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A Jordan Higher Reverse Left (resp. right) Centralizer on Prime Γ-Rings

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Abstract
In this paper, we introduce the concepts of higher reverse left (resp.right) centralizer, Jordan higher reverse left (resp. right) centralizer, and Jordan triple higher reverse left (resp. right) centralizer of Γ-rings. We prove that every Jordan higher reverse left (resp. right) centralizer of a 2-torsion free prime Γ-ring M is a higher reverse left (resp. right) centralizer of M.

Keywords: prime Γ-ring , higher reverse left (resp. right) centralizer , Jordan higher reverse left (resp. right) centralizer

تمركزات جوردان العليا العكسية اليسارية (اليمينية) على الحلقات الأولية من النمط - Γ

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الخلاصة
في هذا البحث قدمنا المفاهيم الآتية: التمركزات العليا العكسية اليسارية (اليمينية) , تمركزات جوردان العليا العكسية اليسارية (اليمينية) و تمركزات جوردان الثلاثية العليا العكسية اليسارية (اليمينية) على الحلقات الأولية من النمط − Γ .
وكذلك برهنا ان كل تمركزات جوردان العليا العكسية اليسارية (اليمينية) في حلقة أولية طليقة الألتواء من النمط 2 هي التمركزات العليا العكسية اليسارية (اليمينية) في حلقة من نمط −Γ .

INTRODUCTION

Let M and Γ be two additive Abelian groups. Suppose that there is a mapping from $M \times \Gamma \times M \longrightarrow M$, where the image of (x, α, y) is denoted by $x \alpha y$, where $x, y \in M$ and $\alpha \in \Gamma$, satisfying the following properties for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

(i) $(x + y) \alpha z = x \alpha z + y \alpha z$

$x(\alpha+\beta) z = x \alpha z + x \beta z$

$x\alpha(y + z) = x \alpha y + x \alpha z$

(ii) $(x \alpha y) \beta z = x \alpha (y \beta z)$.

Then M is called a Γ-ring [1,2] .

M is called a prime if $x \Gamma M \Gamma y = (0)$ implies that $x = 0$ or $y = 0$, where $x, y \in M$ [3, 4].

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M is called a semiprime if $x \Gamma M \Gamma x = (0)$ implies that $x = 0$, where $x \in M$ [4].
M is called a 2-torsion free if $2x = 0$ implies that $x = 0$, for all $x \in M$ [3, 4].
If M is a Γ-ring **,** then $[x,y]_\alpha = x\alpha y - y\alpha x$, for all $x,y \in M$ and $\alpha \in \Gamma$, is known as a commutator [2, 4].
An additive mapping $d : M \longrightarrow M$ is called a derivation if the following holds :

$$d(x\alpha y) = d(x)\alpha y + x\alpha d(y), \text{ for all } x,y \in M \text{ and } \alpha \in \Gamma \text{ [5] .}$$

Additionally, d is called a Jordan derivation if the following property holds :

$$d(x\alpha x) = d(x)\alpha x + x\alpha d(x), \text{ for all } x \in M \text{ and } \alpha \in \Gamma \text{ [5] .}$$

In the following M.Sapanci and A.Nakajima [6] gave the conditions which makes a Jordan derivation is a derivation on Γ- ring .
If M is a 2-torision free completely prime Γ- ring such that for any $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, we have $z \beta a \beta [x,y]_\alpha = 0$ or $[x,y]_\alpha \beta a \beta z = 0$. Implies that $z = 0$, $x \in M$. Then every Jordan derivation is a derivation on M .
Let $D = (d_i)_{i \in N}$ be a family of additive mappings of a Γ-ring M into itself such that $d_0 = id_M$. Then D is called higher derivation if

$$d_n(x\alpha y) = \sum_{i+j=n} d_i(x)\alpha\, d_j(y), \text{ for all } x,y \in M, \alpha \in \Gamma \text{ and } n \in N \text{ [7] .}$$

Consequently , D is called a Jordan higher derivation if
$$d_n(x\,\alpha\, x) = \sum_{i+j=n} d_i(x)\alpha\, d_j(x), \text{ for all } x \in M, \alpha \in \Gamma \text{ and } n \in N \text{ [7].}$$

Also, proved that Let M be a 2-torsion free prime Γ-ring and U be an admissible Lie ideal (U not contained in Z(M) and $u\alpha u \in U$, for every $u \in U$ and $\alpha \in \Gamma$) of M. Then every Jordan higher derivation of U into M is a higher derivation of U into M [7].
A left (resp. right) centralizer of a Γ-ring M is an additive mapping $t : M \longrightarrow M$ which satisfies the following equation
$t(x\alpha y) = t(x)\alpha y$ (resp. $t(x\alpha y) = x\alpha t(y)$) , for all $x, y \in M$ and $\alpha \in \Gamma$.
t is called a centralizer of M if it is both a left and right centralizer [8] .
A left (resp. right) Jordan centralizer of a Γ-ring M is an additive mapping $t : M \longrightarrow M$ which satisfies the following equation

$$t(x\alpha x) = t(x)\alpha x \text{ (resp. } t(x\alpha x) = x\alpha t(x)\text{), for all } x \in M \text{ and } \alpha \in \Gamma.$$

t is called a Jordan centralizer of M if it is both a left and right Jordan centralizer [8] .
And proved that every Jordan centralizer of a 2-torsion free semiprime Γ-ring M satisfying $x\alpha y\beta x = x\beta y\alpha x$ is a centralizer [8] .
Let $t = (t_i)_{i\in N}$ be a family of additive mappings of a ring R into itself . Then t is called a higher left centralizer, we have that

$$t_n(xy) = \sum_{i=1}^{n} t_i(x)\, t_{i-1}(y), \text{ for all } x,y \in R \text{ and } n \in N \text{ [9].}$$

