fuzzy module of an R -module M and A be a fuzzy submodule of X . If A is simple , then A_* is a simple submodule in X_*

Proof :

Suppose that A_* is not simple , then there exists submodule $N < A_*$.

Define $B: M \rightarrow [0,1]$ defined by $B(x) = \begin{cases} A(x) & , x \in N \\ 0 & \text{otherwise} \end{cases}$

It is clear that $B < A$, but A is a simple fuzzy submodule , which is a contradiction . Thus A_* is a simple submodule.

Remark 2.4

The converse of proposition 2.3 is not true in general , as the following example shows:

Let $M = Z_6$ as Z - module . Define $X: M \rightarrow [0,1]$ such that

$$X(x) = \begin{cases} 1 & \text{if } x \in \langle \bar{2} \rangle \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

It is clear that X is fuzzy module . Also, let

$$A(x) = \begin{cases} 1 & \text{if } x \in \langle \bar{2} \rangle \\ 1/2 & \text{otherwise} \end{cases}$$

where A is a fuzzy submodule of X and

$A_* = \langle \bar{2} \rangle < \bar{2} \rangle$ is a simple submodule in Z_6 . But A is not simple since there exists fuzzy submodule B of A such that :

$$B: M \rightarrow [0,1] \text{ defined by: } B(x) = \begin{cases} 1 & \text{if } x \in \langle \bar{2} \rangle \\ 0 & \text{otherwise} \end{cases}$$

Thus, A is not a simple fuzzy submodule of X .

Definition 2.5 [3]

A fuzzy module X is called semisimple if X is the sum of simple fuzzy submodules of X .

Next , we need the following lemma.

Lemma 2.6

Let X be a fuzzy module of an R - module M and let A be a fuzzy direct summand in X ,then A_* is a direct summand in X_* .

Proof:

As A is a fuzzy direct summand of X , then $A \oplus B = X$, for some $B \leq X$.

Then $A_* \oplus B_* = X_*$.And so $A_* \oplus B_* = X_*$ by Lemma1.14. Therefore, A_* is a direct summand of X_* .

Proposition 2.7

Let X be a fuzzy module of an R - module M . If X is a fuzzy semisimple module , then X_* is a semisimple module .

Proof:

Let $N \leq X_*$, and let $A : M \rightarrow [0,1]$, defined by:

$$A(x) = \begin{cases} 1 & \text{if } x \in N, \\ 0 & \text{otherwise} \end{cases}$$

Hence $N = A_*$. As $A < X$,hence A is a direct summand of X .

Then A_* is a direct summand of X_* .

So N is a direct summand of X_* . Thus, X_* is semisimple.

Proposition 2.8

Let X be a fuzzy module of an R -module M , if X_* is semisimple, then X is semisimple when Condition (*) holds.

Proof:

Let $A \leq X$, then $A_* \leq X_*$. But X_* is semisimple, so $A_* \oplus K = X_*$, for some $K \leq X_*$. By putting:

$$B(x) = \begin{cases} 1 & \text{if } x \in K \\ 0 & \text{otherwise} \end{cases}$$

then $B_* = K$.

Thus, $A_* \oplus B_* = X_*$,

so $A \oplus B = X$ by Condition (*) and A is a direct summand of X .

Therefore, X is a semisimple fuzzy module.

Corollary 2.9

Any fuzzy submodule of semisimple fuzzy module is semisimple.

Proof:

Let X be a fuzzy semisimple module and let $A \leq X$. Then X_* is semisimple module by Proposition 2.8. But $A_* \leq X_*$, so A_* is semisimple submodule. Thus A is a fuzzy semisimple submodule of X by Prop 2.8.

Proposition 2.10

Let X be a fuzzy module of an R -module M , where Condition (*) holds. Then the following statements are equivalent:

- 1- X is semisimple
- 2 - X has no proper essential fuzzy submodule.
- 3- Every fuzzy submodule of X is a direct summand of X .

