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# Wang-Ball Polynomials for the Numerical Solution of Singular Ordinary Differential Equations

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**Abstract**
This paper presents a new numerical method for the solution of ordinary differential equations (ODE). The linear second-order equations considered herein are solved using operational matrices of Wang-Ball Polynomials. By the improvement of the operational matrix, the singularity of the ODE is removed, hence ensuring that a solution is obtained. In order to show the employability of the method, several problems were considered. The results indicate that the method is suitable to obtain accurate solutions.

**Keywords:** Boundary value problems, Singular boundary value problems, Wang-Ball Polynomial, Matrix, and Differential Equation**.**

## Introduction

This study considers singular ordinary differential equations of the form

$$q(t)u''(t)+\frac{\gamma}{t}u'(t)p(t)+u(t)=r(t), \quad 0<t\le 1 \quad \text{and} \quad \gamma>0 \tag{1}$$

with either of the initial conditions (ICs)

$$y'(0)=\alpha_2, \quad y(1)=\beta \tag{2}$$

and either of the boundary conditions (BCs)

$$u(0)=\alpha_1, \quad u(1)=\beta \tag{3}$$

where $p(t)$, $q(t)$ and $r(t)$ are analytic in $t\in(0,1)$ and $\alpha_1,\alpha_2,\beta_1,\beta_2$ and $\gamma$ are finite constants. It is observed that problem (1) has singularity at the initial point $t = 0$. Hence, the main difficulty arises in the singularity of the equations at $t = 0$. Generally, singular features exhibited by differential equations of the form (1) at this point incurs difficulty when considering its numerical solution. The solution of singular ordinary differential equations has been explored by various studies. Specifically, the solution of singular initial value problems was considered [1] using the residual-power series method (RPSM) to obtain efficient analytical numerical solutions for a class of nonlinear systems of initial value problems with finitely many singularities. Likewise, a new algorithm was proposed to solve singular initial value problems of Emden Fowler type equations [2], while another work [3] presented a numerical method where the operational matrix with the Tau method is utilized to transform the differential equation into a system of algebraic equations. Another study [4] derived solutions with iterative methods, including Daftardar-Jafari Method (DJM), Adomian Decomposition Method (ADM), and Differential Transform Method (DTM).

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Solutions of singular boundary value problems, on the other hand, are evident in various studies, such as an earlier work [5] which solved the problem of non-linear singular boundary values by introducing the Variational Iteration Method (VIM). Also, the collocation and Galerkin approach was presented [6] and the solution of the singular Dirichlet type boundary value problem was explored using the SincGalerkin method [7]. Likewise, singular boundary value problems were deeply elaborated on producing kernel space [8, 9]. A method known as the parametric spline was proposed [10]. Numerous works have been tested in various regions; for example, mass exchange, fractional Maxwell fluid [11,12], and the Oldroyd-B fluid model [13].

Our method is different from the above described methods, since it employs Wang-Ball operational matrix of differentiation to obtain the solution of singular initial and boundary value problems. Wang-Ball polynomial is restricted to the interval $t \in [0, 1]$ which is pertinent in a generalized form, such as that of the Bezier curves [14, 15]. Many constructive properties are known to be associated with Wang- Ball polynomials, such as the recursive relations, positivity, symmetry, continuity, and the unity partition of set over intervals. Polynomials in their operational matrix form have been utilised to solve differential equations. A previous work [16] solved the Korteweg-de Vries (KdV) differential equation using the modified Bernstein polynomials, while a novel method was presented to solve high order linear differential equations with initial and boundary condition imposed [17]. Similar attempts have been made by several other studies [18-22]. These separate works covered Bernstein polynomial, Bernstein operational matrix of integration, and a combination of Ritz Galerkin method with Neumann and integral condition, all of which being dedicated for the solution of differential equations.

This article considers the solution of singular initial and boundary value problems using Wang-Ball operational matrix for differentiation. The proposed algorithm involves a transformation using an unknown coefficient of algebraic equations into the singular ODEs. MATLAB, Maple, and Mathematic software were utilized to obtain the required solutions when adopting the method to numerical examples. The subsequent sections of this article give more details on the method and its implementation. In Section 2, Wang-Ball Polynomial basic properties are reviewed and developed for operational matrix derivative, while Section 3 details the implementation of the proposed method for solving the singular initial and boundary value problems. Section 4 displays examples of numerical applications of the proposed algorithm, with the conclusion and discussion of results given in Section 5.

