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Dedekind Multiplication Semimodules

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Abstract

The aim of this paper is to introduce the concept of Dedekind semimodules and study the related concepts, such as the class of D_1 semimodules, and Dedekind multiplication semimodules . And thus study the concept of the embedding of a semimodule in another semimodule.

Keywords: Semirings, semimodules, invertible subsemimodules, Dedekind semirings, Dedekind semimodules, multiplication semimodules.

شبه المقاسات الديديكاندية الجدائية

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الخلاصة

الغرض من هذا البحث هو تقديم مفهوم شبه المقاسات الديديكاندية ودراسة المفاهيم المرتبطة به مثل صنف شبه المقاسات الديديكاندية من النمط D_1, و شبه المقاسات الديديكاندية الجدائية. وبالتالي دراسة مفهوم الانغمار لشبه المقاس في شبه مقاس أخر.

Introduction

In ring theory, an ideal I of a commutative ring with identity R is said to be invertible if $I'I = R$ where $I' = \{ x \in R_S : xI \subseteq R \}$ and R_S is the total quotient ring of R. The concept of an invertible submodule was introduced by Naoum and Al-Alwan [1] as a generalization of the concept of an invertible ideal.

A semiring is a non-empty set R together with two binary operations addition(+) and multiplication $(\cdot)$ such that $(R, +)$ is a commutative monoid with identity element 0; $(R, \cdot)$ is a monoid with identity element $1 \neq 0$; $r0 = 0r = 0$ for all $r \in R$; $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for every $a, b, c \in R$. We say that R is a commutative semiring if the monoid $(R, \cdot)$ is commutative. Let $(M, +)$ be an additive abelian monoid with additive identity 0_M. Then M is called an R-semimodule if there exists a scalar multiplication $R \times M \longrightarrow M$ denoted by $(r, m) \mapsto rm$, such that $(rr')m = r(r'm)$; $r(m + m') = rm + rm'$; $(r + r')m = rm + r'm$; $1m = m$ and $r0_M = 0_M = 0m$ for all $r, r' \in R$ and all $m, m' \in M$.

Throughout this paper R will denote a commutative semiring with identity, M is unitary (left) R-semimodule. This paper consists four sections. Section 1 is devoted to introducing the concept of invertible subsemimodules of semimodule as a generalization of the concept of an invertible ideal in semiring. We will also find out some properties of this invertible subsemimodules. A non-zero

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semimodule M is a Dedekind semimodule if each non-zero subsemimodule of M is invertible.

Section 2 argues multiplication semimodules. We show that every multiplicatively cancellative multiplication semimodule is finitely generated.

Section 3 discusses Dedekind multiplication semimodules. We show that if M is a faithful multiplication R-semimodule, then M is a Dedekind semimodule iff R is a Dedekind semiring.

Let A and B be R-semimodules, and $H = \mathrm{Hom}_R(A, B)$. Here's a question that shows : when does H contain a monomorphism?. If H contains a monomorphism we say that A is embeds in B.

It was proved by Low and Smith [2] that if A is a torsionless multiplication R-module then A embeds in R iff $\exists \beta \in A^* = \mathrm{Hom}_R(A, R)$ such that $\mathrm{ann}(R\beta) = \mathrm{ann}(A^*)$.

Indeed if A is not a multiplication semimodule then this condition is not sufficient see Remark 3.2.

Here the importance of the invertible subsemimodules in obtaining the sufficient condition for the existence of a monomorphism.

In the last section we establish that if A is any semimodule, with $\bigcap_{\beta \in H} \ker \beta = (0)$ and $T_H \subseteq T_B$, and if there is a cyclic invertible subsemimodule Rf in H, then f is a monomorphism.

1. Invertible Subsemimodules and Invertible Ideals

In this section we introduce the concept of invertible subsemimodule of a semimodule as a kind of generalization of the concept of invertible ideal in semiring.

Remark (1.1): Let R be a commutative semiring with identity 1. A set $S \subseteq R$ is said to be a multiplicatively closed set of R provided that If $a, b \in S$, then $ab \in S$. The localization of R at S (R_S) is defined in the following way:-

First define the equivalence relation $\sim$ on $R \times S$ by $(a, b) \sim (c, d)$, if $sad = sbc$ for some $s \in S$. Then put R_S the set of all equivalence classes of $R \times S$ and define addition and multiplication on R_S respectively by $[a, b] + [c, d] = [ad + bc, bd]$ and $[a, b] \cdot [c, d] = [ac, bd]$, where $[a, b]$ also denoted by a/b, we mean the equivalence class of (a, b). It is, then, easy to see that R_S with the mentioned operations of addition and multiplication on R_S in above is a semiring [3, 4].

Definition (1.2): In Remark 1.1, if S is the set of all not zero-divisors of R. Then, the total quotient semiring $Q(R)$ of the semiring R is defined as the localization of R at S. Note that $Q(R)$ is also an R-semimodule. If R is a semidomain one can define the semifield of fractions F(R) of R as the localization of R at $R - \{0\}$ [5, 6].

Definition (1.3): Let M be an R-semimodule. In Remark 1.1, if S is the set of all not zero-divisors of R, and $T = T_M = \{s \in S | sm = 0 \text{ for some } m \in M \text{ implies } m = 0\}$. The total quotient semiring $Q_T(R)$ of the semiring R is defined as the localization of R at T. Note that $Q_T(R)$ is also an R-semimodule.

Consider $R = \mathbb{N}$ and $M = \mathbb{Q}^+/\mathbb{N}$. Then $T = \{1\}$ and so $Q_T(R) = \{ \frac{n}{1} : n \in \mathbb{N}\}$.

Similar to that in modules see [1], we give the following remark.

Remark (1.4): Let M be an R-semimodule and let N be a non-zero subsemimodule of M. Suppose that $N' = \{x \in Q_T(R) | \, xN \subseteq M\}$. Then N' is an R-subsemimodule of $Q_T(R)$, $R \subseteq N'$, and $N'N \subseteq M$.

Definition (1.5): Let M be an R-semimodule. A **subtractive** subsemimodule (or k-subsemimodule) N is a subsemimodule of M such that if $x, x + y \in N$, then $y \in N$. A **prime** subsemimodule of M is a proper subsemimodule P of M in which $x \in P$ or $rM \subseteq P$ whenever $rx \in P$, [5]. We define k-ideals and prime ideals of a semiring R in a analogous manner [5].

