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Subclasses of Analytic Functions of Complex Order Involving Generalized Jackson's (p, q)-derivative

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Abstract
This paper aims at introducing a new generalized differential operator and new subclass of analytic functions to obtain some interesting properties like coefficient estimates and fractional derivatives.

Keywords: Differential operator, analytic functions, coefficient inequality, fractional integrals, fractional derivatives.

1. Introduction

Let A represents the class of functions of the form

$$f(z)=z+\sum_{m=2}^{\infty}c_m z^m, \tag{1.1}$$

which are analytic in the open unit disk $U=\{z\in \mathbb{C}: |z|<1\}$.

We also consider T as a subclass of A consisting of functions of the form

$$f(z)=z-\sum_{m=2}^{\infty}c_m z^m, c_m\geq 0, z\in U \tag{1.2}$$

The theory of $q-$analysis was introduced and studied by Silverman [1]. The theory of $q-$analysis has a significant role in different areas of mathematics, such as ordinary fractional calculus, $q-$difference, and q-integral equation. It is a fact that the $q-$difference operator plays a critical role in the theory of hypergeometric series (for more information on $q-$calculus, see [2-8]). Some concepts and definitions of $q-$calculus used in this paper are illustrated.

For $p=0$ and $q=1$, the Jackson`s $(p,q)-$derivative of a function $f\in A$ is given as follows [9]:

$$D_{p,q}f(z)=\begin{cases}\frac{f(pz)-f(qz)}{(p-q)z} & for \quad z\neq 0\\ f'(0) & for \quad z=0\end{cases} \tag{1.3}$$

From (1.1), we have

$$D_q f(z)=1+\sum_{m=2}^{\infty}[m]_{p,q}c_m z^{m-1}$$

where

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$$[m]_{p,q}=\frac{p^m-q^m}{(p-q)z}$$

is called $(p,q)-$ bracket. Obviously, for a function $h(z)=z^m$, we obtain $D_{p,q}h(z)=D_{p,q}z^m=\frac{p^m-q^m}{p-q}z^{m-1}=[m]_{p,q}z^{m-1}$.
We note that for $\mathrm{p}=1$, the Jackson $\mathrm{p,q}-$ derivative will be reduced to the Jackson $q-$derivative [8].
For $f\in A$, we illustrate the Salagean $(p,q)-$differential operator as follows:

$$D^0_{p,q}f(z)=f(z)$$

$$D^1_{p,q}f(z)=zD^n_{p,q}f(z)$$

$$D^n_{p,q}f(z)=zD^n_{p,q}(D^{n-1}_{p,q}f(z))$$

$$D^n_{p,q}f(z)=z+\sum_{m=2}^{\infty}[m]^n_{p,q}c_mz^m,\ \ (n\in\mathrm{N}_0,z\in U)$$

Note that if $p=1$ and $\lim_q\to1^-$, we get the Salagean derivative [5]:

$$D^nf(z)=z+\sum_{m=2}^{\infty}m^nc_mz^m,\ \ (n\in\mathrm{N}_0,z\in U)$$

Now, in order to define our generalized operator, let

$$D^0f(\mathrm{z})=\mathrm{D}^n_{p,q}\ f(\mathrm{z})$$

$$\mathrm{D}^{1,n}_{p,q}\ f(\mathrm{z})=(1-\beta(\lambda-\alpha))\,\mathrm{D}^n_{p,q}\ f(\mathrm{z})+\beta(\lambda-\alpha)z\left(\mathrm{D}^n_{p,q}\ f(\mathrm{z})\right)'+\kappa z^2\left(\mathrm{D}^n_{p,q}\ f(\mathrm{z})\right)''$$

$$\mathrm{D}^{2,n}_{p,q}\ f(\mathrm{z})=(1-\beta(\lambda-\alpha))\,\mathrm{D}^{1,n}_{p,q}\ f(\mathrm{z})+\beta(\lambda-\alpha)z\left(\mathrm{D}^{1,n}_{p,q}\ f(\mathrm{z})\right)'+\kappa z^2\left(\mathrm{D}^{1,n}_{p,q}\ f(\mathrm{z})\right)''$$