In addition, t is called a Jordan higher left centralizer if the following equation holds :

$$t_n(x^2) = \sum_{i=1}^{n} t_i(x)\, t_{i-1}(x), \text{ for all } x \in R \text{ and } n \in N \text{ [9].}$$

And proved that every Jordan higher left centralizer of a 2-torsion free prime ring R is a higher left centralizer of R and let $t = (t_i)_{i\in N}$ be a Jordan higher left centralizer of a 2-torsion free ring R . Then t is a Jordan triple higher left centralizer of R .

In this paper we define and study the concept of higher reverse left (resp. right) centralizer of prime Γ-rings and we present some properties about higher reverse left (resp. right) centralizers one of these theorems is :
Let $t = (t_i)_{i\in N}$ be a Jordan higher reverse left (resp. right) centralizer of a 2-torsion free Γ-ring M , such that $x\alpha y\beta x = x\beta y\alpha x$, for all $x,y \in M$ and $\alpha, \beta \in \Gamma$. Then t is a Jordan triple higher reverse left (resp. right) centralizer of M .

Jordan Higher Reverse Left (resp. right) Centralizer on Prime Γ-Rings

In this section we will introduce the concept of higher reverse left (resp. right) centralizer on prime Γ-rings.

Definition (2.1)

Let $t = (t_i)_{i\in N}$ be a family of additive mappings of a Γ-ring M into itself .Then t is called a higher reverse left (resp. right) centralizer if for all $x, y \in M$, $\alpha \in \Gamma$ and $n \in N$,

$$t_n(x\alpha y) = \sum_{i=1}^{n} t_i(y)\alpha t_{i-1}(x) \text{ (resp. } t_n(x\alpha y) = \sum_{i=1}^{n} t_{i-1}(y)\alpha t_i(x))$$

Definition (2.2)

Let $t = (t_i)_{i\in N}$ be a family of additive mappings of a Γ-ring M into itself .Then t is called a Jordan higher reverse left (resp. right) centralizer if for all $x \in M$, $\alpha \in \Gamma$ and $n \in N$, the following equation holds :

$$t_n(x\alpha x) = \sum_{i=1}^{n} t_i(x)\alpha t_{i-1}(x) \text{ (resp. } t_n(x\alpha x) = \sum_{i=1}^{n} t_{i-1}(x)\alpha t_i(x)).$$

Definition (2.3)

Let $t = (t_i)_{i\in N}$ be a family of additive mappings of a Γ-ring M into itself .Then t is called a Jordan triple higher reverse left (resp. right) centralizer , if for all $x, y \in M$, $\alpha, \beta \in \Gamma$ and $n \in N$, the following equation holds :

$$t_n(x\alpha y\beta x) = \sum_{i=1}^{n} t_i(x)\beta t_{i-1}(y)\alpha t_{i-1}(x)$$

(resp. $t_n(x\alpha y\beta x) = \sum_{i=1}^{n} t_{i-1}(x)\beta t_{i-1}(y)\alpha t_i(x)$).

Lemma (2.4)

Let $t = (t_i)_{i\in N}$ be a Jordan higher reverse left (resp. right) centralizer of a Γ-ring M .Then for all $x, y, z \in M$, $\alpha, \beta \in \Gamma$ and $n \in N$, the following equations hold :

(i) $t_n(x\alpha y + y\alpha x) = \sum_{i=1}^{n} t_i(y)\alpha t_{i-1}(x) + \sum_{i=1}^{n} t_i(x)\alpha t_{i-1}(y)$

(resp. $t_n(x\alpha y + y\alpha x) = \sum_{i=1}^{n} t_{i-1}(y)\alpha t_i(x) + \sum_{i=1}^{n} t_{i-1}(x)\alpha t_i(y)$)

(ii) $t_n(x\alpha y\beta x + x\beta y\alpha x) = \sum_{i=1}^{n} t_i(x)\beta t_{i-1}(y)\alpha t_{i-1}(x) + \sum_{i=1}^{n} t_i(x)\alpha t_{i-1}(y)\beta t_{i-1}(x)$

(resp. $t_n(x\alpha y\beta x + x\beta y\alpha x) = \sum_{i=1}^{n} t_{i-1}(x)\beta t_{i-1}(y)\alpha t_i(x) + \sum_{i=1}^{n} t_{i-1}(x)\alpha t_{i-1}(y)\beta t_i(x)$)

(iii) $t_n(x\alpha y\beta z + z\alpha y\beta x) = \sum_{i=1}^{n} t_i(z)\beta t_{i-1}(y)\alpha t_{i-1}(x) + \sum_{i=1}^{n} t_i(x)\beta t_{i-1}(y)\alpha t_{i-1}(z)$

(resp. $t_n(x\alpha y\beta z + z\alpha y\beta x) = \sum_{i=1}^{n} t_{i-1}(z)\beta t_{i-1}(y)\alpha t_i(x) + \sum_{i=1}^{n} t_{i-1}(x)\beta t_{i-1}(y)\alpha t_i(z)$)