Remark (1.6): Let M be an R-semimodule, we say that M is a torsion-free semimodule whenever $r \in R$ and $m \in M$ with $rm = 0$ implies that either $m = 0$ or $r = 0$. If N is a subsemimodule of M, then $[N: M] = \{r \in R : rM \subseteq N\}$ and $\mathrm{ann}(M) = [0: M] = \{r \in R : rM = 0\}$ are k-ideals of R, [5].

Proposition (1.7): Let M be a non-zero R-semimodule, and let T be the set defined as in Definition 1.3, then T has the following properties:

1) $T \bigcap \mathrm{ann}(M)$ is the empty set.
2) T is a multiplicative subset of S and $1 \in T$.
3) If M is torsion-free then $T = S$.

Proof: For (1) from the definition of T we have $T \bigcap \mathrm{ann}(M) = \emptyset$. For (2) first observe that $1 \in T$. Let $s_1, s_2 \in T$, and $s_1 s_2 m = 0$ for some $m \in M$, then since $s_1, s_2 \in T$, then $s_2 m = 0$ and hence $m = 0$, therefore $s_1 s_2 \in T$. Thus T is a multiplicative subset of S. For (3) from definition of T, then $T \subseteq S$. Now, assume that M is torsion-free. Let $s \in S$ and $sm = 0$ for some $m \in M$, since M is torsion-free then $m = 0$, and hence $s \in T$. Thus $S \subseteq T$. This completes the proof.

Definition (1.8): [4] A subset I of the total quotient semiring $Q(R)$ of R is called **fractional** ideal of a semiring R, if the following hold:

1. I is an R-subsemimodule of $Q(R)$, that is, if $a, b \in I$ and $r \in R$, then $a + b \in I$ and $ra \in I$.

2. There exists a not zero-divisor element $d \in R$ such that $dI \subseteq R$.

Let I, J be two fractional ideals of a semiring R. Then

$$IJ = \{a_1b_1 + a_2b_2 + \cdots + a_nb_n : a_i \in I, b_i \in J, \ \forall i, 1 \leq i \leq n, n \in \mathbb{N}\}.$$

By Frac(R), we mean the set of all nonzero fractional ideals of a semiring R. It is easy to check that Frac(R) equipped with the above multiplication of fractional ideals is an abelian monoid [4]. It is clear that each ideal I of R is fractional ideal of a semiring R since (1) and (2) holds for $d = 1$, $1I \subseteq R$.

Definition (1.9): [4] Let I be a fractional ideal of a semiring R, then I is called **invertible** if there exists a fractional ideal J of R such that $IJ = R$. Note that J is unique and will be denoted that by I^{-1}. The set of all invertible fractional ideals of R is an abelian group.

Example(1.10): Let $\mathbb{N}$ be the set of all non-negative integers. Clearly $\mathbb{Q}^+$ its semifield of fractions. Let n be a positive integer. The set $I = \frac{1}{n}\mathbb{N} = \{\frac{m}{n} : m \in \mathbb{N}\}$ is a fractional ideal of $\mathbb{N}$. It is clear I as an $\mathbb{N}$-subsemimodule of $\mathbb{Q}^+$ is generated by $\frac{1}{n}$ and $nI \subseteq \mathbb{N}$. While $J = < \frac{1}{2^n} >$, where n runs over all positive integers. Since there is no positive integer d such that $dJ \subseteq \mathbb{N}$, J is not a fractional ideal of $\mathbb{N}$.

Let R be a semidomain, F(R) its semifield of fractions, A and B R-subsemimodules of F(R). Then the residual quotient of A by B is defined as $[A : B] = \{x \in F(R) : xB \subseteq A\}$, see [6].

Proposition(1.11): Let R be a semidomain, A and B some fractional ideals of R. Then the following statements hold:

(1) $[AB : A]A = AB$.

(2) $[R : A]$ is a fractional ideal of R.

(3) If A is invertible, then $A^{-1} = [R : A]$.

(4) If A is an invertible ideal of R, then A is finitely generated.

Proof: (1): Suppose that $t \in AB$, then $t = \sum_{i=1}^{n} a_ib_i$, where $a_i \in A, b_i \in B, \forall i$. Now $b_iA \subseteq AB$, so $b_i \in [AB: A]$, $\forall i$. Therefore $t \in [AB : A]A$, and $AB \subseteq [AB : A]A$. By similar way we prove that $[AB : A]A \subseteq AB$. Thus $[AB : A]A = AB$.

(2): R is fractional and A an R-semimodule, 1 is a common denominator of R. Choose a non-zero t in $A \cap R$. Clearly, for any $x \in [R : A]$, then $xt \in R$. Therefore, t is a common denominator of [R: A] and hence $[R : A]$ is fractional.

(3): In the formula, $[AB : A]A = AB$, put $AB = R$.

(4) Let A be an invertible ideal of R. So, there is a fractional ideal B of R such that $AB = R$. This implies that $1 = \sum_{i=1}^{n} x_iy_i$, for some $x_1, x_2, \cdots, x_n \in A$ and $y_1, y_2, \cdots, y_n \in B$. Clearly, the set $\{x_i\}_{i=1}^{n}$ generates A in R.

Now we can give our definition of invertible subsemimodule, as in modules theory [1].

Definition (1.12): Let M be a non-zero R-semimodule and N be a subsemimodule of M. If $N'N = M$, then we say that N is an **invertible** subsemimodule of M. Note that if N is invertible then $N \neq 0$. It is clear that M is invertible in M.

The following proposition is useful for testing the invertibility of subsemimodules.

Proposition (1.13): Let M be a non-zero R-semimodule.

1) A non-zero subsemimodule N of M is invertible of M iff $\forall\, m \in M$, $\exists\, \frac{r_i}{t_i} \in N'$, $n_i \in N, 1 \leq i \leq k$ such that $m = \sum_{i=1}^{k} \frac{r_i}{t_i} n_i$.

2) If N is invertible subsemimodule in M, then $\forall m \in M, \exists t \in T$ such that $tm \in N$.