Proof: (1) \rightarrow (2)

Since X is semisimple, then X_* is semisimple by prop 2.7, and X_* has no proper essential submodule [11]. Now suppose that X has

a fuzzy proper essential submodule, say A . Then A_* is essential submodule in X_* . But X_* has no proper essential submodule, so that $A_* =$

X_* . Then, by condition (*), $A = X$, which is a contradiction.

(2) \rightarrow (3)

Let $A \leq X$ then $A_* \leq X_*$, but X is a fuzzy semisimple module, which implies that X_* is a semisimple module, so that $A \oplus N = X$, for some $N \leq X$.

Let $B: M \rightarrow [0,1]$ defined by: $B(x) = \begin{cases} 1 & \text{if } x \in N \\ 0 & \text{otherwise} \end{cases}$

Then, $B_* = N$ and $B \leq X$. Hence $A_* \oplus B_* = X_*$, which implies that

$$(A \oplus B)_* = X_*$$

, so $A \oplus B = X$, for some $B < X$. Then, $A \cap B = 0_1$ and hence

A is not essential in X .

(3) \rightarrow (1) If every fuzzy submodule in X is a fuzzy direct summand, then every submodule of X_* is a direct summand, by Lemma 2.7.

Hence X_* is semisimple and, so, X is semisimple.

(3) \rightarrow (1)

If every fuzzy submodule in X is a fuzzy direct summand, then every submodule of X_* is a direct summand, by prop 2.6.

Hence X_* is semisimple and, so, X is semisimple by prop 2.7.

Proposition 2.11

Let X be a fuzzy module of an R -module M . Then the followings are equivalent:

1. Every fuzzy submodule of X is a sum of fuzzy simple submodules.
2. X is a direct sum of fuzzy simple submodules of X .
3. Every fuzzy submodule of X is a direct summand of X .

Proof: It is easy.

Proposition 2.1 2

The following are equivalent:

1. Every fuzzy submodule of semisimple fuzzy module is semisimple.
2. Every epimorphic image of semisimple fuzzy module is semisimple .
3. Every sum of semisimple fuzzy modules is semisimple .

Proof: It is easy.

S. 3 Fuzzy maximal submodules

In this section, we introduce the concept of a fuzzy maximal submodule as a generalization of ordinary concept maximal submodules. Some characterizations of fuzzy maximal submodules are introduced.

Definition 3.1

Let A be a fuzzy submodule of fuzzy module X , A is called fuzzy maximal if X/A is a simple fuzzy module

Proposition 3.2

Let A be fuzzy submodule of fuzzy module X , A is a fuzzy maximal submodule if and only if whenever $A \leq B < X$, then $A = B$.

Proof:

Assume that $A \leq B < X$, then $B/A \leq X/A$. But, A is maximal, so X/A is simple. Hence, $B/A = 0_1$. That is $A = B$.

To prove that A is fuzzy maximal, we must show that X/A is simple. Suppose there exists $B/A < X/A$. Then $A \leq B < X$.

Hence, by assumption, $A = B$. That is, $B/A = 0_1$.

Therefore, X/A is simple.

Proposition 3.3

Let X be fuzzy module of an R -module M , and let A be a fuzzy submodule of X . If A is a fuzzy maximal submodule of X , then A_* is maximal submodule of X_* .

Proof:

If A is a fuzzy maximal submodule of X , then X/A is simple and hence $(X/A)_*$ is simple by Prop 1.3. But $X_*/A_* \leq (X/A)_*$ by Lemma 1.21.

That is $X_*/A_* = (X/A)_*$. Hence X_*/A_* is simple and, so, A_* is a maximal submodule of X_* .

Proposition 3.4

Let X be fuzzy module of an R -module M , and let A be a fuzzy submodule X . If A_* is a maximal submodule of X_* , then A is a fuzzy maximal submodule of X , provided that Condition (*) holds.

Proof:

Let A_* be a maximal submodule in X_* . To prove that A is a fuzzy maximal submodule in X , if $A \leq B < X$, where B is a fuzzy submodule of X , then

$A_* \leq B_* < X_*$. So that $A_* = B_*$ and, so, by condition(*), $A = B$.

Proposition 3.5

Let A be fuzzy submodule of fuzzy module X . If A is fuzzy maximal, then $A + \langle x_t \rangle = X$, for all $x_t \notin A$, provided that Condition (*) holds.

