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Essential-Small M-Projective Modules

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Abstract

In this paper, we introduce the concept of e-small M-Projective modules as a generalization of M-Projective modules.

Keywords: e-small M-projective modules, M-projective modules

مقاسات جوهرية صغيرة اسقاطية من النمط–M

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الخلاصة

في هذا البحث قدمنا مفهوم مقاسات جوهرية صغيرة اسقاطية من النمط–M كأعمام لمفهوم المقاسات الأسقاطية من النمط–M .

1. Introduction

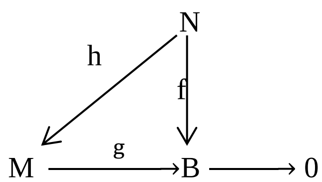
In this paper, all rings are associative and all modules are right and unitary. A submodule L of a module A is called small (for short $L \ll A$) if L+K=A then K=A, for any submodule K of A, Rad(M) is the sum of all small submodules of A. A module A is called B-projective if for each epimorphism $g : B \longrightarrow N$ and each homomorphism $f : A \longrightarrow N$, there exists a homomorphism $h : A \longrightarrow B$ such that $g \circ h = f$. For the previous terminologies see [1]. A submodule N of A is called e-small in A (denoted by $N \ll_e A$) if N+L=A with L is essential submodule in A implies that N=A; $Rad_e(A)$ is the sum of all e-small submodules of A. An epimorphism with e-small kernel is called e-small epimorphism [2]. In an indecomposable module, A proper submodule is e-small if and only if it is small [3]. A module M is said to be e-hollow if every proper submodule N of M is e-small [4].

2. e-small M-Projective

In this section, we introduce the concept of e-small M-projective modules and give some characterization of this concept.

Definition 2.1

A module N is called e-small M-projective, if there is a homomorphism h such that the following diagram commute.



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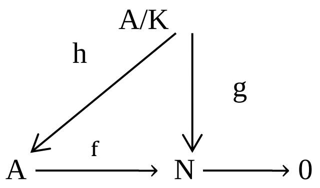
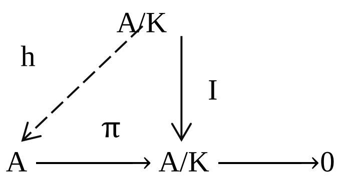
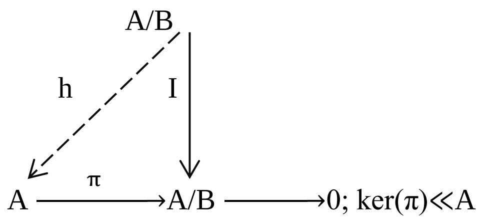
where g is an e-small epimorphism and f is a homomorphism.
Clearly an M-projective module is e-small M-projective, but the reverse is not true in general.
Proposition 2.2 In an indecomposable module A the following are equivalent:
a. Rad(A)=(0).
b. A/K is an e-small A-projective module, where K is a nonzero proper submodule of A. Furthermore A/K can't be A-projective module.
Proof a $\longrightarrow$ b Let Rad(A)=(0) and K is a nonzero proper submodule of A, consider the following diagram:



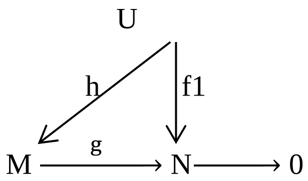
where f: A $\longrightarrow$ N is an e-small epimorphism and g :A/K $\longrightarrow$N is a homomorphism. Since A is an indecomposable module. Then $Rad_e(A)$ = Rad(A). By using (a) we get $Rad_e(A)=0$. Then ker(f)=0, hence f is an isomorphism. Define h: A/K $\longrightarrow$ A by $h=f^{-1}\circ g$. So $f\circ h=f\circ f^{-1}\circ g=I_N\circ g=g$. Thus A/K is e-small A-projective which is not A-projective, since if A/K is A-projective, then we have the following commutative diagram:



Where $\pi\circ h=I$, thus π is split, therefore A=K⊕Im(h) , which is a contradiction.
b $\longrightarrow$ a Let B be a nonzero small submodule of A, by (b) we have the following commutative diagram:



Thus π is split, therefore A=B⊕Im(h), but B≪A, therefore A=Im(h) which means that B=(0), so Rad(A)=(0).
Example 2.3
Z as Z-module is indecomposable module and Rad(Z) =0, by (2.2) Z_n is e-small Z-projective which is not Z-projective for each integer n >1.
Proposition 2.4
Let U and M be modules. Then the following statements are equivalent:
a. U is an e-small M-projective module;
b. For every short exact sequence $0\longrightarrow K\xrightarrow{f} M\xrightarrow{g} N\longrightarrow 0$, where g is e-small epimorphism, the sequence
$0\longrightarrow Hom(U, K)\xrightarrow{Hom(I,f)} Hom(U, M)\xrightarrow{Hom(I,g)} Hom(U, N)\longrightarrow 0$
is short exact;
c. For every e-small submodule K of M, every homomorphism
h: U $\longrightarrow$ M/K factor through the epimorphism π: M $\longrightarrow$ M/K.
Proof a $\longrightarrow$ b) It is enough to show that, Hom(I,g) is an epimorphism.
Let f_1 ∈ Hom(U, N) and consider the following diagram:



Since g is an e-small epimorphism and U is an e-small M-projective module, there exists a homomorphism h:U ⟶M such that g∘h=f_1.

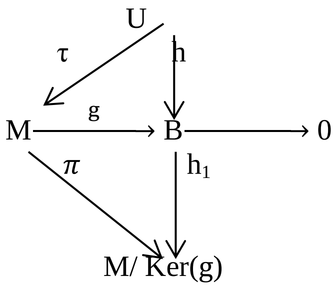
b ⟶ c) Let K be an e-small submodule of M and let h:U ⟶ M/K be an epimorphism. Consider the following exact sequence:

$0 \longrightarrow K \xrightarrow{i} M \xrightarrow{\pi} M/K \longrightarrow 0$

where i is the inclusion homomorphism and π is the natural epimorphism.

By (b) the Hom(I,π):Hom(U, M) ⟶Hom(U, M/K) is an epimorphism. This implies, the existence of a homomorphism f ∈ Hom(U,M) such that h= Hom(I,π)(f) = π∘f.

c ⟶ a) Let g: M ⟶B be an e-small epimorphism and let h:U⟶ B be any homomorphism. Consider the following diagram:



where π: M ⟶ M/Ker(g) is the natural epimorphism and h_1:B⟶M/Ker(g) is the usual isomorphism. By (c), there exists a homomorphism τ : U ⟶M such that π∘τ = h_1∘h. One can easily check that h_1∘g = π. Now, h_1∘g∘τ = π∘τ = h_1∘h. Thus g∘τ = h since h_1 is an isomorphism.

Definition 2.5

Let A and B be modules and Ψ: A ⟶ B be an e-small epimorphism, Ψ is called e-small M-epimorphism, if there exists a homomorphism h: A ⟶ M, such that Ker(Ψ)∩Ker(h)=(0).

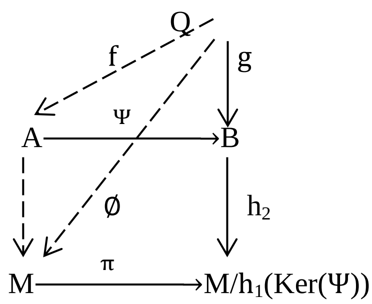
Proposition 2.6

Let M, Q be modules, then the following are equivalent:

a. Q is e-small M-projective.

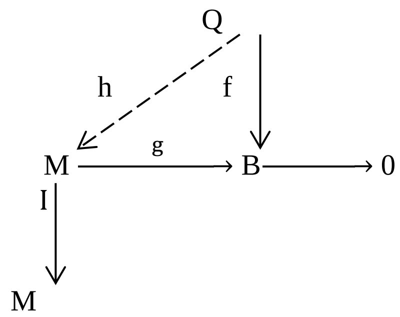
b. Given any e-small M-epimmorphism Ψ: A ⟶ B and a homomorphism g: Q ⟶B, there exists f: Q ⟶ A such that Ψ ∘ f = g.

Proof a⟶b) Let Ψ: A ⟶ B be an e-small M-epimorphism and let g: Q ⟶ B be a homomorphism, consider the following diagram:



Since Ψ is e-small M-epimorphism, so, there exists a homomorphism h_1: A ⟶M such that Ker(Ψ)∩Ker(h_1)=(0), where π:M⟶M/h_1(Ker(Ψ)) is the natural epimorphism.