Proof: The proof of (1) is an immediate consequence of the Definition 1.12. For (2) Since $N'N = M$, then $\forall\, m \in M$, $\exists\, \frac{r_i}{t_i} \in N'$, $n_i \in N, 1 \leq i \leq k$, such that $m = \sum_{i=1}^{k} \frac{r_i}{t_i} n_i$, where $r_i \in R$, $t_i \in T$. Put $t = t_1t_2 \ldots t_k$, and $q_i = r_i \prod_{j \neq i} t_j$, $1 \leq i \leq k$, then $tm = \sum_{i=1}^{k} q_in_i \in N$.

As a special case of Proposition 1.13 we obtain.

Corollary (1.14): A non-zero cyclic subsemimodule Rn of M is invertible in M iff $\forall m \in M, \exists\, t \in T$, $r \in R$ such that $tm = rn$, r depends on m.

Proposition (1.15): If N is a non-zero invertible subsemimodule of R-semimodule M. Then $M = \sum_{\phi \in H} \phi(N)$, where the sum is taken over all $\phi \in H = Hom(N, M)$.

Proof: Since $N'N = M$. Hence each element of N' can be thought of as an R-homomorphism in $Hom(N, M)$. In fact, $\forall\, m \in M$, $m = \sum_{i=1}^{k} q_i n_i$, $q_i \in N'$, $n_i \in N, 1 \le i \le k$. i.e. $m = \sum_{i=1}^{k} \phi_{q_i}(n_i)$, where if $q \in N'$, then $\phi_q(n) = qn, \forall n \in N$. This completes the proof. ■

Definition(1.16): A non-zero R-semimodule M is called a **Dedekind semimodule**(or **D semimodule**), if each non-zero subsemimodule of M is invertible in M, and M is called a $\mathbf{D_1}$ **semimodule** if each non-zero cyclic subsemimodule of M is invertible in M. It is clear that every D semimodule is D_1 semimodule.

Example (1.17): Here some examples to explain invertible subsemimodules and D semimodules:-

1) Let $R = \mathbb{Z}_8$ as a semiring, and let $I = R\bar{2} = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$. So $T = T_I = \{\bar{1}, \bar{3}, \bar{5}, \bar{7}\}$. Let $H = R\bar{4}$. $H' = \{ x \in Q(R) \mid xH \subseteq I \}$. It is easy to check that $Q(R) = R$, and hence $H' = R$. Then $H'H = H \neq I$. Thus H is not invertible in I.

2) Let $\mathbb{N}$ be the semiring of non-negative integer numbers and $0 \neq a \in \mathbb{N}$. Let $I = a\mathbb{N}$, since the set S of all not zero-divisors of $\mathbb{N}$ is $\mathbb{N} - \{0\}$, hence

$$T = T_I = \{s \in \mathbb{N} - \{0\} \mid sa \neq 0\} = \mathbb{N} - \{0\}.$$

Therefore, $(a\mathbb{N})' = I' = \{ x \in \mathbb{Q}^+ \mid x(a\mathbb{N}) \subseteq \mathbb{N} \} = \frac{1}{a}\mathbb{N}$, where $\mathbb{Q}^+$ is the semifield of non-negative rational numbers. Then it is clear that $I' = I^{-1}$. Since I is an invertible ideal in $\mathbb{N}$, we have $I^{-1}I = I'I = \mathbb{N}$, and I is an invertible as subsemimodule. Now let $H = 4\mathbb{N}$ as a subsemimodule of the $\mathbb{N}$-semimodule $2\mathbb{N}$. Then $H' = \{x \in \mathbb{Q}^+ \mid x(4\mathbb{N}) \subseteq 2\mathbb{N}\}$.

One can check that $H' = \frac{1}{2}\mathbb{N}$, therefore $H'H = (\frac{1}{2}\mathbb{N})(4\mathbb{N}) = 2\mathbb{N}$, i.e., $4\mathbb{N}$ is an invertible subsemimodule in $2\mathbb{N}$.

3) Consider $\mathbb{Q}^+$ as an $\mathbb{N}$-semimodule. Suppose that N be a non-zero subsemimodule of $\mathbb{Q}^+$. Since $\mathbb{Q}^+$ is torsion-free, then $T = S = \mathbb{N} - \{0\}$, and $Q_T(R) = Q(R) = \mathbb{Q}^+$. Thus $N' = \{\frac{x}{y} \in \mathbb{Q}^+ \mid (\frac{x}{y})N \subseteq \mathbb{Q}^+\}$. It is clear that $N' = \mathbb{Q}^+$, and we obtain $\mathbb{Q}^+N = \mathbb{Q}^+$, hence $\mathbb{Q}^+$ is a Dedekind $\mathbb{N}$-semimodule.

4) Consider $\mathbb{Z}_n$ as a $\mathbb{Z}$-semimodule, where n is any positive integer >1, which is not prime number. Let N be a non-zero proper subsemimodule of $\mathbb{Z}_n$. Now $T = \{m \in \mathbb{Z} \mid \gcd(m, n) = 1\}$. $Q_T(\mathbb{Z}) = \{\frac{r}{m} \in \mathbb{Q} \mid r, m \in \mathbb{Z},\ \gcd(m, n) = 1\}$. Hence it is clear that, $N' = \{ x \in Q_T(\mathbb{Z}) \mid xN \subseteq \mathbb{Z}_n\} = Q_T(\mathbb{Z})$. Therefore $N'N = Q_T(\mathbb{Z})N = N \neq \mathbb{Z}_n$. Hence N is not an invertible subsemimodule in $\mathbb{Z}_n$. While, if n is a prime number, then $\mathbb{Z}_n$ is simple semimodule; $\mathbb{Z}_n$ has no non-zero proper subsemimodule, hence is a D semimodule. Thus $\mathbb{Z}_n$ is a D semimodule iff n is a prime number.

5) Let p be a prime number, and let $\mathbb{N}_{(p)}$ be the set of rationals of the form m/n, with m and n are in $\mathbb{N}$ and n is not divisible by p. Then $\mathbb{N}_{(p)}$ is a subsemigroup of $\mathbb{Q}^+$. $\mathbb{N}_{p^\infty} = \mathbb{Q}^+/\mathbb{N}_{(p)}$ is an $\mathbb{N}$-semimodule. It is known that each proper non-zero subsemigroup of $\mathbb{N}_{p^\infty}$ is cyclic of the form $\mathbb{N}_{p^n}$. Note that since each element of $f(\mathbb{N}_{p^n})$, where $f \in Hom(\mathbb{N}_{p^n}, \mathbb{N}_{p^\infty})$ is of order less than or equal to p^n. Thus $\mathbb{N}_{p^\infty} \neq \sum_{f \in H} f(\mathbb{N}_{p^n})$, where $f \in Hom(\mathbb{N}_{p^n}, \mathbb{N}_{p^\infty})$. Hence by Proposition 1.15, we have $\mathbb{N}_{p^\infty}$ has no proper invertible subsemimodule.