**Glance on Ball polynomial**

The Ball polynomial was introduced by A. A. Ball in his well-known aircraft design system CONSURF [23]. It is described as a cubic polynomial and defined mathematically as:

$$(1-t)^2, 2t(1-t)^2, 2t^2(1-t), t^2, \quad 0 \le t \le 1. \tag{4}$$

In further research, several studies discussed Ball polynomial’s high generalization and its properties. For instance, two different Ball polynomials of arbitrary degree called Said-Ball and Wang-Ball, were reported in the 1980s [24], while another generalization of Ball polynomial called DP-Ball was reported.

**Wang-Ball Polynomial Representation**

Wang-Ball polynomial $W_i^m(t)$ of degree $m$ can be defined by [24]:

$$W_i^m(t)=\begin{cases}(1-t)^{2+i}(2t)^i & ,0\le i\le \frac{m-3}{2}\\ (1-t)^{\frac{1+m}{2}}(2t)^{\frac{m-1}{2}} & ,i=\frac{m-1}{2}\\ (2(1-t))^{\frac{m-1}{2}}t^{\frac{1+m}{2}} & ,i=\frac{1+m}{2}\\ (2(1-t)t)^{m-i}t^{m+2-i} & ,\frac{m+3}{2}\le i\le m\end{cases} \quad (5)$$

when m is odd, and

$$W_i^m(t)=\begin{cases}(1-t)^{2+i}(2t)^i & ,0\le i\le \frac{m}{2}-1\\ (2t(1-t))^{\frac{m}{2}} & ,i=\frac{m}{2}\\ (2(1-t))^{m-i}t^{m+2-i} & ,\frac{m+3}{2}\le i\le m\end{cases} \quad (6)$$

when m is even.

**Definition of Wang-Ball Monomial Form**

A Wang-Ball curve of degree m denoted by $A_m(t)$ is given together with $m+1$ control points, denoted by $\{w_i\}_{i=0}^m$. The degree m Wang-Ball $W_i^m(t)$ is given in the form of power basis

$$W_i^m(t)=\sum_{k=0}^m\sum_{l=0}^m w_{k,l}t^l,0\le t\le 1 \quad (7)$$

where

$$w_{kl}=\begin{cases}(-1)^{(l-k)}2^k\binom{k+2}{l-k}, & \text{for } 0\le k\le \left\lfloor \frac{m}{2}\right\rfloor-1,\\ (-1)^{(l-k)}2^k\binom{n-k}{l-k}, & \text{for } k=\left\lfloor \frac{m}{2}\right\rfloor,\\ (-1)^{(l-k)}2^{n-k}\binom{n-k}{l-k}, & \text{for } k=\left\lceil \frac{m}{2}\right\rceil,\\ (-1)^{(l-n+k)}2^{n-k}\binom{n-k}{l-n+k-2}, & \text{for } \left\lceil \frac{m}{2}\right\rceil+1\le k\le n.\end{cases} \quad (8)$$

where $\lfloor x\rfloor$ represents $GI\le x$ and $\lceil x\rceil$ represents $LI\ge x$, where $GI$ and $LI$ are the greatest integer and least integer, respectively. The Wang-Ball monomial matrix is

$$\mathrm{A} = \begin{bmatrix} w_{00} & w_{01} & \cdots & \cdots & w_{0m} \\ w_{10} & w_{11} & \cdots & \cdots & w_{1m} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & \cdots & w_{mm} \end{bmatrix}_{(m+1)\times(m+1)} \tag{9}$$

where $w_{kl}$ is given as in (8).

The Wang-Ball basis function satisfies the following properties:

i. The Wang-Ball basis function is non-negative, that is,

$$W_i^m(t) \geq 0, \forall i = 0,1,\cdots,m. \tag{10}$$

ii. The partition of unity, that is,

$$\sum_{i=0}^{m} W_i^m(t) = 1. \tag{11}$$

In general, we approximate any function $u(t)$ with the first $(m+1)$ Wang-Ball polynomials as:

$$u(t) \approx \sum_{i=0}^{m} c_i' W_i^m(t) = C'^T \Omega(t) = C'^T \mathrm{A} H_m(t) \tag{12}$$