(iv) In particular, if M is a 2-torsion free commutative Γ-ring, then

$$t_n(x\alpha y\beta z) = \sum_{i=1}^{n} t_i(z)\beta t_{i-1}(y)\alpha t_{i-1}(x)$$

(resp. $t_n(x\alpha y\beta z) = \sum_{i=1}^{n} t_{i-1}(z)\beta t_{i-1}(y)\alpha t_i(x)$)

(v) $t_n(x\alpha y\alpha z + z\alpha y\alpha x) = \sum_{i=1}^{n} t_i(z)\alpha t_{i-1}(y)\alpha t_{i-1}(x) + \sum_{i=1}^{n} t_i(x)\alpha t_{i-1}(y)\alpha t_{i-1}(z)$

(resp. $t_n(x\alpha y\alpha z + z\alpha y\alpha x) = \sum_{i=1}^{n} t_{i-1}(z)\alpha t_{i-1}(y)\alpha t_i(x) + \sum_{i=1}^{n} t_{i-1}(x)\alpha t_{i-1}(y)\alpha t_i(z)$)

Proof

(i) $t_n((x+y)\alpha(x+y)) = \sum_{i=1}^{n} t_i(x+y)\alpha t_{i-1}(x+y)$

$$= \sum_{i=1}^{n} t_i(x)\,\alpha\, t_{i-1}(x) + \sum_{i=1}^{n} t_i(x)\,\alpha\, t_{i-1}(y) + \sum_{i=1}^{n} t_i(y)\,\alpha\, t_{i-1}(x) + \sum_{i=1}^{n} t_i(y)\,\alpha\, t_{i-1}(y) \ldots (1)$$

Meanwhile , we have that

$$t_n((x+y)\,\alpha\,(x+y)) = t_n(x\alpha x + x\alpha y + y\alpha x + y\alpha y)$$

$$= \sum_{i=1}^{n} t_i(x)\,\alpha\, t_{i-1}(x) + \sum_{i=1}^{n} t_i(y)\,\alpha\, t_{i-1}(y) + t_n(x\,\alpha\, y + y\,\alpha\, x) \ \ldots (2)$$

We obtain the following equation by Comparing equations (1) and (2)

$$t_n(x\,\alpha\, y + y\,\alpha\, x) = \sum_{i=1}^{n} t_i(y)\alpha t_{i-1}(x) + \sum_{i=1}^{n} t_i(x)\alpha t_{i-1}(y)$$

(ii) By substituting $x\,\beta\, y + y\,\beta\, x$ for y in (i) , we have that

$$= \sum_{i=1}^{n} t_i(y)\,\beta\, t_{i-1}(x)\alpha\, t_{i-1}(x) + t_i(x)\,\beta\, t_{i-1}(y)\,\alpha\, t_{i-1}(x) +$$

$$t_i(x)\,\alpha\, t_{i-1}(y)\,\beta\, t_{i-1}(x) + t_i(x)\,\alpha\, t_{i-1}(x)\,\beta\, t_{i-1}(y) \ldots (1)$$

In addition , we obtain that

$$t_n(x\,\alpha\,(x\beta y + y\beta x) + (x\beta y + y\beta x)\,\alpha\, x)$$

$$= t_n(x\,\alpha\, x\,\beta\, y) + t_n(y\,\beta\, x\alpha\, x) + t_n(x\,\alpha\, y\,\beta\, x + x\,\beta\, y\alpha\, x)$$

$$= \sum_{i=1}^{n} t_i(y)\,\beta\, t_{i-1}(x)\alpha\, t_{i-1}(x) + t_i(x)\,\alpha\, t_{i-1}(x)\,\beta\, t_{i-1}(y) + t_n(x\,\alpha\, y\,\beta\, x + x\,\beta\, y\alpha\, x) \ldots (2)$$

We get the required result by Comparing equations (1) and (2) .

(iii) By substituting x + z for x in Definition (2.3) , we have

$$= \sum_{i=1}^{n} t_i(x)\,\beta\, t_{i-1}(y)\alpha\, t_{i-1}(x) + t_i(x)\,\beta\, t_{i-1}(y)\,\alpha\, t_{i-1}(z) +$$

$$t_i(z)\,\beta\, t_{i-1}(y)\,\alpha\, t_{i-1}(x) + t_i(z)\,\beta\, t_{i-1}(y)\,\alpha\, t_{i-1}(z) \ldots (1)$$

Moreover,

$$t_n((x+z)\alpha\, y\,\beta(x+z)) = t_n(x\,\alpha\, y\,\beta x + x\,\alpha\, y\,\beta z + z\,\alpha\, y\,\beta\, x + z\,\alpha\, y\,\beta z)$$

$$= \sum_{i=1}^{n} t_i(x)\,\beta\, t_{i-1}(y)\alpha\, t_{i-1}(x) + t_i(z)\,\beta\, t_{i-1}(y)\,\alpha\, t_{i-1}(z) + t_n(x\,\alpha\, y\,\beta\, z + z\,\alpha\, y\,\beta\, x) \ldots (2)$$

The following equation is obtained by comparing equations (1) and (2)

$$t_n(x\,\alpha\, y\,\beta\, z + z\,\alpha\, y\,\beta\, x) = \sum_{i=1}^{n} t_i(z)\,\beta\, t_{i-1}(y)\,\alpha\, t_{i-1}(x) + \sum_{i=1}^{n} t_i(x)\,\beta\, t_{i-1}(y)\,\alpha\, t_{i-1}(z)$$

(iv) Using Lemma (2.4)(iii) and the fact that M is a commutative Γ-ring , we have that

$$t_n(x\,\alpha\, y\,\beta\, z + x\,\alpha\, y\,\beta\, z) = 2\, t_n(x\,\alpha\, y\,\beta\, z)$$

$$= 2\sum_{i=1}^{n} t_i(z)\,\beta\, t_{i-1}(y)\,\alpha\, t_{i-1}(x)$$

We obtain the required result, by utilizing the fact that M is a 2-torsion free .