In general, we have

$$\begin{aligned}D^{\tau,n}_{\lambda,\beta,\alpha,p,q}f(z)&=(1-\beta(\lambda-\alpha))D^{\tau-1,n}_{\lambda,\beta,\alpha,p,q}f(z)+\beta(\lambda-\alpha)z\left(D^{\tau-1,n}_{\lambda,\beta,\alpha,p,q}f(z)\right)'\\&\quad+\kappa z^2\left(D^{\tau-1,n}_{\lambda,\beta,\alpha,p,q}f(z)\right)''\\&=z+\sum_{m=2}^{\infty}[m]^n_{p,q}\left[1+(m-1)\left(\beta(\lambda-\alpha)+m\kappa\right)\right]^{\tau}c_mz^m\end{aligned}\tag{1.4}$$

where $\alpha\geq0$, $\beta\geq0$, $z\in U,\lambda>0,\lambda\neq\alpha,\tau\in\mathrm{N}_0$.
It is noted that $D^{0,0}_{\lambda,\beta,\alpha,p,q}f(z)=f(z)$ and $D^{1,0}_{\lambda,\beta,\alpha,p,q}f(z)=zf'(z)$. Also, it is noted that when $p=1$, $\beta=1$ and $\alpha=0$, we get the differential operator $D^{\tau,n}_{\lambda,p,q}f(z)$ which was defined and studied by Al-Hawary et al. [10] . When $p=1$ and $\lim_{q\to1^-}$,we obtain the differential operator

$$D^{\tau,n}_{\lambda,\beta,\alpha,p,q}f(z)=z+\sum_{m=2}^{\infty}m^n\left[1+(m-1)\left(\beta(\lambda-\alpha)+m\kappa\right)\right]^{\tau}c_mz^m\ (\lambda,\beta>0,\ \ \alpha\geq0\ and\ \ \tau\in\mathrm{N}_0)$$

It is notable that when $m=\kappa=0$, the differential operator D^{τ}, defined by Ibrahim and Darus [11], is obtained. Also, when $\beta=1$ and $\alpha=0$, Al-Oboudi operator [12] is obtained and when $\tau=0$, Salagean differential operator [13] is obtained.
A function $f(z)$ that belongs to A is said to be in the class $S^{\tau,n}_{\lambda,\beta,\alpha,p,q}(b,\gamma)$ if it satisfies

$$\operatorname{Re}\left\{1+\frac{1}{b}\left(\frac{z\left(\left(D^{\tau,n+1}_{\lambda,\beta,\alpha,p,q}f(z)\right)\right)'}{D^{\tau,n}_{\lambda,\beta,\alpha,p,q}f(z)}+1-2\gamma\right)\right\}$$
$$>\zeta\left|1+\frac{1}{b}\left(\frac{z\left(\left(D^{\tau,n+1}_{\lambda,\beta,\alpha,p,q}f(z)\right)\right)'}{D^{\tau,n}_{\lambda,\beta,\alpha,p,q}f(z)}-1\right)\right|,(z\in U) \tag{1.5}$$

where $0<\gamma\le 1,\zeta\ge 0,\lambda,\beta>0,\alpha\ge 0,n,\tau\in \mathrm{N}_0$ and $b\in \mathrm{C}^*=\mathrm{C}-\{0\}$.

Further, we shall define the class $ST^{\tau,n}_{\lambda,\beta,\alpha,p,q}(b,\gamma)$ by

$$ST^{\tau,n}_{\lambda,\beta,\alpha,p,q}(b,\gamma)=S^{\tau,n}_{\lambda,\beta,\alpha,p,q}(b,\gamma)\cap T \tag{1.6}$$

In this paper, some properties like coefficient inequalities, distortion, and closure theorems for functions belonging to the class $ST^{\tau,n}_{\lambda,\beta,\alpha,p,q}(b,\gamma)$ are identified and proved.

2. Coefficient Inequalities

In this section, the coefficient inequalities for the class $ST^{\tau,n}_{\lambda,\beta,\alpha,p,q}(b,\gamma)$ are attained.

Theorem 2.1. Let $f(z)$ be defined by *(1.2)* and $f(\mathrm{z})\in ST^{\tau,n}_{\lambda,\beta,\alpha,p,q}(\mathrm{b},\gamma)$

$$\sum_{m=2}^{\infty}[(m[m]_{p,q}+|b|)(1-\zeta)+2\gamma-\zeta+1]\left[1+(m-1)\left(\beta(\lambda-\alpha)+m\kappa\right)\right]^{\tau}[m]^{n}_{p,q}|c_m|\le 2-2\gamma+|b|(1-\zeta) \tag{2.1}$$

where $0<\gamma\le 1,\zeta\ge 0$ and $b\in \mathrm{C}^*=\mathrm{C}-\{0\}$.

