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Intuitionistic Fuzzy Ideals of KU-Semigroups

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Abstract
In this work, we introduce an intuitionistic fuzzy ideal on a KU-semigroup as a generalization of the fuzzy ideal of a KU-semigroup. An intuitionistic fuzzy *k*-ideal and some related properties are studied. Also, a number of characteristics of the intuitionistic fuzzy *k*-ideals are discussed. Next, we introduce the concept of intuitionistic fuzzy *k*-ideals under homomorphism along with the Cartesian products.

Keywords: KU-Semigroup, Intuitionistic Fuzzy *K*-Ideal, Intuitionistic Fuzzy S-Ideal.

المثاليات الضبابية الحدسية لـ-KU شبه الزمر

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الخلاصة
في هذا العمل, نقدم المثالي الضبابي الحدسي في الجبر – KU وشبه الزمرة كتعميم للمثالي الضبابي للجبر – KU وشبه الزمرة. فقد تم دراسة المثالي الضبابي الحدسي– k وبعض الخصائص المرتبطة به . وايضا تم مناقشة بعض مميزات المثاليات الضبابي الحدسي –k . وكذلك قدمنا المثاليات الضبابية الحدسية – k تحت شرط التشاكل والضرب الديكارتي.

1. Introduction

A fuzzy set was introduced by Zadeh in 1956 [1]. Many papers studied this concept in different branches of mathematics, such as vector space, group theory, topological space, ring theory and module theory. An intuitionistic fuzzy set was exhibited by Atanassov [2] in 1986 as a generalization of the fuzzy set. An intuitionistic fuzzy set δ over a non-empty set $\aleph$ is a set that is defined by $\delta=\{(\chi,\mu_\delta(\chi),\beta_\delta(\chi)):\chi\in\aleph\}$, where $\mu_\delta:\aleph\to[0,1]$ is the degree of membership of each element χ in $\aleph$ to δ, while $\beta_\delta:\aleph\to[0,1]$ is the degree of non membership of each element χ in $\aleph$ to δ, such that $0\le\mu_\delta(\chi)+\beta_\delta(\chi)\le 1$, $\forall\chi\in\aleph$.

In another two joint articles [3, 4], Jun and Kim studied the intuitionistic fuzzy ideals of near rings and those on BCK-algebras. Shum and Akram [5] studied the intuitionistic (T, S)-fuzzy ideals of near rings. Many authors introduced intuitionistic fuzzy sets by different ways and applied them in many structures [6, 7, 8]. In this paper, an intuitionistic fuzzy *k*-ideal in a KU-semigroup is studied and some important properties of the intuitionistic fuzzy *k*-ideals are discussed. By applying a homomorphism, we could prove some results about the intuitionistic fuzzy *k*-ideal.

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2. Basic concepts

Some basic concepts that are necessary for the main part of the paper are included in this section.

Definition 2.1 [9,10]. A KU-algebra $(\aleph,*,0)$ is a set $\aleph$ with a binary operation $*$ and a constant 0, and it satisfies the following conditions, for all $\chi,\gamma,\tau\in\aleph$

(ku_1) $(\chi*\gamma)*[(\gamma*\tau))*(\chi*\tau)]=0$,

(ku_2) $\chi*0=0$,

(ku_3) $0*\chi=\chi$,

(ku_4) $\chi*\gamma=0$ and $\gamma*\chi=0$ implies $\chi=\gamma$ and

(ku_5) $\chi*\chi=0$.

A binary relation $\leq$ on $\aleph$ is defined by $\chi\leq\gamma\Leftrightarrow\gamma*\chi=0$. It follows that $(\aleph,\leq)$ is a partially ordered set. Then $(\aleph,*,0)$ satisfies the following statements

For all $\chi,\gamma,\tau\in\aleph$,

$(ku_{1'})$ $(\gamma*\tau)*(\chi*\tau)\leq(\chi*\gamma)$,

$(ku_{2'})$ $0\leq\chi$,

$(ku_{3'})$ $\chi\leq\gamma,\gamma\leq\chi$ implies $\chi=\gamma$

$(ku_{4'})$ $\gamma*\chi\leq\chi$.

Theorem 2.2 [11] The following axioms are fulfilled in a KU-algebra $(\aleph,*,0)$, for all $\chi,\gamma,\tau\in\aleph$,

(1) $\chi\leq\gamma$ implies $\gamma*\tau\leq\chi*\tau$,

(2) $\chi*(\gamma*\tau)=\gamma*(\chi*\tau)$,

(3) $((\gamma*\chi)*\chi)\leq\gamma$.

Example 2.3 [10] Let $\aleph=\{0,a,b,c\}$ be a set and $*$ a binary operation defined by the following

$*$	0	a	b	c
0	0	a	b	c
a	0	0	0	b
b	0	b	0	a
c	0	0	0	0

Then $(\aleph,*,0)$ is a KU-algebra.

Definition 2.4 [9] Let $(\aleph,*,0)$ be a KU-algebra and $\phi\neq I\subseteq\aleph$, then I is called an ideal of $\aleph$ if for any $\chi,\gamma\in\aleph$

(1) $0\in I$,

(2) if $\chi*\gamma\in I$, $\chi\in I$ implies $\gamma\in I$.

Definition 2.5[11]. Let $(\aleph,*,0)$ be a KU-algebra and $\phi\neq I\subseteq\aleph$, then I is named a KU-ideal if for any $\chi,\gamma,\tau\in\aleph$, then

(I_1) $0\in I$ and

(I_2) $\forall\chi,\gamma,\tau\in\aleph$, if $(\chi*(\gamma*\tau))\in I$ and $\gamma\in I$ imply $\chi*\tau\in I$.

Definition 2.6[12] A KU-semigroup is a non empty set $\aleph$ with two binary operations $*,\circ$ and a constant 0 satisfying the following

(I) $(\aleph,*,0)$ is a KU-algebra,

(II) $(\aleph,\circ)$ is a semigroup,

(III) $\chi\circ(\gamma*\tau)=(\chi\circ\gamma)*(\chi\circ\tau)$ and $(\chi*\gamma)\circ\tau=(\chi\circ\tau)*(\gamma\circ\tau)$, for all $\chi,\gamma,\tau\in X$.

Example 2.7[12]: Let $\aleph=\{0,1,2,3\}$ be a set. Define $*$-operation and $\circ$-operation by the following tables

$*$	0	1	2	3
0	0	1	2	3
1	0	0	0	2
2	0	2	0	1
3	0	0	0	0

$\circ$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

Then $(\aleph,*,\circ,0)$ is a KU-semigroup.

Definition 2.8[12] A nonempty subset A of $\aleph$ is called a *sub* KU-*semigroup* of $\aleph$, if $\chi*\gamma,\chi\circ\gamma\in A$, for all $\chi,\gamma\in A$.

Definition 2.9[12] Let $(\aleph,*,\circ,0)$ be a KU-semigroup and $\phi\neq A\subseteq\aleph$, then A is said to be an *S-ideal* of $\aleph$, if

1) A nonempty subset A is an ideal of a KU-algebra $(\aleph,*,0)$,
2) For all $\chi\in\aleph$, $a\in A$, we have $\chi\circ a\in A$ and $a\circ\chi\in A$.