(v) The substitution β for α in (iii), we get the required result

Definition (2.5)

Let $t = (t_i)_{i\in N}$ be a Jordan higher reverse left (resp. right) centralizer of a Γ-ring M. Then for all $x, y \in M$, $\alpha \in \Gamma$ and $n \in N$, we define

$$G_n(x,y)_\alpha = t_n(x\,\alpha\, y) - \sum_{i=1}^{n} t_i(y)\alpha\, t_{i-1}(x)$$

$$\text{(resp. } G_n(x,y)_\alpha = t_n(x\,\alpha\, y) - \sum_{i=1}^{n} t_{i-1}(y)\alpha\, t_i(x)\text{)}$$

Lemma (2.6)

Let $t = (t_i)_{i\in N}$ be a Jordan higher reverse left (resp. right) centralizer of a Γ-ring M .Then for all $x, y, z \in M$, $\alpha, \beta \in \Gamma$ and $n \in N$, we have that the following equations hold :

(i) $G_n(x,y)_\alpha = -G_n(y,x)_\alpha$

(ii) $G_n(x+y,z)_\alpha = G_n(x,z)_\alpha + G_n(y,z)_\alpha$
(iii) $G_n(x,y+z)_\alpha = G_n(x,y)_\alpha + G_n(x,z)_\alpha$
(iv) $G_n(x,y)_{\alpha+\beta} = G_n(x,y)_\alpha + G_n(x,y)_\beta$

Proof:

(i) By applying Lemma (2.4) (i) , we have that

$$t_n(x\alpha y + y\alpha x) = \sum_{i=1}^{n} t_i(y)\alpha t_{i-1}(x) + \sum_{i=1}^{n} t_i(x)\alpha t_{i-1}(y)$$

$$t_n(x\alpha y) - \sum_{i=1}^{n} t_i(y)\alpha t_{i-1}(x) = -(t_n(y\alpha x) - \sum_{i=1}^{n} t_i(x)\alpha t_{i-1}(y))$$

$G_n(x,y)_\alpha = - G_n(y,x)_\alpha$

(ii) $G_n(x+y,z)_\alpha = t_n((x+y)\alpha z) - \sum_{i=1}^{n} t_i(z)\alpha t_{i-1}(x+y)$

$$= t_n(x\alpha z + y\alpha z) - \sum_{i=1}^{n} t_i(z)\alpha t_{i-1}(x) - \sum_{i=1}^{n} t_i(z)\alpha t_{i-1}(y)$$

$$= t_n(x\alpha z) - \sum_{i=1}^{n} t_i(z)\alpha t_{i-1}(x) + t_n(y\alpha z) - \sum_{i=1}^{n} t_i(z)\alpha t_{i-1}(y)$$

$= G_n(x,z)_\alpha + G_n(y,z)_\alpha$

(iii) $G_n(x,y+z)_\alpha = t_n(x\alpha(y+z)) - \sum_{i=1}^{n} t_i(y+z)\alpha t_{i-1}(x)$

$$= t_n(x\alpha y + x\alpha z) - \sum_{i=1}^{n} t_i(y)\alpha t_{i-1}(x) - \sum_{i=1}^{n} t_i(z)\alpha t_{i-1}(x)$$

$$= t_n(x\alpha y) - \sum_{i=1}^{n} t_i(y)\alpha t_{i-1}(x) + t_n(x\alpha z) - \sum_{i=1}^{n} t_i(z)\alpha t_{i-1}(x)$$

$= G_n(x,y)_\alpha + G_n(x,z)_\alpha$

(iv) $G_n(x,y)_{\alpha+\beta} = t_n(x(\alpha+\beta)y) - \sum_{i=1}^{n} t_i(y)(\alpha+\beta) t_{i-1}(x)$

$$= t_n(x\alpha y) - \sum_{i=1}^{n} t_i(y)\alpha t_{i-1}(x) + t_n(x\beta y) - \sum_{i=1}^{n} t_i(y)\beta t_{i-1}(x)$$

$= G_n(x,y)_\alpha + G_n(x,y)_\beta$

Remark (2.7)

It is noteworthy that $t = (t_i)_{i\in N}$ is a higher reverse left (resp. right) centralizer of a Γ-ring M if and only if $G_n(x,y)_\alpha = 0$, for all $x, y \in M$, $\alpha \in \Gamma$ and $n \in N$.