Proof. Suppose that the inequality (2.1) holds. Then, for $z\in U$, we have

$$\operatorname{Re}\left\{1+\frac{1}{b}\left(\frac{z\left(\left(D^{\tau,n+1}_{\lambda,\beta,\alpha,p,q}f(z)\right)\right)'}{D^{\tau,n}_{\lambda,\beta,\alpha,p,q}f(z)}+1-2\gamma\right)\right\}$$
$$>\zeta\left|1+\frac{1}{b}\left(\frac{z\left(\left(D^{\tau,n+1}_{\lambda,\beta,\alpha,p,q}f(z)\right)\right)'}{D^{\tau,n}_{\lambda,\beta,\alpha,p,q}f(z)}-1\right)\right|$$

$$\operatorname{Re}\left\{1+\frac{1}{b}\left(\frac{z(2-2\gamma)-\sum_{m=2}^{\infty}(m[m]_{p,q}+2\gamma+1)[m]^{n}_{p,q}\left[1+(m-1)\left(\beta(\lambda-\alpha)+m\kappa\right)\right]^{\tau}c_m z^m}{z-\sum_{m=2}^{\infty}[m]^{n}_{p,q}\left[1+(m-1)\left(\beta(\lambda-\alpha)+m\kappa\right)\right]^{\tau}c_m z^m}\right.\right.$$
$$\left.\left.+1-2\gamma\right)\right\}$$
$$>\zeta\left|1+\frac{1}{b}\left(\frac{\sum_{m=2}^{\infty}(m[m]_{p,q}-1)[m]^{n}_{p,q}\left[1+(m-1)\left(\beta(\lambda-\alpha)+m\kappa\right)\right]^{\tau}c_m z^m}{z-\sum_{m=2}^{\infty}[m]^{n}_{p,q}\left[1+(m-1)\left(\beta(\lambda-\alpha)+m\kappa\right)\right]^{\tau}c_m z^m}\right)\right|$$

By letting $z\to 1^-$, we haveThis implies that

$$1+\frac{1}{b}\left(\frac{(2-2\gamma)-\sum_{m=2}^{\infty}(m[m]_{p,q}+2\gamma+1)[m]_{p,q}^{n}\left[1+(m-1)\left(\beta(\lambda-\alpha)+m\kappa\right)\right]^{\tau}|c_m|}{1-\sum_{m=2}^{\infty}[m]_{p,q}^{n}\left[1+(m-1)\left(\beta(\lambda-\alpha)+m\kappa\right)\right]^{\tau}|c_m|}+1-2\gamma\right)$$

$$>\zeta\left[1+\frac{1}{b}\left(\frac{\sum_{m=2}^{\infty}(m[m]_{p,q}-1)[m]_{p,q}^{n}\left[1+(m-1)\left(\beta(\lambda-\alpha)+m\kappa\right)\right]^{\tau}|c_m|}{1-\sum_{m=2}^{\infty}[m]_{p,q}^{n}\left[1+(m-1)\left(\beta(\lambda-\alpha)+m\kappa\right)\right]^{\tau}|c_m|}\right)\right]$$

Hence, some simple computations lead to

$$\sum_{m=2}^{\infty}[(m[m]_{p,q}+|b|)(1-\zeta)+2\gamma-\zeta+1]\left[1+(m-1)\left(\beta(\lambda-\alpha)+m\kappa\right)\right]^{\tau}[m]_{p,q}^{n}|c_m|$$
$$\leq 2-2\gamma+|b|(1-\zeta)$$

Conversely, suppose that (2.1) is true for $z\in U$. Then

$$\operatorname{Re}\left\{1+\frac{1}{b}\left(\frac{z\left(\left(D_{\lambda,\beta,\alpha,p,q}^{\tau,n+1}f(z)\right)\right)'}{D_{\lambda,\beta,\alpha,p,q}^{\tau,n}f(z)}+1-2\gamma\right)\right\}$$

$$>\zeta\left|1+\frac{1}{b}\left(\frac{z\left(\left(D_{\lambda,\beta,\alpha,p,q}^{\tau,n+1}f(z)\right)\right)'}{D_{\lambda,\beta,\alpha,p,q}^{\tau,n}f(z)}-1\right)\right|>0$$

if

$$1+\frac{1}{b}\left(\frac{(2-2\gamma)-\sum_{m=2}^{\infty}(m[m]_{p,q}+2\gamma+1)[m]_{p,q}^{n}\left[1+(m-1)\left(\beta(\lambda-\alpha)+m\kappa\right)\right]^{\tau}|c_m|}{1-\sum_{m=2}^{\infty}[m]_{p,q}^{n}\left[1+(m-1)\left(\beta(\lambda-\alpha)+m\kappa\right)\right]^{\tau}|c_m|}+1-2\gamma\right)-$$

$$\zeta\left[1+\frac{1}{b}\left(\frac{\sum_{m=2}^{\infty}(m[m]_{p,q}-1)[m]_{p,q}^{n}\left[1+(m-1)\left(\beta(\lambda-\alpha)+m\kappa\right)\right]^{\tau}|c_m|}{1-\sum_{m=2}^{\infty}[m]_{p,q}^{n}\left[1+(m-1)\left(\beta(\lambda-\alpha)+m\kappa\right)\right]^{\tau}|c_m|}\right)\right]$$
$$>0$$