where $C'^T = [c_0', c_1', \ldots, c_m']$, $H_m(t) = [1 \quad t \quad t^2 \quad \ldots \quad t^m]^T$ and $\mathrm{A}$ is the monomial matrix form given in (9). The $m+1$ by $m+1$ operational matrix of derivative of the Wang-Ball polynomials set $\Omega(t)$ is given by:

$$\frac{d\Omega(t)}{dt} = \mathrm{D}'^{(1)}\Omega(t)$$

$$= \frac{d}{dt}\mathrm{A} H_m(t)$$

$$= \mathrm{A}\frac{d}{dt} H_m(t)$$

$$= \mathrm{A}\begin{bmatrix} 0 \\ 1 \\ 2t \\ \vdots \\ mt^{m-1} \end{bmatrix} = \mathrm{A}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & m & 0 \end{bmatrix}\begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^m \end{bmatrix}$$

$$= \mathrm{A}\Lambda H_m(t)$$

$$= \mathrm{A}\Lambda\mathrm{A}^{-1}\mathrm{A} H_m(t)$$

$$= \mathrm{A}\Lambda\mathrm{A}^{-1}\Omega(t)$$

$$= \mathrm{D}'^{(1)}\Omega(t)$$

where

$$\Lambda = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & m & 0 \end{bmatrix}.$$

Hence

$$u'(t) = C'^T \mathrm{D}'^{(1)}\Omega(t) \tag{13}$$

we can generalize Equation (13) as

$$\frac{d^n}{dx^n}\Omega(t)=\frac{d^{n-1}}{dx^{n-1}}\left(\frac{d}{dx}\Omega(t)\right)=\frac{d^{n-1}}{dx^{n-1}}\left(\mathrm{D}'^{(1)}\Omega(t)\right)=\cdots=\left(\mathrm{D}'^{(1)}\right)^n\Omega(t)=\mathrm{D}'^{(n)}\Omega(t),\ n=1,2,\ldots$$

**Applications of the Operational Matrix of Derivative**

In this section, we present the derivation of the method for solving differential equation of the form (1) by Wang-Ball Polynomials, as follows

$$C'^T\mathrm{D}'^{(2)}\Omega(t)+\frac{\gamma}{t}C'^T\mathrm{D}'^{(1)}\Omega(t)+C'^T\Omega(t)=R^T\Omega(t) \qquad (14)$$

where $R^T=[r_0,r_1,\cdots,r_m]$. We can write the residual $\Re_n(t)$ for Equation (14) as

$$\Re(t)=C'^T\mathrm{D}'^{(2)}\Omega(t)+\frac{\gamma}{t}C'^T\mathrm{D}'^{(1)}\Omega(t)+C'^T\Omega(t)-R^T\Omega(t). \qquad (15)$$

We first collocate (15) at $(m-1)$ points. For suitable points, we use $t_i=\frac{2i-1}{2m},\ i=1,2,\cdots,m-1.$

These equations generate $(m+1)$ nonlinear equations which can be solved using Newton's iteration method. Consequently, $u(t)$ can be calculated.

**Numerical Examples**

**Problem 1**

Consider the second order singular boundary value problem given in [20]

$$u''(t)+\frac{1}{t}u'(t)+u(t)=4-9t+t^2-t^3, \qquad (16)$$

with (BCs)

$$u(0)=0,\quad u(1)=0. \qquad (17)$$

and the exact solution $u(t)=t^2-t^3$.

To solve (16), we use the proposed method with $m=3$, hence obtaining the approximate solution as

$$\begin{aligned} u(t)\approx u_3(t)= &\ C'^T\Omega(t)=[c'_0,c'_1,c'_2,c'_3][W_0^3(t),W_1^3(t),W_2^3(t),W_3^3(t)]^T \\ = &\ c'_0W_0^3(t)+c'_1W_1^3(t)+c'_2W_2^3(t)+c'_3W_3^3(t), \end{aligned} \qquad (18)$$

We apply (13) to obtain,

$$\mathrm{D}'^{(1)}=\begin{bmatrix}-2&-1&-1&0\\2&-2&-2&0\\0&2&2&-2\\0&1&1&2\end{bmatrix},\quad \mathrm{D}'^{(2)}=\begin{bmatrix}2&2&2&2\\-8&-2&-2&4\\4&-2&-2&-8\\2&2&2&2\end{bmatrix}. \qquad (19)$$