Lemma (1.18): Let M_1 and M_2 be torsion-free R-semimodules and f be an R-epimorphism from M_1 to M_2. If N is an invertible subsemimodule of M_1 then $f(N)$ is an invertible subsemimodule of M_2.

Proof: Suppose N is invertible subsemimodule in M_1. Then $N'N = M_1$, $N' = \{x \in Q_T(R) \mid xN \subseteq M_1\}$.
If $x \in N'$ then $xN \subseteq M_1$ and so $xf(N) = f(xN) \subseteq M_2$.
So $N' \subseteq (f(N))' = \{x \in Q_T(R) \mid xf(N) \subseteq M_2\}$.
Take $m \in M_2$. Let $m' \in M_1$ be such that $f(m') = m$.
Then $m' = x_1 n_1 + \cdots + x_k n_k$ for some $k \in \mathbb{N}$, $x_i \in N'$ and $n_i \in N$.
Then $m = f(m') = x_1 f(n_1) + \cdots + x_k f(n_k)$, and therefore $M_2 = N' f(N) \subseteq (f(N))' f(N) \subseteq M_2$.
Thus $f(N)$ is an invertible subsemimodule in M_2. ■

Corollary (1.19): Every homomorphic image of a Dedekind semimodule is again Dedekind. ■

Remark (1.20): If N is a non-zero proper direct summand of an R-semimodule M, then N is not invertible subsemimodule in M.

Proof: Let N be invertible subsemimodule in M; thus $N'N = M$, where $N' = \{x \in Q_T(R) | xN \subseteq M\}$, and $T = \{ s \in S | sm = 0$ for some $m \in M$ implies $m = 0 \}$. Since N is a direct summand of M, i.e. there is a subsemimodule K of M such that $M = N \oplus K$. If $0 \neq k \in K$, since N is invertible in M, then by Proposition 1.13, $\exists\ t \in T$ with $tk \in N$, but $tk \in K$, hence $tk \in N \cap K = (0)$, and since $t \in T$, then $k = 0$, which is a contradiction, then N is not invertible in M.

Corollary (1.21): It easy checked that if $M = N \oplus K$, and N is an invertible subsemimodule in M, then $M = N$.

Proposition (1.22): Let R be a semiring and I be a non-zero ideal of R, then I is an invertible ideal in R if and only if I is an invertible R-subsemimodule in $_R$R.

Proof: Let S be the set of all not zero-divisors of R. Then $T = T_I = \{ s \in S \mid sa = 0$ for some $a \in I$ implies $a = 0 \}$. So that $T = S$. Thus $Q_T(R)$ is the total quotient semiring $Q(R)$. Hence $I' = I^{-1}$. i.e. $I'I = I^{-1}I$, and so I is an invertible ideal in R if and only if I is invertible R-subsemimodule in $_R$R.

A semiring R is **semidomain** if $ab = ac$ implies $b = c$ for all $b, c \in R$ and all non-zero $a \in R$ [6]. We say that a semidomain R is said to be a **Dedekind semidomain** if every non-zero ideal of R is invertible in R [6]. According to the equivalent conditions explained on page 143 in Narkiewicz' book [7], a Dedekind domain is a domain in which non-zero fractional ideals form a group under multiplication. Inspired by this, we give the following definition: We define a semidomain R to be a Dedekind semidomain if every non-zero fractional ideal of R is invertible. Hence R is a Dedekind semidomain if and only if Frac(R) is an abelian group.

Corollary (1.23): Let R be a semiring. Then

1) R is Dedekind R-semimodule if and only if R is a Dedekind semidomain.

2) R is D_1 semimodule if and only if R is a semidomain, i.e. each non-zero principal ideal of R is invertible as a subsemimodule in R if and only if it is generated by not a zero-divisor.

The following remark shows that D_1 semimodule may not be D semimodule.

Remark (1.24): Let R be a semidomain, and R_1 the polynomial semiring $R[x, y]$ in two independent variables x and y. Then R_1 is a semidomain. By Corollary 1.21, R_1 is a D_1 semimodule. But if we take the ideal I generated by x and y, it is clear that I is not invertible subsemimodule of R_1. Thus R_1 is not a D R_1 −semimodule.

Next, we defined the notion of "**essential**" subsemimodule. In Golan book's [8], it was proposed the following definitions. An R-monomorphism $f: M \longrightarrow M'$ of R-semimodules is essential if for any R-homomorphism $g : M' \longrightarrow M''$, $g \circ f$ is a monomorphism implies that g is a monomorphism.

A subsemimodule N of an R-semimodule M is essential (or large) in M if the inclusion mapping $i_N: N \longrightarrow M$ is an essential R -monomorphism. Note that $f: M \longrightarrow M'$ is an essential R-homomorphism if and only if $f(M)$ is a large subsemimodule of M' [8].

Another way for defining the notion of "essential" is proposed in [9] as follows. A subsemimodule N of M is said to be semi-essential in M , written as $N \lhd_s M$, if for every subsemimodule H of M : $N \cap H = 0 \Rightarrow H = 0$. A monomorphism $f: M \longrightarrow M'$ of R-semimodules is said to be semi-essential if: $f(M) \lhd_s M'$.

In [9], we have the following characterization of semi-essential subsemimodules.

Lemma (1.25): A subsemimodule N of an R-semimodule M is a semi-essential if and only if for each $0 \neq m \in M$, there exists $r \in R$ such that $0 \neq rm \in N$.

Lemma (1.26): Every invertible subsemimodule of M is a semi-essential subsemimodule of M.

Proof: Let N be invertible subsemimodule of M. Let $0 \neq m \in M$. By Proposition 1.13, $\exists\ t \in T$ such that $0 \neq tm \in N$ and hence N is essential

Proposition (1.27): Let M be a D_1 semimodule. Then ann(Rm)=ann(M), for each $0 \neq m \in M$.