That is, if

$$\sum_{m=2}^{\infty}[(m[m]_{p,q}+|b|)(1-\zeta)+2\gamma-\zeta+1]\left[1+(m-1)\left(\beta(\lambda-\alpha)+m\kappa\right)\right]^{\tau}[m]_{p,q}^{n}|c_m|\leq 2-2\gamma+|b|(1-\zeta)$$

then this gives the required condition.

Corollary 2.2. Let the function $f(z)$, defined by (1.2), be in the class $ST_{\lambda,\beta,\alpha,p,q}^{\tau,n}(b,\gamma)$. Then we have

$$|c_m|\leq\frac{(2-2\gamma)+|b|(1-\beta)}{[(m[m]_{p,q}+|b|)(1-\zeta)+2\gamma-\zeta+1]\left[1+(m-1)\left(\beta(\lambda-\alpha)+m\kappa\right)\right]^{\tau}[m]_{p,q}^{n}},$$

where $m\geq 2,-1\leq\gamma<1,\zeta\geq 0,$ and $b\in\mathbb{C}^{*}=\mathbb{C}-\{0\},$ with equality for

$$f(z)=\frac{(2-2\gamma)+|b|(1-\beta)}{[(m[m]_{p,q}+|b|)(1-\zeta)+2\gamma-\zeta+1]\left[1+(m-1)\left(\beta(\lambda-\alpha)+m\kappa\right)\right]^{\tau}[m]_{p,q}^{n}}z^{m}.$$

3. Integral Means Inequality

We need the following lemma to prove some inequalities in integral means for functions belonging to the class $ST^{\tau,n}_{\lambda,\beta,\alpha,p,q}(b,\gamma)$.

Lemma 3.1 [11]. If f and g are analytic in U with $f(z) \prec g(z)$, then for $\tau > 0$ and $z = re^{i\theta}(0 < r < 1)$,

$$\int_0^{2\pi} |f(z)|^\tau d\theta \le \int_0^{2\pi} |g(z)|^\tau d\theta.$$

Theorem 3.2. Let

$$\left\{[(m[m]_{p,q} + |b|)(1-\zeta) + 2\gamma - \zeta + 1]\left[1 + (m-1)\left(\beta(\lambda-\alpha) + m\kappa\right)\right]^\tau [m]^n_{p,q}\right\}_{m=2}^\infty$$

be a non-decreasing sequence. If $f \in ST^{\tau,n}_{\lambda,\beta,\alpha,p,q}(b,\gamma)$, then

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^\tau d\theta \le \int_0^{2\pi} \left|g(re^{i\theta})\right|^\tau d\theta \;\; (0 < r < 1, \tau > 0) \tag{3.1}$$

where

$$f_2(z) = z - \frac{(2-2\gamma) + |b|(1-\beta)}{[(2[2]_{p,q} + |b|)(1-\zeta) + 2\gamma - \zeta + 1][1 + \beta(\lambda-\alpha) + 2\kappa]^\tau [2]^n_{p,q}} z^2$$

Proof. Let

$$f(z) = z - \sum_{m=2}^\infty c_m z^m = \left(1 - \sum_{m=2}^\infty c_m z^{m-1}\right)$$

and

$$f_2(z) = z - \frac{(2-2\gamma) + |b|(1-\beta)}{[(2[2]_{p,q} + |b|)(1-\zeta) + 2\gamma - \zeta + 1][1 + \beta(\lambda-\alpha) + 2\kappa]^\tau [2]^n_{p,q}} z^2$$

then we prove that

$$\int_0^{2\pi} \left|1 - \sum_{m=2}^\infty c_m z^{m-1}\right|^\tau d\theta$$

$$\le \int_0^{2\pi} \left|1 - \frac{(2-2\gamma) + |b|(1-\beta)}{[(2[2]_{p,q} + |b|)(1-\zeta) + 2\gamma - \zeta + 1][1 + \beta(\lambda-\alpha) + 2\kappa]^\tau [2]^n_{p,q}} z\right|^\tau d\theta$$