Proof:

If $a_1 \notin A$, thus $A(a) < 1$. Therefore, $a \notin A_*$. But A is a fuzzy maximal submodule of X . Hence A_* is maximal submodule of X_* by prop.2.3. Then $A_* + \langle a \rangle = X_*$ [11].

On the other hand,

$$\langle a_1 \rangle(x) = \begin{cases} 1 & \text{if } x \in \langle a \rangle \\ 0 & \text{otherwise.} \end{cases}$$

Hence $(\langle a_1 \rangle)_* = \langle a \rangle$.

Hence $(A + \langle a_1 \rangle)_* = X_*$. Thus $(A + \langle a_1 \rangle) = X$

Since $a_1 \notin A$, then $(A + \langle a_1 \rangle) \geq A$ and A is a fuzzy maximal submodule of X , so $(A + \langle a_1 \rangle) = X$.

Corollary 3.6

Let X be a fuzzy module of an R -module, and let $X(x) = 1, \forall x \in M$. Let A be a fuzzy submodule of X ,. If K is a maximal submodule in X_* , then there exists a fuzzy submodule A in X such that $A_* = K$.

Proof:

Let K be a maximal submodule in X_* . Let $A: M \rightarrow [0,1]$, defined by:

$$A(x) = \begin{cases} 1 & \text{if } x \in K \\ 0 & \text{otherwise} \end{cases}$$

It is clear that A is a fuzzy submodule of X and $A_* = K$. To prove that A is a maximal fuzzy submodule in X ,

let $B \leq X$, and $A \leq B \leq X$. Then $A_* \leq B_* \leq X_*$, which implies that $K \leq B_* \leq M$. Hence $B_* = K$, since K is a maximal in X_* . Thus, $A_* = B_*$ and, so, $A = B$. Thus A is a fuzzy maximal submodule of X .

Proposition 3.7

If A and B are fuzzy maximal submodules of fuzzy module X , then $A + B = X$.

Proof:

Since $A + B > A$ and A is a fuzzy maximal submodule and since $A + B > A$ and A is a fuzzy maximal submodule of X , then $A + B = X$.

Proposition 3.8

Let $f: X \rightarrow Y$ be fuzzy epimorphism and $\ker f$ is a fuzzy maximal submodule in X , then Y is simple.

Proof:

Since f is epimorphism and $X/\ker f \cong Y$, by the first fundamental theorem, thus Y is simple.

Proposition 3.9

If A is fuzzy maximal submodule of fuzzy X , then A is a prime fuzzy submodule of X .

Proof:

Let r_t be a fuzzy singleton of R and x_s be a fuzzy singleton of X such that $r_t x_s \in A$, and assume that $x_s \notin A$.

Then $\langle x_s + A \rangle$

$$\geq A \quad A. \text{ But } A \text{ is a fuzzy maximal submodule}$$

So X/A is simple.

On the other hand, $\langle x_s + A \rangle \leq X/A$, so that $\langle x_s + A \rangle = X/A$.

Since $r_t \cdot \langle x_s + A \rangle = r_t \cdot X/A$,

hence $\langle r_t x_s + A \rangle = r_t \cdot X/A$, which implies that $r_t X + A/A = 0_{X/A}$. Then $X + A = A$, so $r_t X \leq A$. Therefore, $r_t \in (A: X)$.

Remark 3.10

The converse of proposition 3.8 is not true in general. The following example shows that:

Example

Let $X: Z \rightarrow [0,1]$ defined by $X(x) = 1, \forall x \in Z$ and $0_1(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$

0_1 is a prime fuzzy submodule of X , but is not a fuzzy maximal submodule since there exists a fuzzy submodule A of X , such that:

$$A(x) = \begin{cases} 1 & \text{if } x \in 2Z \\ 0 & \text{otherwise} \end{cases}$$

Remark 3.11

The intersection of two fuzzy maximal submodules is not necessarily fuzzy maximal, as shown in the following example.

