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The Local Bifurcation of an Eco-Epidemiological Model in the Presence of Stage- Structured with Refuge

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ABSTRACT

In this paper, we establish the conditions of the occurrence of the local bifurcations, such as saddle node, transcritical and pitchfork, of all equilibrium points of an eco-epidemiological model consisting of a prey-predator model with SI (susceptible-infected) epidemic diseases in prey population only and a refuge-stage structure in the predators. It is observed that there is a transcritical bifurcation near the axial and free predator equilibrium points, near disease-free equilibrium point is a saddle-node bifurcation and near positive (coexistence) equilibrium point is a saddle-node bifurcation, a transcritical bifurcation and a pitchfork bifurcation. Further investigations for Hopf bifurcation near coexistence equilibrium point are carried out. Finally, numerical simulations are used to illustrate the occurrence of the local bifurcations of this model.

Keywords: Prey- predator, Eco-epidemiological, SI epidemic disease function, Local bifurcation. model, Refuge, Stag- structure, Lyapunove

التفرع المحلي لنموذج بيئي وبائي بوجود مراحل عمرية وملجأ

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الخلاصه

في هذه البحث تمت دراسة ظروف حدوث التفرع المحلي مثل (عقد سرجية, تحول حرج, مذراة العشب) بالقرب من كل نقطة من نقاط التوازن لنموذج بئي وبائي متضمن فريسة ومفترس بوجود مرض معد في مجتمع الفريسة من نوع SI مع ملجأ للفريسة ومراحل عمرية في مجتمع المفترس. لوحظ وجود تفرع من النوع (تحول حرج) قرب نقاط التوازن $E1$ و $E2$ وبينما بالقرب من نقطة التوازن $E3$وجد تفرع من النوع (عقد سرجية), ولكن بالقرب من النقطة الموجبة $E4$ تم حدوث تفرع من النوع (عقدة السرج , تحول حرج, مذراة العشب), كذلك تم دراسة تفرع هوبف بالقرب من النقطة الموجبة$E4$ فقط. واخيرا تم استخدام المحاكاة العددية لتأكيد النتائج التحليلية.

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1. Introduction

Bifurcation theory is the mathematical study of changes in the qualitative asymptotic structure of a dynamical system [1, 2]. It attempts to explain various phenomena that have been discovered in the natural sciences over the centuries. Performing a bifurcation analysis is often a powerful way to analyze the properties of such systems. The prey and predator model is an important topic at present, as it is used to solve many problems in ecology, nature and other sciences. The prey system includes several interactions, including competition co-existence and stage-structured [3, 4]. The ecological models of the age stage are more logical than models that do not contain phase structure. In addition, there are several factors that affect the system, for example, refuge, disease, shelter and others. Sometimes, differences in any parameter in the system can lead to complex behaviors that lead to system instability, causing a bifurcation that is the main qualitative change in the behavior of a dynamic system as a result of changing one of its coefficients. The bifurcation is divided into two principal classes, local and global. Local bifurcation can be analyzed through changes in the local stability properties of equilibria or periodic orbits. While global bifurcation occurs when periodic orbits collide with equilibria. This causes changes in the topology of the trajectories in phase space which cannot be confined to a small neighborhood, as is the case with local bifurcation. These bifurcations happen when one varies a single parameter [5-9]. On the other hand, Perko [10] established the conditions of the occurrence of local bifurcation, such as saddle-node, transcritical, pitchfork, period -doubling, and Hopf bifurcation near coexistence equilibrium point. The Hopf bifurcation is a local bifurcation in which the equilibrium point of a dynamical system loses stability, as a pair of complex conjugate eigenvalues of the linearization around the equilibrium point cross the imaginary axis of complex plane. This type is also known as Poincare Andronov Hopf bifurcation.

In this paper, an application of Sotomayor's theorem [11] for local bifurcation is used to study the occurrence of local bifurcation near the equilibria. Furthermore, Hopf bifurcation near positive equilibrium point conditions is established for a mathematical model previously proposed by Kafi and Majeed [12].