Lemma (2.8)

Let $t = (t_i)_{i\in N}$ be a Jordan higher reverse left (resp. right) centralizer of a Γ-ring M .
Then the following equations hold for all $x, y, z \in M$, $\alpha, \beta \in \Gamma$ and $n \in N$:
(i) $G_n(x,y)_\alpha \beta t_{n-1}(z) \beta [t_{n-1}(x), t_{n-1}(y)]_\alpha = 0$
(ii) $G_n(x,y)_\alpha \alpha t_{n-1}(z) \alpha [t_{n-1}(x), t_{n-1}(y)]_\alpha = 0$
(iii) $G_n(x,y)_\beta \alpha t_{n-1}(z) \alpha [t_{n-1}(x), t_{n-1}(y)]_\beta = 0$

Proof:

(i) The proof is utilizing induction on $n \in N$
If n = 1
Let $w = x\alpha y\beta z\beta y\alpha x + y\alpha x\beta z\beta x\alpha y$
Then , we obtain that

$t(w) = t(x\alpha(y\beta z\beta y)\alpha x + y\alpha(x\beta z\beta x)\alpha y)$
$= t(x)\alpha y\beta z\beta y\alpha x + t(y)\alpha x\beta z\beta x\alpha y$ …(1)

Moreover, we have that

$t(w) = t((x\alpha y)\beta z\beta(y\alpha x) + (y\alpha x)\beta z\beta(x\alpha y))$
$= t(y\alpha x)\beta z\beta y\alpha x + t(x\alpha y)\beta z\beta x\alpha y$ …(2)

The Comparison of equations (1) and (2) yields that

$0 = (t(y\alpha x) - t(x)\alpha y)\beta z\beta y\alpha x + (t(x\alpha y) - t(y)\alpha x)\beta z\beta x\alpha y$
$0 = G(y,x)_\alpha \beta z\beta y\alpha x + G(x,y)_\alpha \beta z\beta x\alpha y$
$0 = -G(x,y)_\alpha \beta z\beta y\alpha x + G(x,y)_\alpha \beta z\beta x\alpha y$
$0 = G(x,y)_\alpha \beta z\beta(x\alpha y - y\alpha x)$
Thus, $G(x, y)_\alpha \beta z\beta[x,y]_\alpha = 0$, for all $x,y,z \in M$ and $\alpha , \beta \in \Gamma$.
Now , we assume the following :
$G_s(x,y)_\alpha \beta\, t_{s-1}(z)\beta\, [t_{s-1}(x),t_{s-1}(y)]_\alpha = 0$, for all $x,y,z \in M$, $s , n \in N$ and $s < n$.

$$t_n(w) = t_n(x\alpha(y\beta z\beta y)\alpha x + y\alpha(x\beta z\beta x)\alpha y)$$
$$= \sum_{i=1}^{n} t_i(x)\,\alpha\, t_{i-1}(y\beta z\beta y)\,\alpha\, t_{i-1}(x) + \sum_{i=1}^{n} t_i(y)\,\alpha\, t_{i-1}(x\beta z\beta x)\,\alpha\, t_{i-1}(y)$$
$$= \sum_{i=1}^{n} t_i(x)\,\alpha\, t_{i-1}(y)\,\beta\, t_{i-1}(z)\beta\, t_{i-1}(y)\,\alpha\, t_{i-1}(x) + \sum_{i=1}^{n} t_i(y)\,\alpha\, t_{i-1}(x)\,\beta\, t_{i-1}(z)\beta\, t_{i-1}(x)\alpha\, t_{i-1}(y)$$
$$= (\sum_{i=1}^{n} t_i(x)\,\alpha\, t_{i-1}(y))\,\beta\, t_{n-1}(z)\beta\, t_{n-1}(y)\,\alpha\, t_{n-1}(x) + \sum_{i=1}^{n-1} t_i(x)\,\alpha\, t_{i-1}(y)\,\beta\, t_{i-1}(z)\,\beta\, t_{i-1}(y)\,\alpha\, t_{i-1}(x) +$$
$$(\sum_{i=1}^{n} t_i(y)\,\alpha\, t_{i-1}(x))\,\beta\, t_{n-1}(z)\,\beta\, t_{n-1}(x)\,\alpha\, t_{n-1}(y) + \sum_{i=1}^{n-1} t_i(y)\,\alpha\, t_{i-1}(x)\,\beta\, t_{i-1}(z)\,\beta\, t_{i-1}(x)\,\alpha\, t_{i-1}(y) \quad \ldots (3)$$

Thus ,

$$t_n(w) = t_n((x\alpha y)\beta z\beta(y\alpha x) + (y\alpha x)\beta z\beta(x\alpha y))$$
$$= \sum_{i=1}^{n} t_i(y\alpha x)\,\beta\, t_{i-1}(z)\,\beta\, t_{i-1}(x\alpha y) + \sum_{i=1}^{n} t_i(x\alpha y)\,\beta\, t_{i-1}(z)\,\beta\, t_{i-1}(y\alpha x)$$
$$= t_n(y\alpha x)\,\beta\, t_{n-1}(z)\,\beta\, t_{n-1}(y)\,\alpha\, t_{n-1}(x) + \sum_{i=1}^{n-1} t_i(y\alpha x)\,\beta\, t_{i-1}(z)\,\beta\, t_{i-1}(y)\,\alpha\, t_{i-1}(x) +$$
$$t_n(x\alpha y)\,\beta\, t_{n-1}(z)\,\beta\, t_{n-1}(x)\,\alpha\, t_{n-1}(y) + \sum_{i=1}^{n-1} t_i(x\alpha y)\,\beta\, t_{i-1}(z)\,\beta\, t_{i-1}(x)\,\alpha\, t_{i-1}(y) \quad \ldots (4)$$