Define h_2:B⟶M/h_1(Ker(Ψ)) by h_2(b)=h_1(a)+h_1(Ker(Ψ)) Where Ψ(a) =b, for all b ∈ B. h_2 is homomorphism, by e-small M-projective of Q, there exists a homomorphism ∅:Q⟶M such that π ∘ ∅=h_2∘ g. Define f: Q ⟶A by f(x) = a + a_1, a ∈ A & a_1 ∈ Ker(Ψ).
So (Ψ ∘ f)(x) = Ψ(f(x)) = Ψ ($a+a_1$) = Ψ(a) = g(x), thus Ψ ∘ f = g.
b⟶a) Let g: M ⟶ B be an e-small epimophism and let f:Q ⟶ B be any homomorphism. Consider the following diagram:



g is an e-small M-epimorphism, since there exists the identity I: M ⟶ M such that Ker(I)∩Ker(g) = (0).
By (b), there exists a homomorphism h:Q ⟶M such that g ∘ h=f.

Corollary 2.7

Let Q, M be modules. If Q is an e-small M-projective. Then any e-small M-epimorphism g:A ⟶ Q splits, where A is R-module. Moreover if A is indecomposable then g is isomorphism.

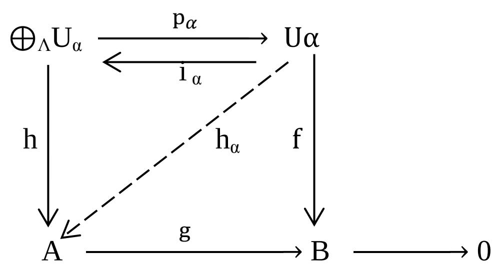
3. Some properties of e-small M-projective Modules

In this section we give some basic properties of e-small M-projective module

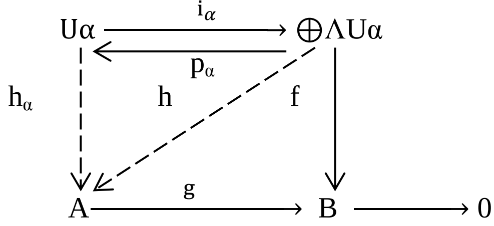
Proposition 3.1

Let A be a module and {U_α | ∝ ∈ Λ} be a family of modules. Then $\bigoplus_\Lambda U_\alpha$ is an e-samll A-projective if and only if every Uα is an e-small A-projective.

Proof ⟹) Let $\bigoplus_\Lambda U_\alpha$ be an e-small A-projective and let $\alpha \in \Lambda$ consider the following diagram:



Where g: A ⟶ B is an e-small epimorphism, f: U_α ⟶ B is any homomorphism, p_α and i_α are the projection and injection homomorphisms respectively. Since $\bigoplus_\Lambda U_\alpha$ is e-small A-projective, then there exists a homomorphism h: $\bigoplus_\Lambda U_\alpha$ ⟶A such that g∘h=f∘p_α.
Define h_α : U_α⟶A by h_α= h∘i_α. So g∘h_α =g∘h∘i_α=f∘p_α∘i_α=f∘I=f.
⟸) Let g : A⟶B be an e-small epimorphism and let
f :$\bigoplus_\Lambda U_\alpha$ ⟶B be a homomorphism. For each α ϵ Λ, consider the following diagram:



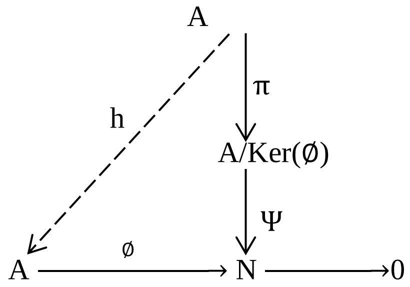
Where i_α and p_α are the injection and projection homomorphism, since U_α is e-small A-projective, for each α ∈ Λ. Therefore there exists a homomorphism h_α: U_α⟶A, such that g∘h_α=f∘i_α for

each $\alpha \in \Lambda$.
Define h: $\bigoplus_{\Lambda} U_{\alpha} \longrightarrow A$ by $h(a) = \sum_{\alpha\in\Lambda} h_{\alpha} \circ p_{\alpha}(a)$, where $a \in A$, clearly $g\circ h=f$. Hence $\bigoplus_{\Lambda} U_{\alpha}$ is an e-small A-projective module.

Proposition 3.2

Let A be e-small A-projective module and let $\emptyset: A \longrightarrow N$be an e-small epimorphism, then there exists $h \in End(A)$ such that $h(Ker(\emptyset)) \leq Ker(\emptyset)$.