Definition 2.10[12] Let $(\aleph,*,\circ,0)$ be a KU-semigroup and A be non-empty subset of $\aleph$. A is said to be a *k-ideal* of $\aleph$, if

i) A is an KU-ideal of a KU-algebra $(\aleph,*,0)$,
ii) For all $\chi\in X$, $a\in A$, we have $\chi\circ a\in A$ and $a\circ\chi\in A$.

Definition 2.11[12] Let $\aleph$ and $\aleph'$ be two KU-semigroups. A mapping $f:\aleph\to\aleph'$ is called a KU-semigroup homomorphism if $f(\chi*\gamma)=f(\chi)*f(\gamma)$ and $f(\chi\circ\gamma)=f(\chi)\circ f(\gamma)$ for all $\chi,\gamma\in\aleph$.

We review some concepts of fuzzy logic.

Let $\aleph$ be the collection of objects, then a fuzzy set in $\aleph$ is defined by $\mu:\aleph\to[0,1]$, where $\mu(\chi)$ is called the membership value of χ in $\aleph$ and $0\leq\mu(\chi)\leq1$. The set $U(\mu,t)=\{\chi\in\aleph:\mu(\chi)\geq t\}$, where $0\leq t\leq1$ is said to be a level set of $\mu(\chi)$.

Definition 2.12[13] Let $\mu(\chi)$ be a fuzzy set in $\aleph$, then $\mu(\chi)$ is called a fuzzy sub KU-semigroup of $\aleph$ if it satisfies the following condition : for all $\chi,\gamma\in\aleph$.

i) $\mu(\chi*\gamma)\geq\min\{\mu(\chi),\mu(\gamma)\}$
ii) $\mu(\chi\circ\gamma)\geq\min\{\mu(\chi),\mu(\gamma)\}$.

Definition 2.13[13] A fuzzy set $\mu(\chi)$ in $\aleph$ is called a fuzzy S-ideal of $\aleph$ if , for all $\chi,\gamma\in\aleph$

(S_1) $\mu(0)\geq\mu(\chi)$,
(S_2) $\mu(\gamma)\geq\min\{\mu(\chi*\gamma),\mu(\chi)\}$
(S_3) $\mu(\chi\circ\gamma)\geq\min\{\mu(\chi),\mu(\gamma)\}$.

Definition 2.14[13] A fuzzy set $\mu(\chi)$ in $\aleph$ is called a fuzzy k-ideal, if it satisfies the following condition: for all $\chi,\gamma,\tau\in\aleph$

(k_1) $\mu(0)\geq\mu(\chi)$,
(k_2) $\mu(\chi*\tau)\geq\min\{\mu(\chi*(\gamma*\tau)),\mu(\gamma)\}$,
(k_3) $\mu(\chi\circ\gamma)\geq\min\{\mu(\chi),\mu(\gamma)\}$.

Example 2.15[13] Let $\aleph=\{0,a,b,c,d\}$ be a set. Define $*$- operation and $\circ$- operation by the following tables

*	0	a	b	c	d
0	0	a	b	c	d
a	0	0	b	c	d
b	0	a	0	c	d
c	0	a	0	0	d
d	0	0	0	0	0

∘	0	a	b	c	d
0	0	0	0	0	0
a	0	0	0	0	0
b	0	0	0	0	b
c	0	0	0	b	c
d	0	a	b	c	d

Then $(\aleph,*,\circ,0)$ is a KU-semigroup. Define a fuzzy set $\mu:\aleph\to[0,1]$ by $\mu(0)=\mu(a)=0.4$, $\mu(b)=\mu(c)=0.2$, $\mu(d)=0.1$. Then it is easy to see that $\mu(\chi),\forall\chi\in\aleph$ is a fuzzy *k*-ideal.

3- Study of Intuitionistic Fuzzy Ideals of a KU-semigroup

In this section, we study an intuitionistic fuzzy ideal and some related properties.

Definition 3.1. Let δ_1 and δ_2 be two intuitionistic fuzzy sets of any set $\aleph$, we can define the following operations:

(1) $\delta_1\subseteq\delta_2$ if and only if $\mu_{\delta_1}(\chi)\le\mu_{\delta_2}(\chi)$ and $\beta_{\delta_1}(\chi)\ge\beta_{\delta_2}(\chi)$, $\forall\chi\in\aleph$,

(2) $\delta^c=\{(\chi,\beta_\delta(\chi),\mu_\delta(\chi)):\chi\in\aleph\}$,

(3) $\delta_1\cap\delta_2=\{(\chi,\min\{\mu_{\delta_1}(\chi),\mu_{\delta_2}(\chi)\},\max\{\beta_{\delta_1}(\chi),\beta_{\delta_2}(\chi)\}:\chi\in\aleph\}$

(4) $\delta_1\cup\delta_2=\{(\chi,\max\{\mu_{\delta_1}(\chi),\mu_{\delta_2}(\chi)\},\min\{\beta_{\delta_1}(\chi),\beta_{\delta_2}(\chi)\}:\chi\in\aleph\}$.

Hasan and Kareem [13] defined the fuzzy *KU*-semigroup and, by extending this idea, we define the intuitionistic fuzzy set (brevity IFS) in a *KU*-semigroup as follows

Definition 3.2 An (IFS) $\delta=(\chi,\mu_\delta,\beta_\delta)$ of $\aleph$ is called an IF- sub KU-semigroup of $\aleph$ if $\forall\chi,\gamma\in\aleph$

(i) $\mu_\delta(\chi*\gamma)\ge\min\{\mu_\delta(\chi),\mu_\delta(\gamma)\}$ and $\beta_\delta(\chi*\gamma)\le\max\{\beta_\delta(\chi),\beta_\delta(\gamma)\}$

(ii) $\mu_\delta(\chi\circ\gamma)\ge\min\{\mu_\delta(\chi),\mu_\delta(\gamma)\}$ and $\beta_\delta(\chi\circ\gamma)\le\max\{\beta_\delta(\chi),\beta_\delta(\gamma)\}$.

Lemma 3. 3 If $\delta=(\chi,\mu_\delta,\beta_\delta)$ is an IF- sub KU-semigroup of $\aleph$,

then $\beta_\delta(0)\le\beta_\delta(\chi)$ and $\mu_\delta(0)\ge\mu_\delta(\chi)$, $\forall\chi\in\aleph$.

Proof. The prove is clear, by applying definition 3.2.

Example3.4. Let $\aleph=\{0,a,b,c\}$ be a set. Define $*$- operation and $\circ$- operation by the following tables

*	0	a	b	c
0	0	a	b	c
a	0	0	0	c
b	0	a	0	c
c	0	0	0	0

∘	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

Then $(\aleph,*,\circ,0)$ is a KU-semigroup. We can define an (IFS) $\delta=(\chi,\mu_\delta,\beta_\delta)$ by the following $\delta=\{(0,1,0),(a,0.8,0.2),(b,0.6,0.4),(c,0.6,0.4)\}$. Then $\delta=(\chi,\mu_\delta,\beta_\delta)$ is an IF- sub KU-semigroup of $\aleph$.