By lemma (3.1), it would be sufficient to show that

$$1 - \sum_{m=2}^\infty |c_m| z^{m-1} \prec 1 - \frac{(2-2\gamma) + |b|(1-\beta)}{[(2[2]_{p,q} + |b|)(1-\zeta) + 2\gamma - \zeta + 1][1 + \beta(\lambda-\alpha) + 2\kappa]^\tau [2]^n_{p,q}} z$$

If the subordination (3.2) holds true, then there exists an analytic function ω with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that

$$1 - \sum_{m=2}^\infty |c_m| z^{m-1} = 1 - \frac{(2-2\gamma) + |b|(1-\beta)}{[(2[2]_{p,q} + |b|)(1-\zeta) + 2\gamma - \zeta + 1][1 + \beta(\lambda-\alpha) + 2\kappa]^\kappa [2]^n_{p,q}} \omega(z)$$

Using (2.1), we have

$$|\omega(z)| = \left| \sum_{m=2}^{\infty} \frac{[(2[2]_{p,q} + |b|)(1-\zeta) + 2\gamma - \zeta + 1][1 + \beta(\lambda - \alpha) + 2\kappa]^{\tau} [2]_{p,q}^{n}}{(2-2\gamma) + |b|(1-\beta)} c_m z^{m-1} \right|$$

$$\leq |z| \sum_{m=2}^{\infty} \frac{[(2[2]_{p,q} + |b|)(1-\zeta) + 2\gamma - \zeta + 1][1 + \beta(\lambda - \alpha) + 2\kappa]^{\tau} [2]_{p,q}^{n}}{(2-2\gamma) + |b|(1-\beta)} |c_m|$$

$$\leq |z| < 1$$

which proves the subordination (3.2).

4. Fractional Calculus

In the literature employed by this study, many significant definitions of fractional calculus were illustrated [14-18]. Some important definitions that were used before by Owa [19] and Srivastava [20] are recalled.

Definition 4.1. The fractional integral of order μ is defined, for a function $f(z)$, by

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(\theta)}{(z-\theta)^{1-\mu}} d\theta$$

where $\mu > 0$, $f(z)$ is an analytic function in a simply-connected region of the $z-$ plane containing the origin, and the multiplicity of $(z-\theta)^{1-\mu}$ is removed by requiring $\log(z-\theta)$ to be real when $z-\theta > 0$.

Definition 4.2. The fractional derivative of order μ is defined, for a function $f(z)$, by

$$D_z^{\mu} f(z) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dz} \int_0^z \frac{f(\theta)}{(z-\theta)^{\mu}} d\theta$$

where $0 \leq \mu < 1$, $f(z)$ is an analytic function in a simply-connected region of the $z-$ plane containing the origin, and the multiplicity of $(z-\theta)^{-\mu}$ is removed as in Definition (4.1) above.

Definition 4.3. Under the hypotheses of definitions (4.2), the fractional derivative of order $m+\mu$ is defined by

$$D_z^{m+\mu} f(z) = \frac{d}{d\theta} D_z^{\mu} f(z)$$

where $0 \leq \mu < 1$, and $m \in \mathrm{N}_0$.

Theorem 4.4. Let $f(z) \in ST_{\lambda,\beta,\alpha,p,q}^{\tau,n}(b,\gamma)$, then

$$\left| D_z^{-\mu} f(z) \right| \geq \frac{|z|^{1+\mu}}{\Gamma(2+\mu)} \left\{ 1 - \frac{(2-2\gamma) + |b|(1-\beta)}{(2+\mu)[(2[2]_{p,q} + |b|)(1-\zeta) + 2\gamma - \zeta + 1][1 + (\beta(\lambda - \alpha) + 2\kappa)]^{\tau} [2]_{p,q}^{n}} \right\} \quad (4.1)$$

and

$$\left| D_z^{-\mu} f(z) \right| \leq \frac{|z|^{1+\mu}}{\Gamma(2+\mu)} \left\{ 1 + \frac{(2-2\gamma) + |b|(1-\beta)}{(2+\mu)[(2[2]_{p,q} + |b|)(1-\zeta) + 2\gamma - \zeta + 1][1 + (\beta(\lambda - \alpha) + 2\kappa)]^{\tau} [2]_{p,q}^{n}} \right\}$$

(4.2)
for $\mu > 0$ and $z \in U$. The results (4.1) and (4.2) are sharp.

