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On Semiannahilator Supplement Submodules

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Abstract
Let R be associative ring with an identity and let *D* be unitary left R- module . In this work we present semiannihilator supplement submodule as a generalization of R-a- supplement submodule, Let U and V be submodules of an R-module D if D=U+V and whenever Y≤ *V* and D=U+Y, then annY≪R . We also introduce the the concept of semiannihilator -supplemented modules and semiannihilator weak supplemented modules, and we give some basic properties of this conseptes.

Keywords- sa- Small Submodule, sa-Hollow And sa -Lifting Modules , sa -supplemented Modules

المقاسات الجزئيه شبه التالفه المكمله

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الخلاصه
لتكن R حلقه تجميعيه ذات عنصر محايد وD مقاس محايد ايسر. المقاس الجزئيN منD يدعى شبه تالف صغيرفي حاله تالف Kصغير فيR حيث K مقاس جزئى من المقاس Dعندما K+V=D.لتكنV وU مقاسات جزئيه من D .نقول ان Vشبه تالف مكمل للمقاس الجزئي U في المقاس D اذا U+V=D كذلك وعندما V<Y وU+Y=D يكون شبه تالفY صغير فيR . الغرض الاساسي من البحث هو تطوير الخواص الاساسيه للمقاسات شبه التالفه المكمله والمقاسات شبه التالفه المكمله الضعيفه.

Introduction

Throughout this paper all rings are associative ring with identity and modules are unitary left modules, D is named a hollow module if every proper submodule is small in D, where *a* submodule *B* of R-module D is named small in D (N << M) if B +K ≠ D for each proper submodule K of D . *A* proper submodule B of R-module D is named an essential in D, if for every non-zero submodule K of M then N∩K ≠0 [1]. The concept of small submodule has been generalized by some researchers, for this see [2,3]. the authors in [4] introduced the concept of R-annihilator small submodules, that is; a submodule B of an R-module *D* is called R-annihilator small, if whenever B+K=D, where K a submodule of D ,implies that ann(K)=0, where ann(K) ={r R: r.K=0}. In [5] sahira introduce the concept of semiannihilator small submodules, in case ann(K) << R where K is a submodule of D whenever B+ K =D. Clear that every R-annihilator small submodule is semiannihilator small, but the convers is not true [5] . Recall that a submodule V of M is called a supplement of U in M. If V is a minimal element in the set of submodules L of M with U+L=M. Let

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M be an R-module, M is called a supplemented module if every submodule of M has a supplement in M. Let U, V be submodules of an R-module M. If M=U+V and U∩V≪ M, then V is called a weak supplement of U in M. Let M be an R-module if every submodule of M has a weak supplement in M, then M is called a weakly supplemented module[6]. In this work we present semiannihilator - supplement submodule , Let U and V be submodules of an R-module D if D=U+V and whenever Y≤ V and D=U+Y, then annY≪R . We also introduce the the concept of semiannihilator supplemented modules and semiannihilator weak supplemented modules, and we give some basic properties of this conseptes.

In section two we introduce the notion of semiannihilator lifting modules and discus some characteristics of this kind of modules.. In part three; we introduce the concept of semiannihilator supplement submodule and basic properties. We show that . If D and D′ be R-modules and let *f*: D→D′ be an epimorphism, if D′ is semiannihilator supplemented module then D is semiannihilator supplemented module. In part four , the concept of semiannihilator weak supplement submodules with some examples and basic properties was introduced .The below lemma award the characteristics of semiannihilator small submodules.

Lemma[5]:

1- Let D be an R-module with submodules A,*N*.such that A⊆*N*. If N ≪sa D then A ≪sa D.

2- Let D be an R-module with submodules A,*N*.such that A⊆*N*, if *A ≪sa N* then A ≪sa D.

3-Let D_1 ,D_2 be an *R*- modules . If N_1≪sa $D_1$and N_2 ≪sa D_1thus N_1⊕N_2 ≪sa D_1 ⊕D_2.

4- let D and S be an *R*- modules and f : D→S be an epimorphism . If H ≪sa S, then f^{-1}(H) ≪sa D.

1. Semiannihilator lifting modules.

An R-module Dis called lifting if for any submodule N of D there exist submodule K of N such that D=K⊕ K′ with K′≤ M and N∩ K′≪ K. In this part we introduce the notion of semiannihilator lifting modules as generalization of R-Annihilator Lifting modules and discus some properties of this kind *of* modules.