Definition 3.5 An (IFS) $\delta=(\chi,\mu_\delta,\beta_\delta)$ of $\aleph$ is called an (IF) S-ideal, if: $\forall\chi,\gamma,\tau\in\aleph$

(if$_1$) $\mu_\delta(0)\ge\mu_\delta(\chi)$, $\beta_\delta(0)\le\beta_\delta(\chi)$

(if$_2$) $\mu_\delta(\gamma)\ge\min\{\mu_\delta(\chi*\gamma),\mu_\delta(\chi)\}$ and $\beta_\delta(\gamma)\le\max\{\beta_\delta(\chi*\gamma),\beta_\delta(\chi)\}$

(if$_3$) $\mu_\delta(\chi\circ\gamma)\ge\min\{\mu_\delta(\chi),\mu_\delta(\gamma)\}$ and $\beta_\delta(\chi\circ\gamma)\le\max\{\beta_\delta(\chi),\beta_\delta(\gamma)\}$.

Definition 3.6. An (IFS) $\delta=(\chi,\mu_\delta,\beta_\delta)$ of $\aleph$ is called an (IF) *k*- ideal, if: $\forall\chi,\gamma,\tau\in\aleph$

(IF$_1$) $\mu_\delta(0) \ge \mu_\delta(\chi)$, $\beta_\delta(0) \le \beta_\delta(\chi)$

(IF$_2$) $\mu_\delta(\chi * \tau) \ge \min\{\mu_\delta(\chi*(\gamma*\tau)), \mu_\delta(\gamma)\}$ and $\beta_\delta(\chi*\tau) \le \max\{\beta_\delta(\chi*(\gamma*\tau)), \beta_\delta(\gamma)\}$

(IF$_3$) $\mu_\delta(\chi \circ \gamma) \ge \min\{\mu_\delta(\chi), \mu_\delta(\gamma)\}$ and $\beta_\delta(\chi\circ\gamma) \le \max\{\beta_\delta(\chi), \beta_\delta(\gamma)\}$.

Example 3.7 Let $\aleph = \{0, a, b, c, d\}$ be a set. Define $*$-operation and $\circ$-operation by the following tables

*	0	a	b	c	d
0	0	a	b	c	d
a	0	0	b	c	d
b	0	a	0	c	d
c	0	a	0	0	d
d	0	0	0	0	0

∘	0	a	b	c	d
0	0	0	0	0	0
a	0	0	0	0	0
b	0	0	0	0	b
c	0	0	0	b	c
d	0	a	b	c	d

Then $(\aleph, *, \circ, 0)$ is a KU-semigroup. We can define an (IFS) $\delta = (\chi, \mu_\delta, \beta_\delta)$ by the following $\delta = \{(0,1,0),(a,0.7,0.1),(b,0.5,0.2),(c,0.4,0.3),(d,0.4,0.3)\}$. Then $\delta = (\chi, \mu_\delta, \beta_\delta)$ is an (IF) *k*-ideal of $\aleph$.

Lemma 3.8 Let $\delta = (\chi, \mu_\delta, \beta_\delta)$ be an (IF) *k*- ideal of a KU-semigroup $\aleph$ and $\chi \le \gamma$, then $\mu_\alpha(\chi) \ge \mu_\alpha(\gamma)$ and $\beta_\alpha(\chi) \le \beta_\alpha(\gamma)$, for all $\chi, \gamma \in \aleph$.

Proof. Let $\chi \le \gamma$, we get $\gamma * \chi = 0$. Now, since $\delta = (\chi, \mu_\delta, \beta_\delta)$ is an (IF) *k*- ideal of a KU-semigroup $\aleph$, then
$$\begin{aligned}\mu_\delta(\chi) &= \mu_\delta(0*\chi) \ge \min\{\mu_\delta(0*(\gamma*\chi)), \mu_\delta(\gamma)\} = \min\{\mu_\delta(0*0), \mu_\delta(\gamma)\}\\ &= \min\{\mu_\delta(0), \mu_\delta(\gamma)\} = \mu_\delta(\gamma)\}\end{aligned}$$
$$\begin{aligned}\beta_\delta(\chi) &= \beta_\delta(0*\chi) \le \max\{\beta_\delta(0*(\gamma*\chi)), \beta_\delta(\gamma)\} = \max\{\beta_\delta(0*0), \beta_\delta(\gamma)\}\\ &= \max\{\beta_\delta(0), \beta_\delta(\gamma)\} = \beta_\delta(\gamma)\end{aligned}.$$

Theorem 3.9 Let $(\aleph, *, \circ, 0)$ be a KU-semigroup. An (IFS) $\delta = (\chi, \mu_\delta, \beta_\delta)$ of $\aleph$ is an (IF) S-ideal if and only if $\delta = (\chi, \mu_\delta, \beta_\delta)$ is an (IF) *k*- ideal.

Proof. $(\Rightarrow)$ Let $\delta = (\chi, \mu_\delta, \beta_\delta)$ be an (IF) S-ideal of $\aleph$, then by definition 3.5, we get $\mu_\delta(\chi*\tau) \ge \min\{\mu_\delta(\gamma*(\chi*\tau)), \mu_\delta(\gamma)\}$ and $\beta_\delta(\chi*\tau) \le \max\{\beta_\delta(\gamma*(\chi*\tau)), \beta_\delta(\gamma)\}$. By Theorem 2.2, we get $\mu_\delta(\chi*\tau) \ge \min\{\mu_\delta(\chi*(\gamma*\tau)), \mu_\delta(\gamma)\}$ and $\beta_\delta(\chi*\tau) \le \max\{\beta_\delta(\chi*(\gamma*\tau)), \beta_\delta(\gamma)\}$. Since $\delta = (\chi, \mu_\delta, \beta_\delta)$ is an (IF) S-ideal of KU-semigroup $\aleph$, then by definition 3.5, it follows that $\delta = (\chi, \mu_\delta, \beta_\delta)$ is an (IF) *k*- ideal of $\aleph$.

$(\Leftarrow)$ Let $\delta = (\chi, \mu_\delta, \beta_\delta)$ be an (IF) *k*- ideal of $\aleph$. If we put $\chi = 0$ in definition 3.6, we get $\mu_\delta(\tau) \ge \min\{\mu_\delta(\gamma*\tau), \mu_\delta(\gamma)\}$ and $\beta_\delta(\tau) \le \max\{\beta_\delta(\gamma*\tau), \beta_\delta(\gamma)\}$. Since $\delta = (\chi, \mu_\delta, \beta_\delta)$ is an (IF) *k* - ideal of KU-semigroup $\aleph$, by definition 3.6, it follows that $\delta = (\chi, \mu_\delta, \beta_\delta)$ is an (IF) S-ideal of $\aleph$.

Lemma 3.10 If $\delta = (\chi, \mu_\delta, \beta_\delta)$ is an (IF) *k* -ideal of $\aleph$, then the sets $H = \{\chi \in \aleph : \mu_\delta(\chi) = \mu_\delta(0)\}$ and
$K = \{\chi \in \aleph : \beta_\delta(\chi) = \beta_\delta(0)\}$ are *k*- ideals of $\aleph$.

Proof. Since $0 \in \aleph$, then $\mu_\delta(\chi) = \mu_\delta(0)$ and $\beta_\delta(\chi) = \beta_\delta(0)$. It follows that $0 \in H$ and $0 \in K$. So H and K are non-empty sets. Let $(\chi*(\gamma*\tau)) \in H$, $\gamma \in H$ implies $\mu_\delta(\chi*(\gamma*\tau)) = \mu_\delta(0)$ and

$\mu_\delta(\gamma)=\mu_\delta(0)$. Since $\mu_\delta(\chi*\tau)\geq \min\{\mu_\delta(\chi*(\gamma*\tau)),\mu_\delta(\gamma)\}=\min\{\mu_\delta(0),\mu_\delta(0)\}=\mu_\delta(0)$, then $(\chi*\tau)\in H$.