Proof. We begin with defining the function $G(z)$ by [10]

$$G(z) = \Gamma(2+\mu) z^{-\mu} D_z^{-\mu} f(z)$$

$$= z - \sum_{m=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2+\mu)}{\Gamma(n+1+\mu)} c_m z^m$$

$$= z - \sum_{m=2}^{\infty} \Omega(m) c_m z^m$$

where

$$\Omega(m) = \frac{\Gamma(m+1)\Gamma(2+\mu)}{\Gamma(m+1+\mu)}, (m \geq 2)$$

If $\varepsilon_2 \geq \varepsilon_1$, implying that $p + \varepsilon_2(n-p) \geq p + \varepsilon_1(n-p)$

$$0 < \Omega(m) \leq \Omega(2) = \frac{\Gamma(3)\Gamma(2+\mu)}{\Gamma(3+\mu)} = \frac{2}{2+\mu} \quad (4.3)$$

Furthermore, it follows from theorem (2.1) that

$$\sum_{m=2}^{\infty} |c_m| \leq \frac{(2-2\gamma) + |b|(1-\beta)}{[(2[2]_{p,q} + |b|)(1-\zeta) + 2\gamma - \zeta + 1][1 + (\beta(\lambda-\alpha) + 2\kappa)]^{\tau} [2]_{p,q}^{n}} \quad (4.5)$$

Then, by using (4.3) and (4.5), we can see that

$$|G(z)| \geq |z| - \Omega(2)|z|^2 \sum_{m=2}^{\infty} |c_m|$$

$$\geq |z| - \frac{2[(2-2\gamma) + |b|(1-\beta)]}{(2+\mu)[(2[2]_{p,q} + |b|)(1-\zeta) + 2\gamma - \zeta + 1][1 + (\beta(\lambda-\alpha) + 2\kappa)]^{\tau} [2]_{p,q}^{n}} |z|^2$$

and

$$|G(z)| \leq |z| + \Omega(2)|z|^2 \sum_{m=2}^{\infty} |c_m|$$

$$\leq |z| + \frac{2[(2-2\gamma) + |b|(1-\beta)]}{(2+\mu)[(2[2]_{p,q} + |b|)(1-\zeta) + 2\gamma - \zeta + 1][1 + \beta(\lambda-\alpha) + 2\kappa]^{\tau} [2]_{p,q}^{n}} |z|^2$$

which completes the proof.

Finally, we can easily see that the results (4.1) and (4.2) are sharp for the function $f(z)$ given by

$$D_z^{-\mu} f(z) \leq \frac{z^{1+\mu}}{\Gamma(2+\mu)}$$

$$\left\{ 1 + \frac{(2-2\gamma) + |b|(1-\beta)}{(2+\mu)[(2[2]_{p,q} + |b|)(1-\zeta) + 2\gamma - \zeta + 1][1 + (\beta(\lambda-\alpha) + 2\kappa)]^{\tau} [2]_{p,q}^{n}} \right\}$$

or

$$f(z) = z - \frac{(2-2\gamma) + |b|(1-\beta)}{(2+\mu)[(2[2]_{p,q} + |b|)(1-\zeta) + 2\gamma - \zeta + 1][1 + (\beta(\lambda-\alpha) + 2\kappa)]^{\tau} [2]_{p,q}^{n}} z^2 \quad (4.6)$$

Theorem 4.5. If $f(z) \in ST^{\tau,n}_{\lambda,\beta,\alpha,p,q}(b,\gamma)$, then

$$\left|D_z^{\mu} f(z)\right| \geq \frac{|z|^{1-\mu}}{\Gamma(2-\mu)} \left\{1-\frac{(2-2\gamma)+|b|(1-\beta)}{(2-\mu)[(2[2]_{p,q}+|b|)(1-\zeta)+2\gamma-\zeta+1]\left[1+\left(\beta(\lambda-\alpha)+2\kappa\right)\right]^{\tau}[2]^{n}_{p,q}}\right\} \quad (4.7)$$

and

$$\left|D_z^{\mu} f(z)\right| \leq \frac{|z|^{1-\mu}}{\Gamma(2-\mu)} \left\{1+\frac{(2-2\gamma)+|b|(1-\beta)}{(2-\mu)[(2[2]_{p,q}+|b|)(1-\zeta)+2\gamma-\zeta+1]\left[1+\left(\beta(\lambda-\alpha)+2\kappa\right)\right]^{\tau}[2]^{n}_{p,q}}\right\} \quad (4.8)$$

for $0 \leq \mu < 1$ and $z \in U$. The results (4.7) and (4.8) are sharp.