Definition 1.1 : An R_1-module D is named semiannihilator lifting (sa- lifting) if for any submodule B of D , there exist submodules *K* , *K′* of D such that D= *K* ⊕ *K′* with *K* ⊆ B and B⋂ *K′* ≪sa *K*

The below theorem given a characterization of following semiannihilator lifting modules.

Theorem 1. For an R-module D the statement are equivalent :

1) *M* is sa- lifting

2) Every submodule N of D, N can be written as N= A⊕ B where A is direct summand of D and B ≪sa D.

3) For every submodule N of D, there exist a direct summand *K* of D s.t *K* ≤ N and N/K≪ sa D/ *K*.

Proof : See proof of lemma2 in [7].

A nontrivial R-module D is called semiannihilator –hollow *(*sa-hollow*)* if every proper submodule of D is sa-small in *M* [5].

Examples 1.2 :

1- Z as Z- module is sa- lifting module but it is not lifting.

2- Z_6 and Z_4 as Z- module are not sa- lifting module.

Not :Every sa- hollow is sa-lifting

Proof: - For a submodule B of D if B ≠ D, B≪sa D, B = (0) ⊕B the result go after, directly by theorem 1

Proposition 1. 3 : Let D be indecomposable module then D is sa- hollow module if and only if D is sa- lifting.

Proof: Let D be sa-hollow then D is sa- lifting ,Conversely suppose that D is sa-lifting and A a proper submodule of D, by Theorem.1,We have A= N⊕K where N is a direct summand in D and K≪sa D, but D is indecomposable . Then either N = (0) or N=D then D=N⊆ A which attend that A=D which is contradiction, if N= (0), so A=K ≪sa D, and D is sa- hollow.

Proposition 1. 4 :Let D=H_1⊕H_2 be duo module .If H_1 and H_2 are sa-lifting modules, then D is sa –lifting module.

Proof: Let H_1 and H_2 be sa-lifting modules, and N submodule of D ,then N= (N∩ H_1) ⊕ (N∩ H_2) .For each i∈ {1,2},there exists a direct summand Ki of Hi ,such that Hi= Ki⊕Li with Di ⊆ N∩ Hi and N∩

Li ≪sa Li then , D= (K_1⊕ L_1) ⊕ (K_1⊕ L_2) = (K_1⊕ K_2) ⊕ (L_1 ⊕ L_2) ,we have (K_1⊕ K_2) ⊆N ,and N∩(L_1⊕ L_2) ≪sa (L_1⊕ L_2) thus D is sa- lifting module.

Corollary 1.5: Let D= H_1⊕ H_2 be a module such that R = ann(H_1) +ann(H_2).If H_1 and H_2 are sa-lifting modules ,then D is sa-lifting module.

Proposition 1.6 : Let D be a multiplication R –module, if D sa-lifting module , then R is sa-lifting ring .

Proof: Assume that D is sa-lifting module . Where I is an ideal in R. D is multiplication Then *N* = ID is a submodule of D, thus there exist submodules K and K' in M with K ⊆ *N*, D = K⊕ K' and (*N* ∩ K')) ≪sa D . D is a multiplication R- module, so there are ideals J and J' of R such that K = JD and K' = J'D. Since K ⊆ *N* then J ⊆ I. We have D =K ⊕K' = JD ⊕J'D = (J ⊕J')D implies that R = J⊕ J '.Now *N* ∩ K' = (ID ∩ J'D) ≪sa D and since (J∩J')D ⊆ ID ∩J'D it follows that (J∩J')D ≪sa D [5] and according to [3] we get [(J∩J')D:D] ≪sa R. But [(J∩J')D:D] = I ∩ J', then (I∩J') ≪sa R and R is sa-lifting ring .

2. Semiannihilator supplemente submodule.

In this section we present the definition of semiannihilator (sa-supplement) supplement class. Then, some basic properties of this class are presented. In addition, several examples are given to illustrate the results.

Definition 2.1: Let V and U be submodules of an R-module D. We say that V is semiannihilator supplement (sa-supplement) of U in D if D=U+V and whenever Y≤ *V* and D=U+Y, then annY ≪ R.

Let D be an R-module. We say that D is semiannihilator supplemented (sa-supplemented) module if every proper submodule of D has sa-supplement. Let R be a commutative ring and let *I* be an ideal of R. We say that R is sa-supplemented ring of R is sa-supplemented as an R-module.