Now, let $\chi\in\aleph, a\in H$, then $\mu_\delta(\chi)=\mu_\delta(0), \mu_\delta(a)=\mu_\delta(0)$. It follows that $\mu_\delta(\chi\circ a)\geq\min\{\mu_\delta(\chi),\mu_\delta(a)\}=\min\{\mu_\delta(0),\mu_\delta(0)\}=\mu_\delta(0)$, hence $\chi\circ a\in H$, and by the same way $a\circ\chi\in H$. So $H=\{\chi\in\aleph:\mu_\delta(\chi)=\mu_\delta(0)\}$ is a *k*- ideal of $\aleph$. Similarly, we can show that $K=\{\chi\in\aleph:\beta_\delta(\chi)=\beta_\delta(0)\}$ is *k*- ideal of $\aleph$.

Definition 3.11 Let $\delta=(\chi,\mu_\delta,\beta_\delta)$ be an (IFS) of a KU-semigroup $\aleph$. Then the set $V(\mu_\delta,\alpha)=\{\chi\in\aleph:\mu_\delta(\chi)\geq\alpha\}$ is said to be an upper α -level of μ_δ and the set $W(\beta_\delta,\theta)=\{\chi\in\aleph:\beta_\delta(\chi)\leq\theta\}$ is said to be a lower θ -level of β_δ.

Theorem 3.12 Let $(\aleph,*,\circ,0)$ be a KU-semigroup. An (IFS) $\delta=(\chi,\mu_\delta,\beta_\delta)$ of $\aleph$ is an (IF) k -ideal if and only if for any $\alpha,\theta\in[0,1]$, the non-empty upper α -level cut $V(\mu_\delta,\alpha)=\{\chi\in\aleph:\mu_\delta(\chi)\geq\alpha\}$ and the non-empty lower θ -level cut $W(\beta_\delta,\theta)=\{\chi\in\aleph:\beta_\delta(\chi)\leq\theta\}$ are *k*-ideals of $\aleph$.

Proof. For any $\alpha,\theta\in[0,1]$, since $V(\mu_I,\alpha)\neq\phi$, for any element $\chi\in V(\mu_\delta,\alpha)$, then $\mu_\delta(\chi)\geq\alpha$. It follows that $\chi=0$, hence $0\in V(\mu_\delta,\alpha)$.

Let $(\chi*(\gamma*\tau))\in\mu_\delta$ and $\gamma\in\mu_\delta$, implies $\mu_\delta(\chi*(\gamma*\tau))\geq\alpha$ and $\mu_\delta(\gamma)\geq\alpha$, and since $\mu_\delta(\chi*\tau)\geq\min\{\mu_\delta(\chi*(\gamma*\tau)),\mu_\delta(\gamma)\}=\min\{\alpha,\alpha\}=\alpha$, then $(\chi*\tau)\in V(\mu_\delta,\alpha)$.

Now, let $a,\chi\in V(\mu_\delta,\alpha)$, then $\mu_\delta(\chi)\geq\alpha$, $\mu_\delta(a)\geq\alpha$. It follows that $\mu_\delta(\chi\circ a)\geq\min\{\mu_\delta(\chi),\mu_\delta(a)\}=\min\{\alpha,\alpha\}=\alpha$, hence $\chi\circ a\in V(\mu_\delta,\alpha)$, and by the same way $a\circ\chi\in V(\mu_\delta,\alpha)$. So $V(\mu_\delta,\alpha)=\{\chi\in\aleph:\mu_\delta(\chi)\geq\alpha\}$ is a *k*- ideal of $\aleph$. By a similar method, we can prove that $W(\beta_\delta,\theta)=\{\chi\in\aleph:\beta_\delta(\chi)\leq\theta\}$ is a *k*- ideal of $\aleph$.

Conversely, assume that $V(\mu_\delta,\alpha)=\{\chi\in\aleph:\mu_\delta(\chi)\geq\alpha\}$ and $W(\beta_\delta,\theta)=\{\chi\in\aleph:\beta_\delta(\chi)\leq\theta\}$ are *k*- ideals of $\aleph$. For any $\alpha,\theta\in[0,1]$, suppose that $\chi,\gamma,\tau\in\aleph$, such that $\mu_\delta(0)<\mu_\delta(\chi)$, $\beta_\delta(0)>\beta_\delta(\chi)$, and let $\alpha_0=\frac{1}{2}[\mu_\delta(0)+\mu_\delta(\chi)]$. It follows that $\alpha_0<\mu_\delta(\chi)$, $0\leq\mu_\delta(0)<\alpha_0<1$, then $\chi\in V(\mu_\delta,\alpha_0)$, and since $V(\mu_\delta,\alpha)$ is a *k*- ideal of $\aleph$, then $0\in V(\mu_\delta,\alpha)\Rightarrow\mu_\delta(0)\geq\alpha_0$, which is a contradiction. Therefore, $\mu_\delta(0)\geq\mu_\delta(\chi)$, for all $\chi\in\aleph$. Similarly, by taking $\theta_0=\frac{1}{2}[\beta_\alpha(0)+\beta_\alpha(\chi)]$, we can show that $\beta_\alpha(0)\leq\beta_\alpha(\chi)$, for any $\chi\in\aleph$.

If $\chi,\gamma,\tau\in\aleph$ such that $\mu_\delta(\chi*\tau)<\min\{\mu_\delta(\chi*(\gamma*\tau)),\mu_\delta(\gamma)\}$.

Put $\alpha_0=\frac{1}{2}[\mu_\delta(\chi*\tau)+\min\{\mu_\delta(\chi*(\gamma*\tau)),\mu_\delta(\gamma)\}]$, implies

$\alpha_0>\mu_\delta(\chi*\tau)$ and $\alpha_0<\min\{\mu_\delta(\chi*(\gamma*\tau)),\mu_\delta(\gamma)\}$, so

$\alpha_0>\mu_\delta(\chi*\tau)$, $\alpha_0<\mu_\delta(\chi*(\gamma*\tau))$ and $\alpha_0<\mu_\delta(\gamma)$, hence

$(\chi*\tau)\notin V(\mu_\delta,\alpha_0)$, $(\chi*(\gamma*\tau))\in V(\mu_\delta,\alpha_0)$ and $\gamma\in V(\mu_\delta,\alpha_0)$, which is a contradiction. Therefore, $\mu_\delta(\chi*\tau)\geq\min\{\mu_\delta(\chi*(\gamma*\tau)),\mu_\delta(\gamma)\}$, $\forall\chi,\gamma,\tau\in\aleph$. Similarly, we can show that $\beta_\delta(\chi*\tau)\leq\max\{\beta_\delta(\chi*(\gamma*\tau)),\beta_\delta(\gamma)\}$, $\chi,\gamma,\tau\in\aleph$.

Now, we must prove that $\mu_\delta(\chi\circ\gamma)\geq\min\{\mu_\delta(\chi),\mu_\delta(\gamma)\}$ and $\beta_\delta(\chi\circ\gamma)\leq\max\{\beta_\delta(\chi),\beta_\delta(\gamma)\}$.