By comparing equations (3) and (4) , we have that

$$0 = (t_n(y\alpha x) - \sum_{i=1}^{n} t_i(x)\,\alpha\, t_{i-1}(y))\,\beta\, t_{n-1}(z)\,\beta\, t_{n-1}(y)\,\alpha\, t_{n-1}(x) +$$
$$(t_n(x\alpha y) - \sum_{i=1}^{n} t_i(y)\,\alpha\, t_{i-1}(x))\,\beta\, t_{n-1}(z)\,\beta\, t_{n-1}(x)\,\alpha\, t_{n-1}(y) +$$
$$\sum_{i=1}^{n-1} (t_i(y\alpha x) - t_i(x)\,\alpha\, t_{i-1}(y))\,\beta\, t_{i-1}(z)\,\beta\, t_{i-1}(y)\,\alpha\, t_{i-1}(x) +$$
$$\sum_{i=1}^{n-1} (t_i(x\alpha y) - t_i(y)\,\alpha\, t_{i-1}(x))\,\beta\, t_{i-1}(z)\,\beta\, t_{i-1}(x)\,\alpha\, t_{i-1}(y)$$

It follows that
$0 = G_n(y,x)_\alpha \beta\, t_{n-1}(z)\,\beta\, t_{n-1}(y)\,\alpha\, t_{n-1}(x) + G_n(x,y)_\alpha \beta\, t_{n-1}(z)\,\beta\, t_{n-1}(x)\,\alpha\, t_{n-1}(y)$
$0 = -G_n(x,y)_\alpha \beta\, t_{n-1}(z)\,\beta\, t_{n-1}(y)\,\alpha\, t_{n-1}(x) + G_n(x,y)_\alpha \beta\, t_{n-1}(z)\,\beta\, t_{n-1}(x)\,\alpha\, t_{n-1}(y)$
$0 = G_n(x,y)_\alpha \beta\, t_{n-1}(z)\,\beta(t_{n-1}(x)\,\alpha\, t_{n-1}(y) - t_{n-1}(y)\,\alpha\, t_{n-1}(x))$
$G_n(x,y)_\alpha \beta\, t_{n-1}(z)\,\beta[t_{n-1}(x), t_{n-1}(y)]_\alpha = 0$, for all $x,y, z \in M$, $\alpha , \beta \in \Gamma$ and $n \in N$
(ii) By substituting β for α in Lemma (2.8)(i) and applying similar arguments as in the proof of Lemma (2.8)(i) , we obtain Lemma (2.8)(ii) .
(iii) We get Lemma (2.8)(iii), by Interchanging α and β in Lemma (2.8)(i) .

Lemma (2.9)

Let $t = (t_i)_{i\in N}$ be a Jordan higher reverse left (resp. right) centralizer of a prime Γ-ring M .
Then for all $x , y , z , u , v \in M$, $\alpha , \beta \in \Gamma$ and $n \in N$
(i) $G_n(x ,y)_\alpha \beta\, t_{n-1}(z)\,\beta\, [t_{n-1}(u) , t_{n-1}(v)]_\alpha = 0$
(ii) $G_n(x ,y)_\alpha \alpha\, t_{n-1}(z)\,\alpha\, [t_{n-1}(u) , t_{n-1}(v)]_\alpha = 0$
(iii) $G_n(x ,y)_\alpha \alpha\, t_{n-1}(z)\,\alpha\, [t_{n-1}(u) , t_{n-1}(v)]_\beta = 0$

Proof

(i) Replacing x + u for x in Lemma (2.8)(i) , we have that

$G_n(x + u , y)_\alpha \beta t_{n-1}(z) \beta[t_{n-1}(x + u) , t_{n-1}(y)]_\alpha = 0$

$G_n(x,y)_\alpha \beta t_{n-1}(z) \beta[t_{n-1}(x) , t_{n-1}(y)]_\alpha + G_n(x,y)_\alpha \beta t_{n-1}(z) \beta[t_{n-1}(u) , t_{n-1}(y)]_\alpha +$

$G_n(u,y)_\alpha \beta t_{n-1}(z) \beta[t_{n-1}(x) , t_{n-1}(y)]_\alpha + G_n(u,y)_\alpha \beta t_{n-1}(z) \beta[t_{n-1}(u) , t_{n-1}(y)]_\alpha = 0$

By Lemma (2.8)(i) , we have that

$G_n(x,y)_\alpha \beta t_{n-1}(z)\beta[t_{n-1}(u),t_{n-1}(y)]_\alpha + G_n(u,y)_\alpha \beta t_{n-1}(z)\beta[t_{n-1}(x),t_{n-1}(y)]_\alpha = 0$

Therefore, we have that

$G_n(x,y)_\alpha \beta t_{n-1}(z)\beta[t_{n-1}(u),t_{n-1}(y)]_\alpha \beta t_{n-1}(z)\beta G_n(x,y)_\alpha \beta t_{n-1}(z)\beta[t_{n-1}(u),t_{n-1}(y)]_\alpha = 0$

$0 = - G_n(x ,y)_\alpha \beta t_{n-1}(z)\beta[t_{n-1}(u), t_{n-1}(y)]_\alpha \beta t_{n-1}(z)\beta G_n(u ,y)_\alpha \beta t_{n-1}(z)\beta[t_{n-1}(x), t_{n-1}(y)]_\alpha$

Since M is a prime Γ-ring , we get

$G_n(x ,y)_\alpha \beta t_{n-1}(z)\beta[t_{n-1}(u) ,t_{n-1}(y)]_\alpha = 0$, for all $x , y , z , u \in M$, $\alpha , \beta \in \Gamma$ and $n \in N$ … (1)