The bellow proposition gives a charactrization of sa-supplement submodule

Proposition 2.2: For submodules U and V of an R-module D. Then V is sa-supplment of U if and only if D=U+*V* and U∩V ≪saV.

Proof: Let V be sa-supplement of U. To show that U∩V sa- small in *V*, let V= (U∩V) +Y. Now D= U+ V = U+ (U∩*V*) +Y= U+Y. But Y≤V, therefore annY ≪R_1 and U∩V ≪saV.

Conversely, Let D=U+V and U∩V ≪sa V . We want to show that V sa-supplement of U. Let Y≤V such that D=U+Y. By (Modular law) , V= (U∩V)+Y. But U∩V ≪sa V , therefore annY ≪ R. Then V is sa-supplement of D.

Examples and Remarks 2.3

1- sa-supplement submodule not supplement submodule to see that consider Z as *Z*-module. For every proper submodule n*Z* of *Z* , *Z*= *nZ*+*Z* and n*Z* ∩ *Z* = nZ≪sa*Z*. Then Z is sa-supplement of nZ. Thus every proper submodule of *Z* has sa-supplement. But it is known that every non trivial submodule of *Z* has no supplement in *Z*. Where Z is indecomposable and {0} is the only small submodule of *Z.*

2- A supplement submodule need not be sa-supplement submodule. For example, Let Z_4 as *Z*-module. Z_4 is a supplement of {0 ,2 }. And Z_4 is not sa-supplement of {0 ,2 },where {0 ,2 }∩ Z_4={0 ,2}is not sa-small in Z_4, since Z_4={0 ,2 } + Z_4 and ann Z_4={n ∈ Z ; n. Z_4 = 0}=4Z not small in Z.

3- Let D be an R-module. Then every sa-small submodule of D has sa-supplement in D. That is if , N be sa-small submodule of D. Then D =N+D and N∩D =N is sa-small submodule of D. Thus D is sa-supplement of N in D.

4-Let U and V be two submodules in an R-module D such that V sa-supplement in U. If D= W+V, where W a submodule in U, then V sa-supplement in W.

Proof: Since V is sa-supplment of U, then D=U+V and U∩V ≪sa V. Since W≤ U. Then W∩V ≪sa V, by prop. (2.4) in [5]. Thus V is sa- supplement of W in D.

Let D be an R_1-module its known that every direct summand of D has a supplment in D. But this is not true for sa-supplement as the below examples shows :

Example2.4: Let Z_6 as Z-module. and U={0 ,2 ,4 } ,V={0 ,3 }, Z_6= U⊕V. U and V are supplement of each others. But each of U and V has no sa-supplement in Z_6, where ann Z_6={n∈Z ; n. Z_6=0}=6Z not small in Z.. Hence Z_6 has no sa-small submodule [5]. Thus every submodule of Z_6 has no sa-supplement in Z_6.

Proposition2.5 : For a finitely generated D in R-module and For submodules U and V in D such that V sa-supplment of U in D. Thus there exists a finitly generated sa-supplement W in U *s.t* W≤ V.
Proof: Let $M=Rx_1+ Rx_2+ . . . +Rxn$, where xi∈D, ∀i=1,2,…,n. Since D=U+V, then xi = ui+ vi, where ui∈U, vi∈V, ∀i=1,2,…,n. Now let $W=Rv_1+Rv_2+ . . . +Rvn$. Clearly that D=U+W. Since V sa-supplment of U and W≤V, then annW ≪ R. is clear that W is sa-supplement of U.
Proposition 2.6: Let *D* and *N* be R_1-modules and let f: *D→N* an epimorphism, if *N* is sa-supplemented module, then *D* is sa-supplmented module.
Proof: For a submodule K of *D*, then f(K) submodule in *N*. Since *N* is sa-supplemented module. Then there exists a submodul L in N s.t N= f(K)+L and f(K)∩L is sa-small in L. $M=f^{-1}(N)=f^{-1}(f(K)+L)= f^{-1}(f(K))+f^{-1}(L)=K+\ker f+f^{-1}(L)=K+f^{-1}(L)$.Claim that $K\cap f^{-1}(L)$ is sa-small submodule of $f^{-1}(L)$. Since f(K) ∩L≪sa L, then by prop (2-7) in [3] , $f^{-1}(f(K) \cap L)$≪sa $f^{-1}(L)$ .But $f^{-1}(f(K) \cap L)=f^{-1}(f(K)) \cap f^{-1}(L)=(K+\ker f) \cap f^{-1}(L) =\ker f+(K\cap f^{-1}(L))$, by(Modular Law) . ker f $+(K\cap f^{-1}(L))$ ≪sa $f^{-1}(L)$. By Prop (2-4)in[5], $K\cap f^{-1}(L)$ ≪sa $f^{-1}(L)$.So $f^{-1}(L)$ is sa-supplment of *K* in *D*. Then *M* is sa- supplemented module.
Proposition 2.7: Let D be a finitely generated faithful multiplication modul over a commutative ring R and let I be an ideal of R. if ID has sa-supplment in D, so I has sa-supplment in R.
Proof: For an ideal I in R_1 such that ID has sa-supplement in D. Then there exists a submodul N in D s.t D=ID+N and ID∩N ≪saN. D a multiplication modul, thus N=JD, for some ideal J of D. Now $D=R_1$ D=ID+JD=(I+J)D. But D is finitely generated faithful multiplication modul and hence R=I+J[8]. ID∩N=ID∩JD=(I∩J)D ≪sa JD. To show that I∩J≪saJ. Let J=(I∩J)+L, where L an ideal of R. Then JD=((I∩J)+L)D=(I∩J)D+LD. Therefore ann (LD)≪R. ann L≤ ann LD. Thus ann L≪R and I∩J≪sa J. Then J is sa- supplement of I.