Suppose that $\mu_\delta(\chi\circ\gamma)<\min\{\mu_\delta(\chi),\mu_\delta(\gamma)\}$ and let $\alpha_0=\frac{1}{2}[\mu_\delta(\chi\circ\gamma)+\min\{\mu_\delta(\chi),\mu_\delta(\gamma)\}]$

$\Rightarrow \mu_\delta(\chi \circ \gamma) < \alpha_0 < \min\{\mu_\delta(\chi), \mu_\delta(\gamma)\}$

$\Rightarrow \mu_\delta(\chi \circ \gamma) < \alpha_0$, $\alpha_0 < \mu_\delta(\chi)$ and $\alpha_0 < \mu_\delta(\gamma)$

$\Rightarrow (\chi \circ \gamma) \notin V(\mu_\delta, \alpha_0)$, $\chi \in V(\mu_\delta, \alpha_0)$ and $\gamma \in V(\mu_\delta, \alpha_0)$, which is a contradiction. Hence $\mu_\delta(\chi \circ \gamma) \geq \min\{\mu_\delta(\chi), \mu_\delta(\gamma)\}$, $\forall \chi, \gamma, \tau \in \aleph$.

Similarly, we can show that $\beta_\delta(\chi \circ \gamma) \leq \max\{\beta_\delta(\chi), \beta_\delta(\gamma)\}$, $\chi, \gamma, \tau \in \aleph$.

Therefore, an (IFS) $\delta = (\chi, \mu_\delta, \beta_\delta)$ of $\aleph$ is an (IF) k-ideal of $\aleph$.

4- On homomorphism of KU-semigroups

Definition 4.1 Let $\aleph$ and $\aleph'$ be two KU-semigroups. A mapping $f : \aleph \to \aleph'$ is called a KU-semigroup homomorphism if for all $\chi, \gamma \in \aleph$, we have

$f(\chi * \gamma) = f(\chi) * f(\gamma)$ and $f(\chi \circ \gamma) = f(\chi) \circ f(\gamma)$.

Definition 4.2 Let a mapping $f : \aleph \to \aleph'$ be a homomorphism of two KU-semigroups $\aleph$ and $\aleph'$. For any (IFS) $\delta = (\chi, \mu_\delta, \beta_\delta)$ of $\aleph$, we can define a new (IFS) $\delta^f = (\chi, \mu_\delta^f, \beta_\delta^f)$ on $\aleph$ as follows: $\mu_\delta^f(\chi) = \mu_\delta(f(\chi))$, $\beta_\delta^f(\chi) = \beta_\delta(f(\chi))$, where $\chi \in \aleph$.

Theorem 4.3 Let $\aleph$ and $\aleph'$ be two KU-semigroups and f be a homomorphism and onto mapping from $\aleph$ into $\aleph'$. Then

(1) If $\delta = (\chi', \mu_\delta, \beta_\delta)$ is an (IF) k-ideal of $\aleph'$, then $\delta^f = (\chi, \mu_\delta^f, \beta_\delta^f)$ is an (IF) k-ideal of $\aleph$.

(2) If $\delta^f = (\chi', \mu_\delta^f, \beta_\delta^f)$ is an (IF) k-ideal of $\aleph'$, then $\delta = (\chi, \mu_\delta, \beta_\delta)$ is an (IF) k-ideal of $\aleph$.

Proof. (1) There exists $\chi \in \aleph$, for any $\chi' \in \aleph'$ such that $f(\chi) = \chi'$. We have

$\mu_\delta^f(0) = \mu_\delta(f(0)) = \mu_\delta(0') \geq \mu_\delta(\chi') = \mu_\delta(f(\chi)) = \mu_\delta^f(\chi)$ and

$\beta_\delta^f(0) = \beta_\delta(f(0)) = \beta_\delta(0') \leq \beta_\delta(\chi') = \beta_\delta(f(\chi)) = \beta_\delta^f(\chi)$. Now, where $\chi, \gamma \in \aleph$ and $\tau' \in \aleph'$, then there exists $\tau \in \aleph$ such that $f(\tau) = \tau'$. Then

$$\mu_\delta^f(\chi * \gamma) = \mu_\delta(f(\chi * \gamma)) = \mu_\delta(f(\chi) * f(\gamma)) \geq \min\{\mu_\delta[f(\chi) * (\tau' * f(\gamma))], \mu_\delta(\tau')\}$$
$$= \min\{\mu_\delta[f(\chi) * (f(\tau) * f(\gamma))], \mu_\delta(f(\tau))\} = \min\{\mu_\delta^f[\chi * (\tau * \gamma)], \mu_\delta^f(\tau)\}.$$

And

$$\beta_\delta^f(\chi * \gamma) = \beta_\delta(f(\chi * \gamma)) = \beta_\delta(f(\chi) * f(\gamma)) \leq \max\{\beta_\delta[f(\chi) * (\tau' * f(\gamma))], \beta_\delta(\tau')\}$$
$$= \max\{\beta_\delta[f(\chi) * (f(\tau) * f(\gamma))], \beta_\delta(f(\tau))\} = \max\{\beta_\delta^f[\chi * (\tau * \gamma)], \beta_\delta^f(\tau)\}.$$

Also

$$\mu_\delta^f(\chi \circ \gamma) = \mu_\delta(f(\chi \circ \gamma)) = \mu_\delta(f(\chi) \circ f(\gamma)) \geq \min\{\mu_\delta(f(\chi)), \mu_\delta(f(\gamma))\}$$
$$= \min\{\mu_\delta^f(\chi), \mu_\delta^f(\gamma)\}$$
$$\beta_\delta^f(\chi \circ \gamma) = \beta_\delta(f(\chi \circ \gamma)) = \beta_\delta(f(\chi) \circ f(\gamma)) \leq \max\{\beta_\delta(f(\chi)), \beta_\delta(f(\gamma))\}$$
$$= \max\{\beta_\delta^f(\chi), \beta_\delta(\gamma)\}.$$

Hence $\delta^f = (\chi, \mu_\delta^f, \beta_\delta^f)$ is an (IF) k-ideal of $\aleph$.

(2) Since $f : \aleph \to \aleph'$ is onto mapping, then $\forall \chi', \gamma', \tau' \in \aleph'$, $\exists \chi, \gamma, \tau \in \aleph$ s.t. $f(\chi) = \chi'$, $f(\gamma) = \gamma'$ and $f(\tau) = \tau'$. So

$$\mu_\delta(\chi' * \gamma') = \mu_\delta(f(\chi) * f(\gamma)) = \mu_\delta(f(\chi * \gamma)) = \mu_\delta^f(\chi * \gamma) \geq \min\{\mu_\delta^f[\chi * (\tau * \gamma)], \mu_\delta^f(\tau)\}$$
$$= \min\{\mu_\delta[f(\chi) * (f(\tau) * f(\gamma))], \mu_\delta(f(\tau))\} = \min\{\mu_\delta[\chi' * (\tau' * \gamma')], \mu_\delta(\tau')\}.$$