Proof. Define the function [10] $H(z)$ by

$$\begin{aligned} H(z) &= \Gamma(2-\mu) z^{\mu} D_z^{\mu} f(z) \\ &= z-\sum_{m=2}^{\infty} \frac{\Gamma(m)\Gamma(2-\mu)}{\Gamma(m+1-\mu)} c_m z^m \\ &= z-\sum_{m=2}^{\infty} \Psi(m) c_m z^m, \end{aligned}$$

where

$$\Psi(m)=\frac{\Gamma(m)\Gamma(2-\mu)}{\Gamma(m+1-\mu)}, (m \geq 2)$$

It is easy to see that

$$0<\Psi(m) \leq \Psi(2)=\frac{\Gamma(2)\Gamma(2-\mu)}{\Gamma(3-\mu)}=\frac{1}{2-\mu} \quad (4.9)$$

Then, by using (4.5) and (4.9), we have

$$\begin{aligned} |H(z)| &\geq |z|-\Psi(2)|z|^2 \sum_{m=2}^{\infty} m|c_m| \\ &\geq |z|-\frac{(2-2\gamma)+|b|(1-\beta)}{(2-\mu)[(2[2]_{p,q}+|b|)(1-\zeta)+2\gamma-\zeta+1]\left[1+\left(\beta(\lambda-\alpha)+2\kappa\right)\right]^{\tau}[2]^{n}_{p,q}}|z|^2 \end{aligned} \quad (4.10)$$

and

$$\begin{aligned} |H(z)| &\leq |z|+\Psi(2)|z|^2 \sum_{m=2}^{\infty} m|c_m| \\ &\leq |z|+\frac{(2-2\gamma)+|b|(1-\beta)}{(2-\mu)[(2[2]_{p,q}+|b|)(1-\zeta)+2\gamma-\zeta+1]\left[1+\left(\beta(\lambda-\alpha)+2\kappa\right)\right]^{\tau}[2]^{n}_{p,q}}|z|^2 \end{aligned} \quad (4.11)$$

Now, (4.7) and (4.8) follow from (4.9) and (4.10), respectively.
Finally, by taking the function $f(z)$ defined by

$$\left|D_z^{\mu} f(z)\right| \le \frac{|z|^{1-\mu}}{\Gamma(2-\mu)}$$

$$\left\{1-\frac{(2-2\gamma)+|b|(1-\beta)}{(2-\mu)[(2[2]_{p,q}+|b|)(1-\zeta)+2\gamma-\zeta+1]\left[1+\left(\beta(\lambda-\alpha)+2\kappa\right)\right]^{\tau}[2]_{p,q}^{n}}\right\}$$

or for the function given by (4.6), the results (4.7) and (4.8) are easily seen to be sharp.

5. Fractional Integral Operator

We need this definition of fractional operator, given by Srivastava *et al*. [20].

Definition 5.1. For real number $\eta>0,\rho>0$ and $\delta>0$, the fractional integral operator $\mathrm{I}_{0,z}^{\eta,\rho,\delta}$ is defined by

$$I_{0,z}^{\eta,\rho,\delta} f(z)=\frac{z^{-\eta-\rho}}{\Gamma(\eta)}\int_0^z (z-t)^{\eta-1}F(\eta+\rho,-\delta,\eta,1-\frac{t}{z})f(t)dt$$

where a function $f(z)$ is analytic in a simply-connected region of the $z-$ plane containing the origin with the order

$$f(z)=O(|z|^{\varepsilon}),(z\to 0)$$

with $\varepsilon>\max\{0,\gamma-\delta\}-1.$

Here, $F(a,b,c,z)$ is the Gauss hypergeometric function defined by

$$F(a,b,c,z)=\sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(c)_m(1)_m}$$

where $(v)_m$ is the Pochhammer symbol defined by

$$(v)_m=\frac{\Gamma(v+m)}{\Gamma(v)}=\begin{cases}1 & m=0\\ v(v+1)(v+2)...(v+m-1) & m\in \mathrm{N}\end{cases}$$

and the multiplicity of $(z-t)^{\eta-1}$ is removed by requiring $\log(z-t)$ to be real when $z-t>0.$

Remark 5.2. For $\rho=-\eta$, we note that

$$I_{0,z}^{\eta,-\eta,\delta} f(z)=D_z^{-\eta} f(z)$$

Lemma 5.3 [21]. If $\eta>0$ and $n>\rho-\delta-1,$ then

$$I_{0,z}^{\eta,\rho,\delta} z^m=\frac{\Gamma(m+1)\Gamma(m-\rho+\delta+1)}{\Gamma(m-\rho+1)\Gamma(m+\rho+\delta+1)}$$

We begin by proving the following.