Now, replacing y + v for y in Lemma (2.8) (i) , we have that

$G_n(x, y + v)_\alpha \beta t_{n-1}(z) \beta[t_{n-1}(x) ,t_{n-1}(y + v)]_\alpha = 0$

$G_n(x,y)_\alpha \beta t_{n-1}(z) \beta[t_{n-1}(x), t_{n-1}(y)]_\alpha + G_n(x,y)_\alpha \beta t_{n-1}(z) \beta[t_{n-1}(x), t_{n-1}(v)]_\alpha +$

$G_n(x,v)_\alpha \beta t_{n-1}(z) \beta[t_{n-1}(x),t_{n-1}(y)]_\alpha + G_n(x,v)_\alpha \beta t_{n-1}(z) \beta[t_{n-1}(x),t_{n-1}(v)]_\alpha = 0$

By Lemma (2.8)(i) , we get

$G_n(x,y)_\alpha \beta t_{n-1}(z)\beta[t_{n-1}(x),t_{n-1}(v)]_\alpha + G_n(x,v)_\alpha \beta t_{n-1}(z)\beta[t_{n-1}(x),t_{n-1}(y)]_\alpha = 0$

Therefore, we have that

$G_n(x,y)_\alpha \beta t_{n-1}(z) \beta [t_{n-1}(x) , t_{n-1}(v)]_\alpha \beta t_{n-1}(z) \beta G_n(x,y)_\alpha \beta t_{n-1}(z) \beta[t_{n-1}(x) , t_{n-1}(v)]_\alpha = 0$

$0 = - G_n(x,y)_\alpha \beta t_{n-1}(z)\beta[t_{n-1}(x) , t_{n-1}(v)]_\alpha \beta t_{n-1}(z)\beta G_n(x ,v)_\alpha \beta t_{n-1}(z) \beta[t_{n-1}(x) ,t_{n-1}(y)]_\alpha$

Hence, by the primness of M :

$G_n(x,y)_\alpha \beta t_{n-1}(z)\beta[t_{n-1}(x),t_{n-1}(v)]_\alpha = 0$, for all $x , y , z , v \in M$, $\alpha , \beta \in \Gamma$ and $n \in N$ … (2)

Now , $G_n(x,y)_\alpha \beta t_{n-1}(z)\beta[t_{n-1}(x + u),t_{n-1}(y+v)]_\alpha = 0$

$G_n(x,y)_\alpha \beta t_{n-1}(z) \beta[t_{n-1}(x),t_{n-1}(y)]_\alpha + G_n(x,y)_\alpha \beta t_{n-1}(z)\beta[t_{n-1}(x),t_{n-1}(v)]_\alpha +$

$G_n(x,y)_\alpha \beta t_{n-1}(z) \beta[t_{n-1}(u), t_{n-1}(y)]_\alpha + G_n(x,y)_\alpha \beta t_{n-1}(z) \beta[t_{n-1}(u),t_{n-1}(v)]_\alpha = 0$

By (1), (2) and Lemma (2.8) (i) , we get

$G_n(x,y)_\alpha \beta t_{n-1}(z) \beta[t_{n-1}(u),t_{n-1}(v)]_\alpha = 0$, for all $x , y , z , u , v \in M$, $\alpha , \beta \in \Gamma$ and $n \in N$

(ii) Replace β for α in (i) , we get (ii) .

(iii) Replacing α + β for α in (ii) , we have that

$G_n(x,y)_{\alpha+\beta} \alpha t_{n-1}(z)\alpha[t_{n-1}(u), t_{n-1}(v)]_{\alpha+\beta} = 0$

$G_n(x,y)_\alpha \alpha t_{n-1}(z) \alpha[t_{n-1}(u),t_{n-1}(v)]_\alpha + G_n(x,y)_\alpha \alpha t_{n-1}(z) \alpha[t_{n-1}(u),t_{n-1}(v)]_\beta +$

$G_n(x,y)_\beta \alpha t_{n-1}(z) \alpha[t_{n-1}(u), t_{n-1}(v)]_\alpha + G_n(x ,y)_\beta \alpha t_{n-1}(z) \alpha[t_{n-1}(u),t_{n-1}(v)]_\beta = 0$

By (i) and (ii) , we have that

$G_n(x,y)_\alpha \alpha t_{n-1}(z) \alpha[t_{n-1}(u),t_{n-1}(v)]_\beta + G_n(x,y)_\beta \alpha t_{n-1}(z) \alpha[t_{n-1}(u),t_{n-1}(v)]_\alpha = 0$

Therefore, we have that

$G_n(x ,y)_\alpha \alpha t_{n-1}(z)\alpha[t_{n-1}(u) , t_{n-1}(v)]_\beta \alpha t_{n-1}(z)\alpha G_n(x ,y)_\alpha \alpha t_{n-1}(z) \alpha [t_{n-1}(u) , t_{n-1}(v)]_\beta = 0$

$0 = - G_n(x ,y)_\alpha \alpha t_{n-1}(z)\alpha[t_{n-1}(u), t_{n-1}(v)]_\beta \alpha t_{n-1}(z)\alpha G_n (x ,y)_\beta \alpha t_{n-1}(z)\alpha[t_{n-1}(u) , t_{n-1}(v)]_\alpha$

Since M is a prime Γ-ring , we get

$G_n(x,y)_\alpha \alpha t_{n-1}(z)\alpha[t_{n-1}(u), t_{n-1}(v)]_\beta = 0$, for all $x , y , z , u , v \in M$,$\alpha \in \Gamma$ and $n \in N$

Theorem (2.10)

Every Jordan higher reverse left (resp. right) centralizer of a 2-torsion free prime Γ-ring M is a higher reverse left (resp. right) centralizer of M .