And

$$\beta_\delta(\chi' * \gamma') = \beta_\delta(f(\chi) * f(\gamma)) = \beta_\delta(f(\chi * \gamma)) = \beta_\delta^f(\chi * \gamma) \leq \max\{\beta_\delta^f[\chi * (\tau * \gamma)], \beta_\delta^f(\tau)\}$$
$$= \max\{\beta_\delta[f(\chi) * (f(\tau) * f(\gamma))], \beta_\delta(f(\tau))\} = \max\{\beta_\delta[\chi' * (\tau' * \gamma')], \beta_\delta(\tau')\}.$$

Also

$$\mu_\delta(\chi'\circ\gamma')=\mu_\delta(f(\chi)\circ f(\gamma))=\mu_\delta(f(\chi\circ\gamma))=\mu_\delta^f(\chi\circ\gamma))\geq\min\{\mu_\delta^f(\chi),\mu_\delta^f(\gamma)\}$$
$$=\min\{\mu_\delta(f(\chi)),\mu_\delta f(\gamma))\}=\min\{\mu_\delta(\chi'),\mu_\delta(\gamma')\}$$
$$\beta_\delta(\chi'\circ\gamma')=\beta_\delta(f(\chi)\circ f(\gamma))=\beta_\delta(f(\chi\circ\gamma))=\beta_\delta^f(\chi\circ\gamma))\leq\max\{\beta_\delta^f(\chi),\beta_\delta^f(\gamma)\}$$
$$=\max\{\beta_\delta(f(\chi)),\beta_\delta(f(\gamma)\}=\max\{\beta_\delta(\chi'),\beta_\delta(\gamma')\}.$$

Hence $\delta=(\chi,\mu_\delta,\beta_\delta)$ is an (IF) k-ideal of $\aleph$.

Definition 4.4 Let $\delta_1=(\chi,\mu_{\delta_1},\beta_{\delta_1})$ and $\delta_2=(\chi,\mu_{\delta_2},\beta_{\delta_2})$ be two intuitionistic fuzzy sets of a KU-semigroup $\aleph$. The Cartesian product $\delta_1\times\delta_2:\aleph\times\aleph\to[0,1]$ is defined by $\delta_1\times\delta_2=[(\chi,\gamma),\mu_{\delta_1}\times\mu_{\delta_2},\beta_{\delta_1}\times\beta_{\delta_2}]$ such that $(\mu_{\delta_1}\times\mu_{\delta_2})(\chi,\gamma)=\min\{\mu_{\delta_1}(\chi),\mu_{\delta_2}(\gamma)\}$ and $(\beta_{\delta_1}\times\beta_{\delta_2})(\chi,\gamma)=\max\{\beta_{\delta_1}(\chi),\beta_{\delta_2}(\gamma)\}$.

Theorem 4.5 Let $\delta_1=(\chi,\mu_{\delta_1},\beta_{\delta_1})$ and $\delta_2=(\chi,\mu_{\delta_2},\beta_{\delta_2})$ be two (IF) S-ideals of a KU-semigroup $\aleph$, then $\delta_1\times\delta_2$ is an (IF) S-ideal of $\aleph\times\aleph$.

Proof . (i) For any $(\chi,\gamma)\in\aleph\times\aleph$, we have

$(\mu_{\delta_1}\times\mu_{\delta_2})(0,0)=\min\{\mu_{\delta_1}(0),\mu_{\delta_2}(0)\}\geq\min\{\mu_{\delta_1}(\chi),\mu_{\delta_2}(\gamma)\}=(\mu_{\delta_1}\times\mu_{\delta_2})(\chi,\gamma)$ and

$$(\beta_{\delta_1}\times\beta_{\delta_2})(0,0)=\max\{\beta_{\delta_1}(0),\beta_{\delta_2}(0)\}\leq\max\{\beta_{\delta_1}(\chi),\beta_{\delta_2}(\gamma)\}=(\beta_{\delta_1}\times\beta_{\delta_2})(\chi,\gamma)$$

(ii) Let $(\chi_1,\chi_2),(\gamma_1,\gamma_2)\in\aleph\times\aleph$, then

$$(\mu_{\delta_1}\times\mu_{\delta_2})(\gamma_1,\gamma_2)=\min\{\mu_{\delta_1}(\gamma_1),\mu_{\delta_2}(\gamma_2)\}$$
$$\geq\min\{\min\{\mu_{\delta_1}(\chi_1*\gamma_1),\mu_{\delta_1}(\chi_1)\},\min\{\mu_{\delta_2}(\chi_2*\gamma_2),\mu_{\delta_2}(\chi_2)\}\}$$
$$\geq\min\{\min\{\mu_{\delta_1}(\chi_1*\gamma_1),\mu_{\delta_2}(\chi_2*\gamma_2)\},\min\{\mu_{\delta_1}(\chi_1),\mu_{\delta_2}(\chi_2)\}\}$$
$$=\min\{(\mu_{\delta_1}\times\mu_{\delta_2})(\chi_1*\gamma_1,\chi_2*\gamma_2),(\mu_{\delta_1}\times\mu_{\delta_2})(\chi_1,\chi_2)\}$$
$$=\min\{(\mu_{\delta_1}\times\mu_{\delta_2})(\chi_1*\gamma_1,\chi_1),(\mu_{\delta_1}\times\mu_{\delta_2})(\chi_2*\gamma_2,\chi_2)\}.$$

And

$$(\beta_{\delta_1}\times\beta_{\delta_2})(\gamma_1,\gamma_2)=\max\{\beta_{\delta_1}(\gamma_1),\beta_{\delta_2}(\gamma_2)\}$$
$$\leq\max\{\max\{\beta_{\delta_1}(\chi_1*\gamma_1),\beta_{\delta_1}(\chi_1)\},\max\{\beta_{\delta_2}(\chi_2*\gamma_2),\beta_{\delta_2}(\chi_2)\}\}$$
$$\leq\max\{\max\{\beta_{\delta_1}(\chi_1*\gamma_1),\beta_{\delta_2}(\chi_2*\gamma_2)\},\max\{\beta_{\delta_1}(\chi_1),\beta_{\delta_2}(\chi_2)\}\}$$
$$=\max\{(\beta_{\delta_1}\times\beta_{\delta_2})(\chi_1*\gamma_1,\chi_2*\gamma_2),(\beta_{\delta_1}\times\beta_{\delta_2})(\chi_1,\chi_2)\}$$
$$=\max\{(\beta_{\delta_1}\times\beta_{\delta_2})(\chi_1*\gamma_1,\chi_1),(\beta_{\delta_1}\times\beta_{\delta_2})(\chi_2*\gamma_2,\chi_2)\}$$

(iii) Let $(\chi_1,\chi_2),(\gamma_1,\gamma_2)\in\aleph\times\aleph$, then we have

$$(\mu_{\delta_1}\times\mu_{\delta_2})(\chi_1\circ\chi_2,\gamma_1\circ\gamma_2)=\min\{\mu_{\delta_1}(\chi_1\circ\chi_2),\mu_{\delta_2}(\gamma_1\circ\gamma_2)\}$$
$$\geq\min\{\min\{\mu_{\delta_1}(\chi_1),\mu_{\delta_1}(\chi_2)\},\min\{\mu_{\delta_2}(\gamma_1),\mu_{\delta_2}(\gamma_2)\}\}$$
$$=\min\{\min\{\mu_{\delta_1}(\chi_1),\mu_{\delta_2}(\gamma_1)\},\min\{\mu_{\delta_1}(\chi_2),\mu_{\delta_2}(\gamma_2)\}\}$$
$$=\min\{(\mu_{\delta_1}\times\mu_{\delta_2})(\chi_1,\gamma_1),(\mu_{\delta_1}\times\mu_{\delta_2})(\chi_2,\gamma_2)\}.$$