Proof : Let $\emptyset: A \longrightarrow N$ be an e-small epimorphism, consider the following diagram :

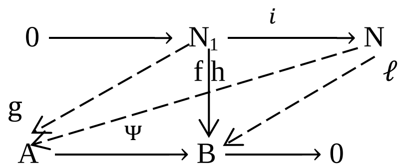


Where $\Psi : A/Ker(\emptyset) \longrightarrow N$ is the usual isomorphism defined by $\Psi(m+ker(\emptyset)) = \emptyset(m)$ for all $m \in A$ and π is the natural epimorphism. Since A is e-small A-projective module, there exists a homomorphism $h: A \longrightarrow A$ such that $\emptyset\circ h = \Psi\circ\pi$. Now it is easy to show that $h(Ker(\emptyset)) \leq Ker(\emptyset)$.
Recall that a submodule B of A is called A-cyclic submodule if it is the image of an element of End(A) [5].

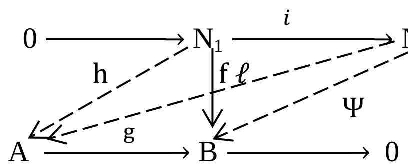
Proposition 3.3

Let A, N be modules. If N is an e-small A-projective and every A-cyclic submodule of A is N-injective, then A is N-injective and every submodule of N is an e-small A-projective. The converse is true if A is e-hollow module.

Proof : Let N be an e-small A-projective and suppose that every A-cyclic submodule is N-injective. Since A is trivially A-cyclic, then A is N-injective. Let $\Psi: A \longrightarrow B$ be a homomorphism, where N_1 is a submodule of N. consider the following diagram:



Where $i: N_1 \longrightarrow N$ is the inclusion homomorphism since B is A-cyclic module, thus by our hypothesis B is N-injective module. Therefore, there exists a homomorphism $\ell: N \longrightarrow B$ such that $\ell \circ i=f$, but N is an e-small A-projective module, so there exists a homomorphism $h : N_1 \longrightarrow A$ such that $\Psi \circ h = \ell$. Define $g: N_1 \longrightarrow A$ by $g = h\circ i$. Now, $\Psi \circ h \circ i = \ell \circ i=f$.
The converse holds if A is e-hollow. Suppose that A is N-injective and every submodule of N is an e-small A-projective. Thus N is an e-small A-projective module. Let B be A-cyclic submodule of A. Consider the following diagram:



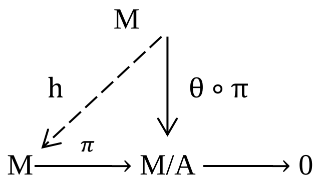
Where $i: N_1 \longrightarrow N$ is the inclusion homomorphism and $f: N_1 \longrightarrow B$ is any homomorphism and $g: A \longrightarrow B$ is the required epimorphism into B, since B is A-cyclic module. Clear that g is an e-small epimorphism. By the assumption, N_1 is an e-small A-projective module. Thus, there exists homomorphism $h: N_1 \longrightarrow A$ such that $g\circ h=f$, but A is N-injective, so there exists a homomorphism ℓ:N⟶A such that $\ell \circ i$=h. Define Ψ:N⟶B by Ψ = g ∘ ℓ. Now, Ψ ∘ i = g ∘ $\ell \circ i$ =g ∘ h=f.

Recall that A submodule N of a module A is called small pseudo stable, if for any epimorphism f:A⟶M, and any small epimorphism g:A⟶M, with N ≤ ker (g)∩ker(f), there exists h ∈ End(A) such that f = g ∘ h, then h(N) ≤ N [6].

Proposition 3.4

If K is a small pseudo stable submodule of a module M, where M is e-small M-projective and A ≪ K, then K/A is a small pseudo submodule of M/A.

Proof : Let f:M/A ⟶B be an e-small epimorphism and let g:M/A⟶B be an epimorphism with K/A ≤ ker(f)∩ker(g), there exists θ ∈ End(M/A) such that g = f ∘ θ.Let π:M ⟶ M/A be the natural epimorphism. Consider the following diagram:



Since M is e-small M-projective, there exists a homomorphism h:M ⟶M, such that π ∘ h= θ ∘ π. Now g ∘ π =f ∘ θ ∘ π = f ∘ π ∘ h .

Hence g ∘ π(K) = g(K/A)=0 and f ∘ π(K) = f(K/A) = 0.

Therefore K ≤ Ker(g ∘ π)∩ker(f ∘ π), but K is a small pseudo stable submodule of M. Hence h(K) ≤ K. Now, θ(K/A) = θ ∘ π(K) = π ∘ h(K) ≤ π(K) = K/A.

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