Proof

Let $t = (t_i)_{i \in N}$ be a Jordan higher reverse left (resp. right) centralizer of a prime Γ-ring M .

Since M is a prime Γ-ring , then by employing Lemma (2.9)(i) , we have that either

$G_n(x ,y)_\alpha = 0$ or $[t_{n-1}(u) , t_{n-1}(v)]_\alpha = 0$, for all $x , y , u , v \in M$, $\alpha \in \Gamma$ and $n \in N$.

If $[t_{n-1}(u) , t_{n-1}(v)]_\alpha \neq 0$, for all $u , v \in M$, $\alpha \in \Gamma$, then $G_n(x ,y)_\alpha = 0$, for all $x , y \in M$ and $n \in N$. Hence , using Remark (2.7) , we obtain that t is a higher reverse left (resp. right) centralizer of M .

If $[t_{n-1}(u) , t_{n-1}(v)]_\alpha = 0$, for all $u , v \in M$ and $n \in N$, then M is a commutative Γ-ring .

By utilizing Lemma (2.4) (i) , we have that

$$t_n(x\,\alpha\,y + x\,\alpha\,y) = 2t_n(x\,\alpha\,y)$$
$$= 2\sum_{i=1}^{n} t_i(y)\alpha\ t_{i-1}(x)$$

Since M is a 2-torsion free Γ-ring , we get that

$$t_n(x\,\alpha\,y) = \sum_{i=1}^{n} t_i(y)\alpha\, t_{i-1}(x)$$

Then t is a higher reverse left (resp. right) centralizer of M .

Proposition (2.11)

Let $t = (t_i)_{i\in N}$ be a Jordan higher reverse left (resp. right) centralizer of a 2-torsion free Γ-ring M , such that $x\,\alpha\,y\,\beta\,x = x\,\beta\,y\,\alpha\,x$, for all $x ,y \in M$ and $\alpha , \beta \in \Gamma$. Then t is a Jordan triple higher reverse left (resp. right) centralizer of M .

Proof

By the substitution of y for $x\,\beta\,y + y\,\beta\,x$ in Lemma (2.4)(i) , we have that

$$= \sum_{i=1}^{n} t_i(y)\ \beta\ t_{i-1}(x)\alpha\ t_{i-1}(x) + t_i(x)\ \beta\ t_{i-1}(y)\ \alpha\ t_{i-1}(x) +$$
$$t_i(x)\ \alpha\ t_{i-1}(y)\ \beta\ t_{i-1}(x) + t_i(x)\ \alpha\ t_{i-1}(x)\ \beta\ t_{i-1}(y)\ldots(1)$$

Moreover , we get that

$$t_n(x\,\alpha\,(x\,\beta\,y + y\,\beta\,x) + (x\,\beta\,y + y\,\beta\,x)\,\alpha\,x)$$
$$= t_n(x\,\alpha\,x\,\beta\,y) + t_n(y\,\beta\,x\,\alpha\,x) + t_n(x\,\alpha\,y\,\beta\,x + x\,\beta\,y\,\alpha\,x)$$
$$= \sum_{i=1}^{n} t_i(y)\ \beta\ t_{i-1}(x)\alpha\ t_{i-1}(x) + t_i(x)\ \alpha\ t_{i-1}(x)\ \beta\, t_{i-1}(y) +$$
$$t_n(x\,\alpha\,y\,\beta\,x + x\,\beta\,y\,\alpha\,x)\ldots(2)$$

By comparing equations (1),(2) and the fact that $x\alpha y\beta x = x\beta y\alpha x$, for all $x , y \in M$ and $\alpha , \beta \in \Gamma$, we have that

$$t_n(x\,\alpha\,y\,\beta\,x + x\,\alpha\,y\,\beta\,x) = 2t_n(x\,\alpha\,y\,\beta\,x)$$
$$= 2\sum_{i=1}^{n} t_i(x)\beta\, t_{i-1}(y)\alpha\, t_{i-1}(x)$$

Since M is a 2-torsion free Γ-ring, we get the required result .

Lemma (2.12)

Let M be a semiprime Γ- ring and $t=(t_i)_{i\in N}$ be a higher reverse left (resp. right) centralizer of M if $t^2_n = 0$. Then $t_n = 0$, for all $n \in N$

Proof

Since $t^2_n = 0$

$$\sum_{i=1}^{n} t_i(t_i(x\alpha x)) = 0$$

$$\sum_{i=1}^{n} t_i(t_i(x)\ \alpha\ t_{i-1}(x)) = 0$$

$$\sum_{i=1}^{n} t_i(t_{i-1}(x))\ \alpha\ t_{i-1}(t_i(x)) = 0$$

Replace $t_{i-1}(x)$ by x , we have that

$$\sum_{i=1}^{n} t_i(x)\ \alpha\ t_{i-1}(t_i(x)) = 0$$

Right multiply by $\alpha\ t_i(x)$, we have that

$$\sum_{i=1}^{n} t_i(x)\ \alpha\ t_{i-1}(t_i(x))\ \alpha\ t_i(x) = 0$$

Since M is a semiprime Γ- ring , we have that

$$\sum_{i=1}^{n} t_i(x) = 0 \text{ , for all } x \in M \Rightarrow t_n = 0 \text{ , for all } n \in N .$$

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