And

$$(\beta_{\delta_1}\times\beta_{\delta_2})(\chi_1\circ\chi_2,\gamma_1\circ\gamma_2)=\max\{\beta_{\delta_1}(\chi_1\circ\chi_2),\beta_{\delta_2}(\gamma_1\circ\gamma_2)\}$$
$$\leq\max\{\max\{\beta_{\delta_1}(\chi_1),\beta_{\delta_1}(\chi_2)\},\max\{\beta_{\delta_2}(\gamma_1),\beta_{\delta_2}(\gamma_2)\}\}$$
$$=\max\{\max\{\beta_{\delta_1}(\chi_1),\beta_{\delta_2}(\gamma_1)\},\max\{\beta_{\delta_1}(\chi_2),\beta_{\delta_2}(\gamma_2)\}\}$$
$$=\max\{(\beta_{\delta_1}\times\beta_{\delta_2})(\chi_1,\gamma_1),(\beta_{\delta_1}\times\beta_{\delta_2})(\chi_2,\gamma_2)\}.$$

Then $\delta_1\times\delta_2$ is an (IF) S-ideal of $\aleph\times\aleph$.

Theorem 4.6 Let $\delta_1 = (\chi, \mu_{\delta_1}, \beta_{\delta_1})$ and $\delta_2 = (\chi, \mu_{\delta_2}, \beta_{\delta_2})$ be two (IF) S-ideals of a KU-semigroup $\aleph$, such that $\delta_1 \times \delta_2$ is an (IF) S-ideal of $\aleph \times \aleph$. Then

(i) Either $\mu_{\delta_1}(0) \geq \mu_{\delta_1}(\chi)$, $\beta_{\delta_1}(0) \leq \beta_{\delta_1}(\chi)$ or $\mu_{\delta_2}(0) \geq \mu_{\delta_2}(\gamma)$, $\beta_{\delta_2}(0) \leq \beta_{\delta_2}(\gamma)$, for all $\chi, \gamma \in \aleph$.

(ii) If $\mu_{\delta_1}(0) \geq \mu_{\delta_1}(\chi)$, $\beta_{\delta_1}(0) \leq \beta_{\delta_1}(\chi)$ for all $\chi \in \aleph$. Then either $\mu_{\delta_2}(0) \geq \mu_{\delta_1}(\chi)$, $\beta_{\delta_2}(0) \leq \beta_{\delta_1}(\chi)$ or $\mu_{\delta_2}(0) \geq \mu_{\delta_2}(\gamma)$, $\beta_{\delta_2}(0) \leq \beta_{\delta_2}(\gamma)$.

(iii) If $\mu_{\delta_2}(0) \geq \mu_{\delta_2}(\chi)$, $\beta_{\delta_2}(0) \leq \beta_{\delta_2}(\chi)$ for all $\chi \in \aleph$, then either $\mu_{\delta_1}(0) \geq \mu_{\delta_1}(\chi)$, $\beta_{\delta_1}(0) \leq \beta_{\delta_1}(\chi)$ or $\mu_{\delta_1}(0) \geq \mu_{\delta_2}(\chi)$, $\beta_{\delta_1}(0) \leq \beta_{\delta_2}(\chi)$.

Proof. (i) Suppose that $\mu_{\delta_1}(0) \leq \mu_{\delta_1}(\chi)$, $\beta_{\delta_1}(\chi) \leq \beta_{\delta_1}(0)$ and $\mu_{\delta_2}(0) \leq \mu_{\delta_2}(\gamma)$, $\beta_{\delta_2}(\gamma) \leq \beta_{\delta_2}(0)$, for some $\chi, \gamma \in \aleph$. Then

$(\mu_{\delta_1} \times \mu_{\delta_2})(\chi, \gamma) = \min\{\mu_{\delta_1}(\chi), \mu_{\delta_2}(\gamma)\} \geq \min\{\mu_{\delta_1}(0), \mu_{\delta_2}(0)\} = (\mu_{\delta_1} \times \mu_{\delta_2})(0,0)$

And $(\beta_{\delta_1} \times \beta_{\delta_2})(\chi, \gamma) = \max\{\beta_{\delta_1}(\chi), \beta_{\delta_2}(\gamma)\} \leq \max\{\beta_{\delta_1}(0), \beta_{\delta_2}(0)\} = (\beta_{\delta_1} \times \beta_{\delta_2})(0,0)$, for all $\chi, \gamma \in \aleph$. This is a contradiction. Therefore, either $\mu_{\delta_1}(0) \geq \mu_{\delta_1}(\chi)$, $\beta_{\delta_1}(0) \leq \beta_{\delta_1}(\chi)$ or $\mu_{\delta_2}(0) \geq \mu_{\delta_2}(\gamma)$, $\beta_{\delta_2}(0) \leq \beta_{\delta_2}(\gamma)$, for all $\chi, \gamma \in \aleph$.

(ii) Suppose that $\mu_{\delta_2}(0) \leq \mu_{\delta_1}(\chi)$, $\beta_{\delta_2}(0) \geq \beta_{\delta_1}(\chi)$ and $\mu_{\delta_2}(0) \leq \mu_{\delta_2}(\gamma)$, $\beta_{\delta_2}(0) \geq \beta_{\delta_2}(\gamma)$ for all $\chi, \gamma \in \aleph$.

Then $(\mu_{\delta_1} \times \mu_{\delta_2})(0,0) = \min\{\mu_{\delta_1}(0), \mu_{\delta_2}(0)\} = \mu_{\delta_2}(0)$

and $(\mu_{\delta_1} \times \mu_{\delta_2})(\chi, \gamma) = \min\{\mu_{\delta_1}(\chi), \mu_{\delta_2}(\gamma)\} \geq \min\{\mu_{\delta_2}(0), \mu_{\delta_2}(0)\} = \mu_{\delta_2}(0) = (\mu_{\delta_1} \times \mu_{\delta_2})(0,0)$

And $(\beta_{\delta_1} \times \beta_{\delta_2})(0,0) = \max\{\beta_{\delta_1}(0), \beta_{\delta_2}(0)\} = \beta_{\delta_2}(0)$

and $(\beta_{\delta_1} \times \beta_{\delta_2})(\chi, \gamma) = \max\{\beta_{\delta_1}(\chi), \beta_{\delta_2}(\gamma)\} \leq \max\{\beta_{\delta_2}(0), \beta_{\delta_2}(0)\} = \beta_{\delta_2}(0) = (\beta_{\delta_1} \times \beta_{\delta_2})(0,0)$

. This is a contradiction. Therefore, either $\mu_{\delta_2}(0) \geq \mu_{\delta_1}(\chi)$, $\beta_{\delta_2}(0) \leq \beta_{\delta_1}(\chi)$ or $\mu_{\delta_2}(0) \geq \mu_{\delta_2}(\gamma)$, $\beta_{\delta_2}(0) \leq \beta_{\delta_2}(\gamma)$.

(ii) The proof is similar to (ii).

The partial converse of Theorem 4.5 is the following.