Theorem 5.4. Let $\eta>0,\rho>2,\eta+\delta>-2,\rho-\delta<2$ and $\rho(\eta+\delta)\le 3\eta.$ If $f(z)\in ST_{\lambda,\beta,\alpha,p,q}^{\tau,n}(b,\gamma)$, then

$$\left|I_{0,z}^{\eta,\rho,\delta} f(z)\right|$$

$$\ge \frac{\Gamma(2-\rho+\delta)|z|^{1-\rho}}{\Gamma(2-\rho)\Gamma(2+\eta+\delta)}$$

$$\left\{1-\frac{2(2-\rho+\delta)\left[(2-2\gamma)+|b|(1-\beta)\right]}{(2-\rho)(2+\rho+\delta)[(2[2]_{p,q}+|b|)(1-\zeta)+2\gamma-\zeta+1]\left[1+\left(\beta(\lambda-\alpha)+2\kappa\right)\right]^{\tau}[2]_{p,q}^{n}}|z|^2\right\} \quad (5.1)$$

and

$$\left|I_{0,z}^{\eta,\rho,\delta} f(z)\right|$$
$$\leq \frac{\Gamma(2-\rho+\delta)|z|^{1-\gamma}}{\Gamma(2-\rho)\Gamma(2+\eta+\delta)}$$
$$\left\{1+\frac{2(2-\rho+\delta)\left[(2-2\gamma)+|b|(1-\beta)\right]}{(2-\rho)(2+\rho+\delta)[(2[2]_{p,q}+|b|)(1-\zeta)+2\gamma-\zeta+1]\left[1+\left(\beta(\lambda-\alpha)+2\kappa\right)\right]^{\tau}[2]_{p,q}^{n}}|z|^{2}\right\} \quad (5.2)$$

Proof. For $z \in U_0$, where

$$U_0=\begin{cases} U & , \quad \rho \leq 1 \\ U-\{0\} & , \quad \rho>1 \end{cases}$$

the equalities in (5.1) and (5.2) are attained for the function f (z) given by (4.6).
By using Lemma 5.3, we have

$$I_{0,z}^{\eta,\rho,\delta} f(z)$$
$$=\frac{\Gamma(2-\rho+\delta)}{\Gamma(2-\rho)\Gamma(2+\eta+\delta)} z^{1-\rho}-$$
$$\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(n-\rho+\delta+1)}{\Gamma(n-\rho+1)\Gamma(n+\eta+\delta+1)} c_m z^{m-\rho}$$

where $z \in U_0$.
Let

$$\Delta(z)=\frac{\Gamma(2-\rho)\Gamma(2+\eta+\delta)}{\Gamma(2-\rho+\delta)} z^{\rho} I_{0,z}^{\eta,\rho,\delta} f(z)$$
$$=z-\sum_{m=2}^{\infty} \varphi(m) c_m z^m$$

where

$$\varphi(m)=\frac{(2-\rho+\delta)_{m-1}(2)_{m-1}}{(2-\rho)_{m-1}(2+\rho+\delta)_{m-1}},(m \geq 2)$$

The function $\varphi(m)$ is non-increasing for $n \geq 2$, then we get

$$|\Delta(z)| \geq |z|-\varphi(2)|z|^2 \sum_{m=2}^{\infty}|c_m|$$
$$\geq |z|-$$
$$\frac{2(2-\rho+\delta)\left[(2-2\gamma)+|b|(1-\beta)\right]}{(2-\rho)(2+\rho+\delta)[(2[2]_{p,q}+|b|)(1-\zeta)+2\gamma-\zeta+1]\left[1+\left(\beta(\lambda-\alpha)+2\kappa\right)\right]^{\tau}[2]_{p,q}^{n}}|z|^{2}$$

and

$$|\Delta(z)| \leq |z|+\varphi(2)|z|^2 \sum_{m=2}^{\infty}|c_m|$$
$$\leq |z|+$$
$$\frac{2(2-\rho+\delta)\left[(2-2\gamma)+|b|(1-\beta)\right]}{(2-\rho)(2+\rho+\delta)[(2[2]_{p,q}+|b|)(1-\zeta)+2\gamma-\zeta+1]\left[1+\left(\beta(\lambda-\alpha)+2\kappa\right)\right]^{\tau}[2]_{p,q}^{n}}|z|^{2}$$

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