Therefore, by collocating Eq. (15), choosing the suitable collocation points $t=\frac{3}{4},\frac{1}{4}$ yields

$$\begin{cases}\frac{67}{64}c'_0+\frac{25}{128}c'_1+\frac{501}{128}c'_2+\frac{219}{64}c'_3+\frac{501}{256}=0,\\ -\frac{55}{64}c'_0-\frac{103}{128}c'_1+\frac{115}{128}c'_2+\frac{65}{64}c'_3+\frac{115}{256}=0,\\ c'_0=0,\\ c'_3=0.\end{cases} \qquad (20)$$

The last two rows come from the BCs. Thus, solving (20) gives $c'_0 = 0, c'_1 = 0, c'_2 = \frac{1}{2}, c'_3 = 0.$ Hence

$$u_3(t) = \left[c'_0, c'_1, c'_2, c'_3\right]\begin{bmatrix}(t-1)^2 \\ 2(t-1)^2 t \\ -2(t-1)t^2 \\ t^2\end{bmatrix} = t^2 - t^3.$$

which is an approximate solution using the proposed method and the same as the given exact solution.

**Problem 2**

Consider the first order ode [22]

$$u'(t) - tu(t) + u^2(t) = e^{t^2}, \tag{21}$$

with the (ICs)

$$u(0) = 1. \tag{22}$$

with the exact solution

$$u(t) = e^{\frac{t^2}{2}}. \tag{23}$$

Here we see that $q(t) = 0.$ The numerical results of the newly proposed method in comparison to [22] are provided in Table-1.

**Table 1-** Comparison of solutions of Problem 2 using the proposed method and the method described previously [22].

| t | Error PM | Error Adomian Decomposition Method [22] |
|---|---|---|
| 0.00 | 0 | 0 |
| 0.01 | 8.28E-12 | 1.750000 E-7 |
| 0.02 | 2.222E-11 | 6.400000 E-7 |
| 0.03 | 3.229E-11 | 1.314000 E-6 |
| 0.04 | 3.538E-11 | 2.123000 E-6 |
| 0.05 | 3.210E-11 | 2.999000 E-6 |
| 0.06 | 2.504E-11 | 3.883000 E-6 |
| 0.07 | 1.724E-11 | 4.720000 E-6 |
| 0.08 | 1.163E-11 | 5.463000 E-6 |
| 0.09 | 1.033E-11 | 6.069000 E-6 |
| 0.1 | 1.468E-11 | 6.501000 E-6 |

**Problem 3**

Consider the differential equation from [3]

$$u''(t) + \frac{2}{t}u'(t) + u(t) = 0, \quad u(0) = 1, u'(0) = 0. \tag{24}$$

The exact solution is given by $u(t) = \frac{sin(t)}{t}$. We solve the above equation when $m = 7$ and $m = 8$. Figure-1 displays the absolute error which shows that the proposed method obtained highly accurate solutions even in large computational intervals.