Proof: It is clear that ann(M) $\subseteq$ ann(Rm), so it is enough to show that ann(Rm) $\subseteq$ann(M). Let $r \in$ann(Rm), then $rm = 0$. Let $a \in M$. Since M is a D_1 semimodule; then Rm is invertible in M, and hence by Corollary 1.14, $\exists t \in T, s \in R$ such that $ta = sm$. Thus $tra = rsm = 0$. Hence $ra = 0$, and ann(Rm) $\subseteq$ann(M). This completes the proof.

From now on, we will put $End_R(M)$, for the semiring of endomorphisms of R-semimodule M.

Lemma (1.28): Let M be a non-zero R-semimodule and $f \in End_R(M)$. If ker$f$ contains an invertible subsemimodule of M then $f = 0$. Therefore if M is a D_1 semimodule then every non-zero element of $End_R(M)$ is a monomorphism.

Proof: Let $N \subseteq \ker f$ is invertible in M. Then by Proposition 1.13, $\forall m \in M, \exists t \in T, and\ n \in N$ such that $tm = n$. So $0 = f(n) = tf(m)$; but $t \in T$ hence $f(m) = 0$ and $f = 0$.

Now assume that M is a D_1 semimodule and $0 \neq f \in End_R(M)$. Let $0 \neq k \in \ker f$, then Rk invertible in M and subset of $\ker f$ from above; we have $f = 0$, which is a contradiction, then $\ker f = 0$, and f is a monomorphism.

For any R-semimodule M, there exists an obvious semiring monomorphism:
$\Phi : R/\text{ann}(M) \longrightarrow End_R(M)$. Hence one may think of as a subsemiring of $End_R(M)$. So we have:

Corollary (1.29): If M is a D_1 semimodule, then $R/\text{ann}(M)$ is a semidomain and thus $\text{ann}(M)$ is a prime ideal.

As a special case, we record the following.

Corollary (1.30): If a semiring R is a D_1 R-semimodule. Then R is a semidomain.

2. Multiplication Semimodules

In this section we study multiplication semimodules. We begin with following definition:

Definition (2.1): Let R be a semiring and M an R-semimodule. Then M is said to be **multiplication semimodule** if for all subsemimodule N of M there exists an ideal I of R such that $N = IM$. In this case it is easy to show that $N = [N : M]M$. For instance, all cyclic R-semimodule are multiplication R-semimodule [10, Example 2].

Note that, if I is an ideal of R, then the set IM consisting of all finite sums of elements $r_i m_i$ with $r_i \in R$ and $m_i \in M$ is a subsemimodule of M.

Example(2.2): Let R be a multiplicatively idempotent semiring. Then all ideals of R are multiplication R-semimodule [11].

An element r of a semiring R is multiplicatively-cancellable (abbreviated as MC),
if $rx = rwy$ implies $x = y$ for all $x, y \in R$. Each non-zero element in a semidomain is an MC element.

Theorem (2.3): Let R be a semiring. An ideal I of R is invertible if and only if it is a multiplication R-semimodule which contains an MC element of R, see [11].

Proposition(2.4): Let R be a semiring. An R-semimodule M is multiplication semimodule if and only if for each m in M there exists an ideal I of R such that $Rm = IM$.

Proof: The necessity is clear. For the sufficiency, assume that for each $m \in M$ there exists an ideal I of R such that $Rm = IM$. Let N be a subsemimodule of M. For each $m \in N$ there exists an ideal I_m such that $Rm = I_m M$. Let $I = \sum_{m \in N} I_m$. Hence $N = \sum_{m \in N} Rm = \sum_{m \in N} I_m M = IM$. Therefore M is a multiplication semimodule. ∎

Theorem (2.5): Let M be a multiplication semimodule over a semiring R. If N is a finitely generated subsemimodule of M, then there exists a finitely generated ideal I of R such that $N = IM$.

Proof: Suppose that $N =< x_1, x_2, \cdots, x_n >$. Since M is a multiplication, we have $N = [N : M]M$. So, there exists $a_{i,j} \in [N : M]$ and $y_{i,j} \in M$ such that $x_i = a_{i,1}y_{i,1} + \cdots + a_{i,r}y_{i,r}$ for $i = 1,2, \cdots, n$ and $j = 1,2, \cdots, r$. Let I be an ideal of R generated by $\{a_{1,1}, \cdots, a_{n,r}\}$. It is easy to see that $I \subseteq [N : M]$ and $IM \subseteq [N : M]M$. On the other hand, since for every $i, x_i \in IM$, we must have $N \subseteq IM$. Hence $N \subseteq IM \subseteq [N : M]M \subseteq N$. Thus $N = IM$ and I is finitely generated. ∎

The following shows that every homomorphic image of a multiplication semimodule is again multiplication [11].

Theorem(2.6): Let M and N R-semimodules and $f: M \longrightarrow N$ a surjective R-homomorphism. If M is a multiplication R-semimodule, then N is a multiplication R-semimodule.

A semiring R is called **yoked** if for all $a, b \in R$, there exists an element $t \in R$ such that $a + t = b$ or $b + t = a$ [8, p. 49]. A semiring is **entire** if $ab = 0$ implies that $a = 0$ or $b = 0$ [8, p. 4].

An R-semimodule M is called **multiplicatively cancellative** (or simply MC) if for any $r, r' \in R$ and $0 \neq m \in M, rm = r'm$ implies $r = r'$ [11]. For example every ideal of a semidomain R is an MC R-semimodule.

Note that if M is an MC R-semimodule, then M is a faithful semimodule. Let $rM = \{0\}$ for some $r \in R$. If $0 \neq m \in M$, then $rm = 0m = 0$. Hence $r = 0$. Thus M is faithful.

An element m of an R-semimodule M is called cancellable if $m + m' = m + m''$ implies that $m' = m''$. The semimodule M is **cancellative** if and only if every element of M is cancellable [8, P. 172].

Lemma (2.7):[11] Let R be a yoked entire semiring and M a cancellative faithful

Multiplication R-semimodule. Then M is an MC semimodule.

Theorem(2.8):[11] Let R be a yoked semidomain and M a cancellative torsion-free R-semimodule. Then M is an MC semimodule.

Lemma(2.9):[11] Let M be an R-semimodule and $\theta(M) = \sum_{m\in M}[Rm : M]$. If M is a multiplication R-semimodule, then $M = \theta(M)M$.