Theorem 4.7 In a KU-semigroup $\aleph$. If $\delta_1 \times \delta_2$ is an (IF) S-ideal of $\aleph \times \aleph$, then $\delta_1 = (\chi, \mu_{\delta_1}, \beta_{\delta_1})$ or $\delta_2 = (\chi, \mu_{\delta_2}, \beta_{\delta_2})$ is an (IF) S-ideal of a KU-semigroup $\aleph$.

Proof. By Theorem 4.6, without loss of generality, we assume that $\mu_{\delta_2}(0) \geq \mu_{\delta_2}(\chi)$, $\beta_{\delta_2}(0) \leq \beta_{\delta_2}(\chi)$ for all $\chi \in \aleph$. It follows from Theorem 4.6 that either $\mu_{\delta_1}(0) \geq \mu_{\delta_1}(\chi)$, $\beta_{\delta_1}(0) \leq \beta_{\delta_1}(\chi)$ or $\mu_{\delta_1}(0) \geq \mu_{\delta_2}(\chi)$, $\beta_{\delta_1}(0) \leq \beta_{\delta_2}(\chi)$. If $\mu_{\delta_1}(0) \geq \mu_{\delta_2}(\chi)$, $\beta_{\delta_1}(0) \leq \beta_{\delta_2}(\chi)$, for all $\chi \in \aleph$.

Then

$$(\mu_{\delta_1} \times \mu_{\delta_2})(0, \chi) = \min\{\mu_{\delta_1}(0), \mu_{\delta_2}(\chi)\} = \mu_{\delta_2}(\chi) \ldots\ldots(1)$$

$$(\beta_{\delta_1} \times \beta_{\delta_2})(0, \chi) = \max\{\beta_{\delta_1}(0), \beta_{\delta_2}(\chi)\} = \beta_{\delta_2}(\chi) \ldots\ldots(2)$$

Since $\delta_1 \times \delta_2$ is an (IF) S-ideal of $\aleph \times \aleph$, then

$$\begin{aligned}(\mu_{\delta_1} \times \mu_{\delta_2})(\gamma_1, \gamma_2) &\geq \min\{(\mu_{\delta_1} \times \mu_{\delta_2})((\chi_1, \chi_2) * (\gamma_1, \gamma_2)), (\mu_{\delta_1} \times \mu_{\delta_2})(\chi_1, \chi_2)\} \\ &= \min\{(\mu_{\delta_1} \times \mu_{\delta_2})(\chi_1 * \gamma_1, \chi_2 * \gamma_2), (\mu_{\delta_1} \times \mu_{\delta_2})(\chi_1, \chi_2)\}\end{aligned}$$

Put $\chi_1 = \gamma_1 = 0$, then we have

$(\mu_{\delta_1} \times \mu_{\delta_2})(0, \gamma_2) \geq \min\{(\mu_{\delta_1} \times \mu_{\delta_2})(0, \chi_2 * \gamma_2), (\mu_{\delta_1} \times \mu_{\delta_2})(0, \chi_2)\}$ and by using equation (1), we have $\mu_{\delta_2}(\gamma_2) \geq \min\{\mu_{\delta_2}(\chi_2 * \gamma_2), \mu_{\delta_2}(\chi_2)\}$. And

$$(\mu_{\delta_1} \times \mu_{\delta_2})((\chi_1, \chi_2) \circ (\gamma_1, \gamma_2)) \geq \min\{(\mu_{\delta_1} \times \mu_{\delta_2})(\chi_1, \chi_2), (\mu_{\delta_1} \times \mu_{\delta_2})(\gamma_1, \gamma_2)\}$$

$$(\mu_{\delta_1} \times \mu_{\delta_2})((\chi_1 \circ \gamma_1, \chi_2 \circ \gamma_2)) \geq \min\{(\mu_{\delta_1} \times \mu_{\delta_2})(\chi_1, \chi_2), (\mu_{\delta_1} \times \mu_{\delta_2})(\gamma_1, \gamma_2)\}$$

Put $\chi_1 = \gamma_1 = 0$, then we have

$(\mu_{\delta_1} \times \mu_{\delta_2})((0, \chi_2 \circ \gamma_2)) \geq \min\{(\mu_{\delta_1} \times \mu_{\delta_2})(0, \chi_2), (\mu_{\delta_1} \times \mu_{\delta_2})(0, \gamma_2)\}$, and by using equation (1), we have $\mu_{\delta_2}(\chi_2 \circ \gamma_2) \geq \min\{\mu_{\delta_2}(\chi_2), \mu_{\delta_2}(\gamma_2)\}$.

Also, we have

$$(\beta_{\delta_1} \times \beta_{\delta_2})(\gamma_1, \gamma_2) \leq \max\{(\beta_{\delta_1} \times \beta_{\delta_2})((\chi_1, \chi_2) * (\gamma_1, \gamma_2)), (\beta_{\delta_1} \times \beta_{\delta_2})(\chi_1, \chi_2)\}$$

$$= \max\{(\beta_{\delta_1} \times \beta_{\delta_2})(\chi_1 * \gamma_1, \chi_2 * \gamma_2), (\beta_{\delta_1} \times \beta_{\delta_2})(\chi_1, \chi_2)\}$$

Put $\chi_1 = \gamma_1 = 0$, then we have

$(\beta_{\delta_1} \times \beta_{\delta_2})(0, \gamma_2) \leq \max\{(\beta_{\delta_1} \times \beta_{\delta_2})(0, \chi_2 * \gamma_2), (\beta_{\delta_1} \times \beta_{\delta_2})(0, \chi_2)\}$ and by using equation (2), we have $\beta_{\delta_2}(\gamma_2) \leq \max\{\beta_{\delta_2}(\chi_2 * \gamma_2), \beta_{\delta_2}(\chi_2)\}$. And

$$(\beta_{\delta_1} \times \beta_{\delta_2})((\chi_1, \chi_2) \circ (\gamma_1, \gamma_2)) \leq \max\{(\beta_{\delta_1} \times \beta_{\delta_2})(\chi_1, \chi_2), (\beta_{\delta_1} \times \beta_{\delta_2})(\gamma_1, \gamma_2)\}$$

$$(\beta_{\delta_1} \times \beta_{\delta_2})((\chi_1 \circ \gamma_1, \chi_2 \circ \gamma_2)) \leq \max\{(\beta_{\delta_1} \times \beta_{\delta_2})(\chi_1, \chi_2), (\beta_{\delta_1} \times \beta_{\delta_2})(\gamma_1, \gamma_2)\}$$

Put $\chi_1 = \gamma_1 = 0$, then we have

$(\beta_{\delta_1} \times \beta_{\delta_2})((0, \chi_2 \circ \gamma_2)) \leq \max\{(\beta_{\delta_1} \times \beta_{\delta_2})(0, \chi_2), (\beta_{\delta_1} \times \beta_{\delta_2})(0, \gamma_2)\}$ and by using equation (2), we have $\beta_{\delta_2}(\chi_2 \circ \gamma_2) \leq \max\{\beta_{\delta_2}(\chi_2), \beta_{\delta_2}(\gamma_2)\}$.

Then, it follows that $\delta_2 = (\chi, \mu_{\delta_2}, \beta_{\delta_2})$ is an (IF) S-ideal of a KU-semigroup $\aleph$.

This completes the proof.

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