3. Semiannihilator weakly supplmented modules .

In this part, we introduce the definition of semiannihilator weakly supplmented modules. And we introduce some basic characterization of this modules .

Definition 3.1: For submodules U and V of an R-module D. We say that V semiannihilator -weak supplment (sa-weak supplement) of U in D if D=U+V and U∩V ≪sa D.
We say that D is semiannihilator weakly (sa-weakly) supplemented module if every submodule of D has sa-weak supplment in D.

Remarks and examples 3.4:
1. semiannihilator weak supplement submodule not be weak supplement submodule. For example, consider Z as Z module. Cleary that Z=2Z+3Z and 2Z∩3Z=6Z≪sa Z . Thus 3Z is sa-weak supplement of 2Z. But {0} is the only small submodule of Z and hence 3Z is not weak supplement of 2Z.
2. Every sa-supplemented module is sa-weakly supplemented module. To show that, let D be an sa-supplemented module and let U be a proper submodule of D , then there exists a submodule V of D, such that D=U+V and U∩V ≪saV. By prop (2-3) in [8] , U∩V ≪sa D. Hence V sa-weak supplement of U. Clearly that D=D+0 and D∩{0}=0≪sa D. So {0}is sa-weak supplement of D . Thus D is sa-weak-supplemented module.
3. sa-weak supplement submodule need not be sa-supplement submodule. For example, let D be a faithful R-module. Then D=D+0 and D∩{0}=0≪saD. Thus {0} is sa-weak supplement of D. Now D∩{0}=0 is not sa-small in 0, when 0=0+0 ann0=R not small in R.
4. Let X and Y be submodules of R_1-module D if X is sa-weak supplement of Y, then Y is sa-weak supplement of X, where D=X+Y and X∩Y ≪sa D.

Proposition 3.5:-For N , K and L submodules in an R-module D such that L≤ N. If K is sa-weak supplement of N and D=L+K, there K is sa-weak supplment of L in D.
Proof: Since K sa-weak supplement in N, then D=N+K and N∩K ≪sa D. Now D=L+K and L∩K≤ N∩K ≪sa D. Hence L∩K ≪sa D by prop (2-4) in [5]. Thus K is sa-weak supplement in L is D.
Proposition 3.6: For submodules N and L of a finitely generated R-module D. if L sa-weak supplment of N, then L contains a finitly generated sa-weak supplment of *N*.
Proof:- Let $D =Rx_1+Rx_2+. . . +Rx_n$, xi ∈ D, for some xi ∈ D, ∀ i=1,2,…,n.
Since M=N+L, then xi=ai+bi , where ai ∈ *N*, bi ∈ K, ∀ i=1,2,…, n. Now let
$L'=Rb_1 +Rb_2+ . . . +Rb_n$, D=N+K′. Clearly that *K′ ≤ K*. But N∩ L′=N∩L ≪sa D, therefore N∩ L'≪sa D, by prop (2-4) in [5].Thus L′ is a finitely generated sa-weak supplement of *N*.

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