Theorem (2.10): [11] Let R be a semiring and M is an MC multiplication R-semimodule. Then M is finitely generated.

By Lemma 2.7, we have the following result.

Corollary (2.11): Let R be an entire yoked semiring and M a cancellative faithful multiplication R-semimodule. Then M is finitely generated.

The next theorems give a characterization of MC multiplication semimodules, for the proof see[11].

Theorem(2.12): If M is an MC multiplication R-semimodule. Then M is a projective R-semimodule.

Theorem (2.13): Let R be a semidomain. If M is an MC multiplication R-semimodule, then M is a torsion-free semimodule.

Theorem (2.14): Let R be a semidomain. If M is an MC multiplication R-semimodule, then M is isomorphic to an invertible ideal in R.

3. Dedekind Multiplication Semimodules

From Remark 2.3 we can say that a semiring R is a Dedekind semidomain iff each non-zero ideal in R is a multiplication ideal which contains a not zero-divisor. In this section we study Dedekind multiplication semimodules. We begin with the following.

Lemma (3.1): Let M be a torsion-free R-semimodule. If N is an invertible subsemimodule of M and I is an invertible ideal in R, then IN is an invertible subsemimodule of M.

Proof: Suppose $H = IN$. But $N'N = M, I^{-1}I = R$, and hence $I^{-1}N'H = (I^{-1}I)N'N = M$. From Proposition 1.7, we have $T_M = S$ and from Proposition 1.22, we have $Q_T(R) = Q(R)$. Hence easy to see that $I^{-1}N' \subseteq H'$. By above we have $I^{-1}N' = H'$, and H is invertible.

Lemma (3.2): Let M be a non-zero R-semimodule and I is invertible ideal in R. Then IM is an invertible subsemimodule of M.

Proof: Suppose $K = IM$. But $I^{-1}I = R$, and hence $I^{-1}K = (I^{-1})IM = (I^{-1}I)M = RM = M$. From Proposition 1.22, we have $Q_T(R) = Q(R)$, thus it follows that $I^{-1} \subseteq K'$. Hence $M = I^{-1}K \subseteq K'K \subseteq M$, so $K'K = M$, and K is invertible.

A subsemimodule N of an R-semimodule M is called **invariant** subsemimodule if $f(N) \subseteq N$, $\forall f \in Hom(M, M)$, [3, 12].

Definition (3.3): A semimodule M is said to be **duo** if each subsemimodule of M is invariant, [12].

In [12], we have the following characterization of duo subsemimodules.

Theorem(3.4): Let R be a yoked semidomain, and M a torsion-free R-semimodule. Then M is duo if and only if for each R-endomorphism f of M, there exists r in R such that $f(m) = rm$ for all $m \in M$.

Remark(3.5): It is clear that any multiplication semimodule is duo. Hence by Theorem 3.4, if M is a multiplication torsion-free semimodule over a yoked semidomain R, then for each $f \in End_R(M)$, $\exists r \in R$, such that $f(m) = rm$ for all $m \in M$.

Corollary (3.6): If M is a torsion-free multiplication semimodule over a yoked semidomain R, then there exists an epimorphism of semirings from R onto $End_R(M)$.

Proof: By Remark 3.5, $\forall f \in End_R(M)$, $\exists r \in R$, such that $f = f_r$ and $f_r(m) = rm$ for all $m \in M$. Hence $\phi: R \longrightarrow End_R(M)$, defined by $\phi(r) = f_r$. It is easily check, that ϕ is an epimorphism of semirings.

Theorem(3.7): If M is a torsion-free multiplication semimodule over a yoked semidomain R, then

$$End_R(M) \cong R/\text{ann}(M)$$

Proof: By Corollary 3.6, $ker\phi = \{r \in R | \phi(r) = 0\} = \{r \in R | f_r = 0\} = \{r \in R | rm = 0 \ \forall m \in M\}$ =ann(M). But $End_R(M) \cong R/ker\phi$, then $End_R(M) \cong R/\text{ann}(M)$.

By Lemma 2.7, Theorem 2.13, and Theorem 3.7 we have.

Theorem(3.8): If M a cancellative faithful multiplication semimodule over a yoked semidomain R. Then $End_R(M) \cong R$.

The following lemma shows the importance of the faithful multiplication semimodules.

Lemma(3.9): Let M be a finitely generated cancellative faithful multiplication semimodule over a yoked semidomain R. If $N = IM$ is an invertible subsemimodule of M for some ideal I of R, then I is an invertible ideal in R..

Proof: Since $N \neq 0$, then $I \neq 0$. By assumption $N'N = M$, hence $M = N'N = N'IM$. It is clear that $N'I$ is an R-subsemimodule of R. Also, it is easy to see that every element of $N'I$ can be considered as an R-endomorphism of M. Now, since M is a faithful multiplication semimodule, then by Theorem 3.8, $End_R(M) \cong R$. Therefore $N'I$ is an ideal in R. As in modules see [13], it follows that $N'I = R$. Hence $N' \subseteq I^{-1}$, so $R = N'I \subseteq I^{-1}I \subseteq R$ which implies $I^{-1}I = R$.

Theorem (3.10): Let M be a cancellative faithful multiplication R-semimodule over a yoked Dedekind semidomain R. Then M is a finitely generated Dedekind R-semimodule.

Proof: Since M is a faithful multiplication semimodule, and R is a semidomain. By Corollary 2.11, we have M is a finitely generated. Now, let N be a non-zero subsemimodule of M. Hence there exists a non-zero ideal I in R such that $N = IM$. Since R is a Dedekind semidomain, thus I is invertible in R, and by Lemma 3.2, N is invertible.

The following theorem is a converse of above theorem:

Theorem (3.11): Let M be a cancellative faithful multiplication semimodule over a yoked semidomain R. If M is a Dedekind semimodule, then R is a Dedekind semidomain.

Proof: By assumption, R is a semidomain. By Corollary 2.11, we get M is a finitely generated. Assume that I is any non-zero ideal of R. Then IM is a non-zero subsemimodule of M, hence IM is invertible. From Lemma 3.9, I is an invertible ideal.

A semidomain R is said to be a **Prüfer semidomain** if every non-zero finitely generated ideal of R is invertible in R [6]. Note that R is a Dedekind semidomain if and only if R is a Noetherian (each of its ideals is finitely generated) Prüfer semidomain.