Example:

Let $M = Z$ as Z -module, and define $X: M \rightarrow [0,1]$ by: $X(x) = 1, \forall x \in M$.

$$\text{Let } A(x) = \begin{cases} 1 & \text{if } x \in \langle 2 \rangle \\ 0 & \text{otherwise} \end{cases}$$

$$B(x) = \begin{cases} 1 & \text{if } x \in \langle 3 \rangle \\ 0 & \text{otherwise} \end{cases}$$

A and B are fuzzy submodules of X and A and B are maximal of X since $A_* = \langle 2 \rangle$ and $B_* = \langle 3 \rangle$

which are maximal of $Z = X_*$
 However, $A_* \cap B_* = (A \cap B)_* = \langle 6 \rangle$ which is not maximal submodule of X_* .
 Thus $A \cap B = \begin{cases} 1 & \text{if } x \in \langle 6 \rangle \\ 0 & \text{otherwise} \end{cases}$
 $A \cap B$ is not maximal fuzzy submodule of X .

Proposition 3.12

If X is a finitely generated fuzzy module of an R -module M , then every proper fuzzy submodule of X is contained in a maximal fuzzy submodule of X .

Proof :

If X is a finitely generated fuzzy module, then X_t is finitely generated as explained in a previous work [13]. So, X_* is a finitely generated module. Hence every proper submodule is contained in a maximal submodule of X_* [14].

Let A be a fuzzy submodule of X . Then, $A_* \leq X_*$.
 Hence, $A_* \leq N < X_*$, where N is a maximal in X_* .

Define $B < X$, such that $B: M \rightarrow [0,1]$ by:

$$B(x) = \begin{cases} 1 & \text{if } x \in N \\ 0 & \text{otherwise} \end{cases}$$

Then $B_* = N$ and, so, $A_* \leq B_* < X_*$.

But by condition (*), $A < B \leq X$. Since $B_* = N$ is a maximal

Submodule and Condition (*) holds, then B is a fuzzy maximal submodule of X . Thus A is contained in a fuzzy maximal submodule of X .

References

1. Kasch, F. **1982**. *Modules and Rings*, Academic press.
2. Inaam M.A.Hahi, A. and Eelwi., A. **2019**. Semi - -maximal sumodules. *Iraqi Juornal of science*, **60**(12): 2725-2731.
3. Maysoun A. **2002**. F-regular fuzzy modules, M. Sc. Thesis, University of Baghdad.
4. Zadeh L. A. **1965**. Fuzzy Sets, *Information and control*, **8**: 338-353, 1965.
5. Zahedi M. M. **1992**. On L-Fuzzy Residual Quotient Modules and P. Primary Submodules, *Fuzzy Sets and Systems*, **51**: 33-344.
6. Zahedi M. M. **1991**. A characterization of L-Fuzzy Prime Ideals, *Fuzzy Sets and Systems*, **44**: 147-160.
7. Martinez L. **1996**. Fuzzy Modules Over Fuzzy Rings in Connection with Fuzzy Ideal of Ring, *J. Fuzzy Math.* **4**: 843-857.
8. Abd alguad Q. **1999**. Some Results On Fuzzy Modules .M . Sc Thesis University of Baghdad.
9. Rabi H. J. **2000**. Prime Fuzzy Submodule and Prime Fuzzy Modules, M. Sc. Thesis, University of Baghdad,).
10. Pan F. **1992**. "The Various Structures of Fuzzy Quotient Modules, *Fuzzy Sets and Systems*, **50**: 187-192.
11. Goodreal K.R. **1976**. *Ring Theory –Non Singular Rings and modules*, Marci-Dekker,New York and Basel.
12. Inaam, M.A., Hahi,F., Shyaa, D. and Shukur N. **2019**. Essentially Second modules, *Iraqi Juornal of science*, 2019. **60**: 1374-1380.
13. Inaam. M. Hadi, maysoun, A. and Hamil, A. **2011**. Cancellation and Weakly Cancellation Fuzzy Modules, *Journal of Basrah Reserchs((sciences))*. **37**(4.D).
14. Burton D.M. **1971**. *Abstract and Linear Algebra*, University of Hampshire