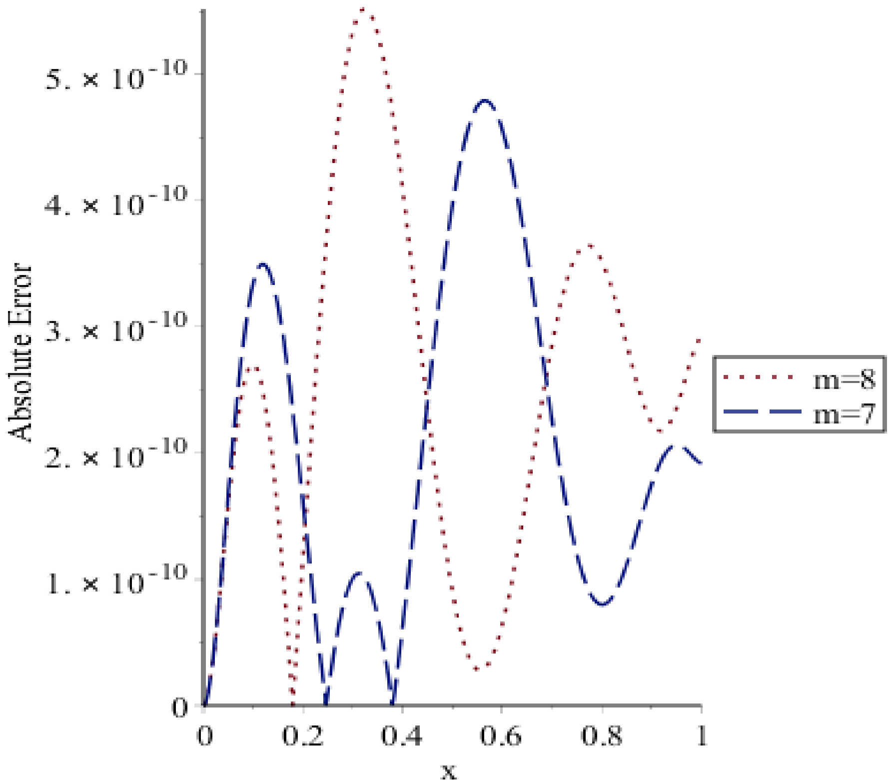


**Figure 1-** Absolute Error Plot of the Proposed Method with $m=7$ and $m=8$ for Problem 3

**Problem 4**

Consider the following ordinary differential equation [22]

$$u''(t)+tu'(t)+t^2u^3(t)-(2+6t^2)e^{(t^2)}-t^2e^{(3t^2)}=0, \tag{25}$$

Subject to IC

$$u(0)=1, \quad u'(0)=0. \tag{26}$$

with the exact solution $u(t)=e^{t^2}$. We apply the above method when $m=12$. Table-2 shows the absolute error for Problem 4.

**Table 2-**Comparison of solutions of Problem 4 using the proposed method and that described earlier [22].

| t | Adomian Decomposition Method Ref[22] | PM m=12 |
|---|---|---|
| 0 | 0 | 0 |
| 0.01 | 2.000000E-9 | 8.12030E-10 |
| 0.02 | 2.900000E-8 | 2.72633E-9 |
| 0.03 | 2.900000 E-8 | 5.17165E-9 |
| 0.04 | 4.450000 E-7 | 7.79530E-9 |
| 0.05 | 1.074000 E-6 | 1.039860E-8 |
| 0.06 | 2.207000 E-6 | 1.288758E-8 |
| 0.07 | 4.057000 E-6 | 1.523544E-8 |
| 0.08 | 6.872000 E-6 | 1.745542E-8 |
| 0.09 | 1.093000 E-5 | 1.958109E-8 |
| 0.1 | 1.654900 E-5 | 2.165294E-8 |

Problem 5

Consider the singular Dirichlet type boundary value problem on the interval [0, 1] given in [7]

$$u''(t)-\frac{1}{t}u'(t)+\frac{1}{t(t+1)}u(t)+t^3=0, \tag{25}$$

with (BCs) $u(0)=u(1)=0$. The exact solution of this problem is

$$u(t)=\frac{1}{144}\left(14\ln(t+1)t+14\ln(t+1)-1tx+6t^2-12t^2\ln(2)-2t^3+\right.$$
$$\left.4t^3\ln(2)+t^4-2t^4\ln(2)+9t^5-18t^5\ln(2))/(-1+2\ln(2)\right).$$

**Table 3-** Comparison of solutions of Equation (26) using the proposed method with $m=12$ and the method reported previously [7].

| t | Error P M | Error Sinc-Galerkin method [7] |
|---|---|---|
| 0.2 | 7.466000000000000E-12 | 1.88415721000000e-10 |
| 0.4 | 2.385900000000000E-11 | 7.13501861405898e-10 |
| 0.6 | 2.385900000000000E-11 | 8.20803253396388e-10 |
| 0.8 | 4.034590000000000E-10 | 5.53448662985227e-10 |

## Conclusions

This article has considered the numerical solution of first and second order DEs. The introduced method is the Wang-Ball operational matrix which was derived as a generalization of the conventional Ball polynomial. For each of the numerical problems considered, the new approach reduces the DEs into a set of linear and non-linear algebraic equations with respect to the property of the DE itself. This reduction renders the DEs to be in a form that is easier to solve, while still obtaining accurate results. Aside from being able to recover the exact solution of certain DEs, the Wang-Ball operational matrix confers a more exact and robust numerical solution than that provided by the classical method. It can be clearly noticed that the proposed method performs well even on a few number of terms of the Wang-Ball polynomials, also showing impressive results as compared to the methods reported by the existing literature. Hence, the proposed method in this article can be adequately adopted to solve any real life scenario model in the form of either first or second order DEs.

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