Let D be a Dedekind domain (D is a ring). By Theorem 3.7 in [4], the semiring of ideals Id(D) of D (the set of all ideals of D) is a Prüfer semidomain. By Theorem 3.7 in [4], Id(D) is subtractive (each of its ideals is subtractive). If Id(D) is also Noetherian, then Id(D) is a Dedekind semidomain. Note that the semiring Id(D) is proper semiring, i.e., it is not a ring. If D is a Dedekind semidomain then the above argument remains true. Note that, each Noetherian Prüfer semidomain is Dedekind.

For a more specific example, we assert that (Id(ℤ),+,·) is a principal ideal semidomain (each of its ideals is principal) [6]. Hence, Id(ℤ) is evidently a Dedekind semidomain. Note that the semiring (Id(ℤ),+,·) is isomorphic to the semiring (ℕ, gcd, ·).

Definition (3.12): A semimodule M is said to be a **Prüfer semimodule** if every non-zero finitely generated subsemimodule of M is invertible in M.

The proof of the following theorem is basically the same as the proof of the last results.

Theorem (3.13): Let M be a cancellative faithful multiplication semimodule over a yoked semiring R. Then M is a $Prüfer$ semimodule if and only if R is a $Prüfer\ semidomain$.

If M is a D_1 semimodule, we have the following remark which is special case of above theorem.

Remark (3.14): Let M be a cancellative faithful multiplication semimodule over a yoked semiring R. Then M is a D_1 semimodule if and only if R is a $semidomain$.

Proof:(⇒) By Corollary 1.29, we get R is a semidomain, so each non-zero principal ideal is invertible.

(⇐) Assume that R is a semidomain. Let now Rm be a non-zero cyclic subsemimodule of M, $Rm = IM$, for some ideal I of R. In this case we can take $I = [Rm{:}M]$, and hence $Rm = [Rm{:}M]M$. By Corollary 2.11, we get M is finitely generated, and thus $[Rm{:}M]$ is a multiplication ideal in R [13]. But R is a semidomain; thus by Theorem 2.3, $[Rm{:}M]$ is an invertible ideal in R. Then by Lemma 3.2, Rm is an invertible subsemimodule of M.

Proposition (3.15): If M is a faithful multiplication Dedekind R-semimodule. Then $M^* = Hom_R(M, R)$ is also a faithful multiplication Dedekind R-semimodule.

Proof: Similarly in the proof of Theorem 3.10, M is a f.g. faithful multiplication semimodule. So as in the modules see Corollary (2) of [2], we obtain that M^* is a f.g. faithful multiplication R-semimodule. By assumption and using Theorem 3.11, we get R is a Dedekind semidomain. Now M^* is a f.g. faithful multiplication R-semimodule over the Dedekind semidomain R, then by Theorem 3.10, M^* is a Dedekind R-semimodule.

4. Embedding of Semimodules

In this section we study "embeddability proplem" , thus we look for necessary and (or) sufficient conditions under which an R-semimodule A is isomorphic to a subsemimodule of the R-semimodule B. Now, put $H = Hom_R(A, B)$, H is an R-semimodule. We start by the following.

Proposition (4.1): Let A and B be R-semimodules. If there exists a monomorphism $f \in H$, then ann(Rf) =ann(H).

Proof: It is clear that ann$(H) \subseteq$ ann(Rf), so it is enough to show that ann$(Rf) \subseteq$ ann(H). Let $r \in$ ann (Rf) , then $0 = rf(A) = f(rA)$. But f is a monomorphism, therefore $rA = (0)$, and $r \in$ann(A). But it is easily seen that ann$(A) \subseteq$ ann(H), thus ann(Rf) =ann(H).

Remark (4.2): The converse of Proposition 4.1 is not true in general.

Proof: Let A be a projective R-semimodule with a non-commutative endomorphisms semiring, $E(A)$ (for example A can be any free semimodule of rank >1, such as $A = \mathbb{Z}\oplus\mathbb{Z}$ as $\mathbb{Z}$-semimodule). Put $B = A\oplus R$. Then $B^* = A^*\oplus R^* \cong A^*\oplus R$, where $B^* = Hom(B, R)$ and $A^* = Hom(A, R)$. If β represents a generator of a semiring R in the last direct sum, hence it is clear that ann$(R\beta) =$ ann$(B^*) = 0$. Whereas B^* does not contain any monomorphism. To prove this, let $f \in B^*$ such that $kerf = 0$. Thus $f(B)$ is a projective ideal of R (since B is projective). And thus by [14], $f(B)$, so also B is a multiplication ideal. By [15], $End_R(B)$ is commutative. By [16, lemma 2.1], we have $End_R(A)$ is commutative, which is a contradiction.

Now, let us observe that if there exists a monomorphism $f: A \longrightarrow B$, for any R-semimodules, A and B, then it is clear that $\bigcap_{\forall g\in H} ker\, g = (0)$.

The following theorem gives a sufficient condition for the existence of a monomorphism in $H = Hom(A, B)$, in the case A is a multiplication R-semimodule.

Theorem(4.3): Let A be a multiplication R-semimodule and B any R-semimodule such that $\bigcap_g ker\, g = (0), \forall\, g \in H = Hom(A, B)$. Then for any $f \in H$, then f is a monomorphism iff ann(Rf) =ann(H).

Proof: ($\Rightarrow$) If f is a monomorphism then by Proposition 4.1, we have ann(Rf) =ann(H).

($\Leftarrow$) Put $N = kerf$. There is an ideal I in R such that $N = IA$. So $(0) = f(N) = f(IA) = If(A)$, which implies $I \subseteq$ ann(Rf). Then $IH = (0)$, hence $I \subseteq kerg, \forall\, g \in H$, and thus $IA = (0)$. Therefore $N = (0)$ and f is a monomorphism.

As a special case of Theorem 4.3, we give the following , comparison with [2, Lemma(1.1)]. We say that an R-semimodule A is called **torsionless** if $\bigcap_g ker\, g = (0), \forall\, g \in A^*$.

Corollary (4.4): Let A be a torsionless multiplication R-semimodule. Then A is embeddable in R iff $\exists \beta \in A^*$ such that ann$(R\beta)$ =ann(A^*).

More generally, we have:

Corollary (4.5): Let A be a torsionless multiplication R-semimodule. Then A is embeddable in R^n iff $\exists$ a f.g. subsemimodule N of A^*, which is generated by a set $\{\beta_1, \beta_2, \dots, \beta_n\}$, where $\beta_i \in A^*, 1 \le i \le n$ and ann(N) =ann(A^*).

Proof: ($\Rightarrow$) Assume that A embeds in R^n, i.e. $\exists\beta: A \longrightarrow R^n$ which is a monomorphism. $\forall i, 1 \le i \le n$ define $\beta_i: A \longrightarrow R$ as follows $\beta_i = \rho_i \circ \beta$, where $\rho_i\ \forall i, 1 \le i \le n$ is the natural projection of R^n onto the ith component. Note, since $Hom(A, R^n)$ is isomorphic to the direct sum of n copies of $A^* = Hom(A, R)$. Therefore ann$(Hom(A, R^n))$ = ann(A^*) and since β is a monomorphism hence, by Proposition 4.1 ann(β) =ann(A^*). Now, ann$(\beta) = \bigcap_{i=1}^{n} ann(\beta_i)$ =ann(N). Thus ann(N) =ann(A^*).

($\Leftarrow$) Assume that $\exists$ a f.g. subsemimodule N of A^*, which is generated by a set $\{\beta_1, \beta_2, \dots, \beta_n\}$, and ann $(N) =$ ann (A^*) . Now let us define an R-homomorphism $\beta: A \longrightarrow R^n$ as follows $\beta(x) = \big(\beta_1(x), \beta_2(x), \dots, \beta_n(\mathrm{x})\big), \forall x \in A$. Now since ann$(Hom(A, R^n))$ = ann(A^*), and by assumption ann(A^*) =ann$(N) = \bigcap_{i=1}^{n} ann(\beta_i)$ =ann(β). Therefore by using Theorem 4.3, we obtain β is a monomorphism in $Hom(A, R^n)$.

From our main results in this section, is that if $\exists \beta \in A^*$ such that $(R\beta)$ is invertible in A^*, and A is torsionless, then β is a monomorphism, and hence A embeds in R, this means A is isomorphic to an ideal of R. But now, let us recall that for any R-semimodule B,

$T_B = \{s \in S \mid$ if $sb = 0$ for some $b \in B$, then $b = 0\}$. Hence, for an R-semimodule $H = Hom(A,\ B)$, $T_H = \{s \in S \mid$ if $s\beta = 0$ for some $\beta \in H$, then $\beta = 0\}$.

Theorem(4.6): Let A and B be any two R-semimodules, with $\bigcap_{\beta\in H} ker\, \beta = (0)$, and $T_H \subseteq T_B$. If there exists a cyclic invertible subsemimodule (Rf) in H, then f is a monomorphism, and hence A embeds in B. Moreover, if $\sum_{\beta\in H} \beta(A) = B$, then $f(A)$ is invertible subsemimodule in B.

Proof: By Corollary 1.14 $\forall\beta \in H, \exists t \in T_H, s \in R$ such that $t\beta = sf$. Put $N = kerf$ and let $x \in N$, then $sf(x) = t\beta(x) = 0$, which implies $x = 0$. Thus f is a monomorphism. Next, by assumption, $\forall b \in B, \exists f_1, f_2, \dots, f_m \in H$ and $a_1, a_2, \dots, a_m \in A$ such that $b = \sum_{i=1}^{m} f_i(a_i)$. Since (Rf) is invertible in H, so by Corollary 1.14 $\forall i, 1 \le i \le m, \exists s_i \in R, t_i \in T_H$ such that $f_i = \frac{s_i}{t_i} f$. Hence $b = \sum_{i=1}^{m} \frac{s_i}{t_i} f(a_i)$, and by Proposition 1.13, we obtain that $f(A)$ is invertible in B.

The following two corollaries are special case of Theorem 4.6.

Corollary(4.7): Let A be a torsionless R -semimodule. If A^* contains a cyclic invertible subsemimodule, then A is isomorphic to an ideal of R. Further if trace$(A) = R$, then A is isomorphic to an invertible ideal, and thus is a faithful multiplication semimodule.

Proof: Since $T_R = S$, where S is the set of all non-zero devisor in R, and hence $T_{A^*} \subseteq T_R$. Let $\alpha \in A^*$ such that $(R\alpha)$ is invertible in A^*. Thus by Theorem 4.6, α is a monomorphism. Since trace$(A) = \sum_{\beta \in A^*} \beta(A) = R$, again by Theorem 4.6, $\alpha(A)$ is an invertible subsemimodule of R. Hence by Proposition 1.20, $\alpha(A)$ is an invertible ideal in R. By Remark 2.3, we obtain $\alpha(A)$, and hence A is a faithful multiplication semimodule

Corollary (4.8): Let A be a torsionless R-semimodule. If A^* contains a f.g. invertible subsemimodule N, and N can be generated by n elements. Then A embeds in R^n.

Proof: Let $\{\beta_1, \beta_2, \dots, \beta_n\}$ be a set of generators of N. Define $\beta: A \longrightarrow R^n$, as follows, $\beta(x) = (\beta_1(x), \beta_2(x), \dots, \beta_n(x)), \forall x \in A$. Now our aim is to show that β is a monomorphism. Since N is invertible in A^*, then by Proposition 1.13, we have $\forall \alpha \in A^*, \exists t \in T_{A^*} \subseteq S$ and $\exists r_i \in R, 1 \le i \le n$ such that $t\alpha = \sum_{i=1}^{n} r_i \beta_i$. Now, let $y \in ker\beta = \bigcap_{i=1}^{n} ker\beta_i$. Thus $t\alpha(y) = \sum_{i=1}^{n} r_i \beta_i(y) = 0$, but $t \in S$, then $\alpha(y) = 0\ \forall \alpha \in A^*$, i.e $y \in \bigcap_{\alpha} ker\alpha = (0)$. Thus $ker\beta = (0)$, and A embeds in R^n.

Theorem (4.9): Let M be a Dedekind semimodule and let m be a non-zero element of M. Then M is isomorphic to the R-subsemimodule $(Rm)'$ of Q(R).

Proof: Since M is a Dedekind semimodule, then $\forall\, m_1 \in M$, $\exists\, z \in (Rm)'$ such that $m_1 = zm$. Define a homomorphism $f: (Rm)' \longrightarrow M$ with $f(z) = zm$ for each $z \in (Rm)'$. It is clear that f is an R-isomorphism.

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