2 .The mathematical models [12]

In this section, an eco-epidemiological model is proposed for study. The model consists of a prey, whose total population density at time T is denoted by $N(T)$, interacting with a stag-structured predator . It is assumed that the prey population is infected by an infectious disease with the prey refuge. Now, the following assumptions are adopted in formulating the basic eco-epidemiology model:

1. There is an SI epidemic disease in the prey population which divides the prey population into two classes, namely $S(T)$ that represents the density of susceptible prey at time T and $I(T)$ which represents the density of infected prey at time T. Therefore, at any time T, we have $N(T) = S(T) + I(T)$. The predator population is divided into two classes, namely $S(T)$ that represents the density of immature predator at time T and $W(T)$ which represents the density of mature predator at time T.
2. It is assumed that only susceptible prey S is capable of reproducing in a logistic growth with a carrying capacity $K > 0$ and an intrinsic growth rate constant $r > 0$. The infected prey I is removed before having the possibility of reproducing. However, the infected prey population I still contribute with S to population growth towards the carrying capacity.
3. The disease is transmitted within the same species by contact with an infected individual at infection rates of $\alpha > 0$ for the prey.
4. The mature predator w(T) consumes the susceptible prey S(T) and the infected prey I(T), according to Holling type-II of functional responses with a maximum attack rate of $a > 0$, and a half saturation rate of $b > 0$ for the susceptible prey , as well as a maximum attack rate of $c > 0$ and a half saturation rate of $d > 0$ for the infected prey .
5. The disease may causes mortality with a constant mortality rate of $d_1 > 0$ for the infected prey.
6. The immature predator depends completely in its feeding on his parents, so that it feeds on the portion of the uptake food by the mature predator from the susceptible and infected prey, with portion rates of $0 < n_1 < 1$ and $0 < n_2 < 1$ associated with uptake rates of $0 < e_1 < 1$ and $0 < e_2 <$

1, respectively. The immature predator individuals grow up and become mature predator individuals with a grown up rate of $u > 0$.

7. There is a type of protection of the prey species from facing predation by the mature predator with refuge rate constants $m_1 \in (0.1)$ and $m_2 \in (0,1)$ for susceptible and infected prey, respectively.

8. Finally, in the absence of the predator facing death with natural death rates of $d_2 > 0$ and $d_3 > 0$ for immature and mature predators, respectively.

Therefore, the dynamics of the above proposed model can be represented by the following set of first order nonlinear differential equations.

$$\left.\begin{aligned}
\frac{dS}{dT} &= r\,S\left(1-\frac{S+I}{k}\right) - \alpha\,SI - \frac{a\,(1-m_1)SW}{b+(1-m_1)S}\\
\frac{dI}{dT} &= \alpha\,SI - \frac{c\,(1-m_2)IW}{d+(1-m_2)I} - d_1\,I\\
\frac{dZ}{dT} &= \frac{e_1 a\,(1-n_1)(1-m_1)\;SW}{b+(1-m_1)S} + \frac{e_2 c\,(1-n_2)(1-m_2)IW}{d+(1-m_2)I} - uZ - d_2 Z\\
\frac{dW}{dT} &= \frac{e_1 a\,n_1(1-m_1)\,SW}{b+(1-m_1)S} + \frac{e_2 c\,n_2(1-m_2)IW}{d+(1-m_2)I} + uZ - d_3\,W
\end{aligned}\right\} \quad (1)$$

. $m_1 = m_2 =$ m For the simplicity of the above model, it is assumed that

Note that the above proposed model has sixteen parameters in all, which makes the analysis difficult. In order to simplify the system, the number of parameters is reduced by using the following dimensionless variables and parameters.

$$t = r\,\mathrm{T},\ a_1 = \frac{\alpha k}{\mathrm{r}},\ a_2 = \frac{\mathrm{a}(1-\mathrm{m})}{\mathrm{r}},\ a_3 = \frac{\mathrm{b}}{\mathrm{k}},\ a_4 = \frac{c\,(1-\mathrm{m})}{r}, a_5 = \frac{\mathrm{d}}{\mathrm{k}},$$
$$a_6 = \frac{d_1}{r},\ a_7 = \frac{e_1 a(1-n_1)\,(1-\mathrm{m})}{r},\ a_8 = \frac{e_2 c(1-n_2)(1-\mathrm{m})}{r},$$
$$a_9 = \frac{u}{r},\ a_{10} = \frac{d_2}{r},\ a_{11} = \frac{e_1 a n_1\,(1-\mathrm{m})}{r},\ a_{12} = \frac{e_2 c n_2(1-\mathrm{m})}{r},$$
$$a_{13} = \frac{d_3}{r},\quad s = \frac{S}{k}, i = \frac{I}{k}, z = \frac{Z}{k}, w = \frac{W}{k}.$$

Then dimensional form of system (1) can be written as:

$$\left.\begin{aligned}
\frac{ds}{dt} &= s\left[1 - s - (1+a_1)i - \frac{a_2 w}{a_3+(1-m)s}\right] = f_1(s,i,z,w)\\
\frac{di}{dt} &= i\left[a_1 s - \frac{a_4\,w}{a_5+(1-m)i} - a_6\right] = f_2(s,i,z,w)\\
\frac{dz}{dt} &= \frac{a_7 sw}{a_3+(1-m)s} + \frac{a_8\,iw}{a_5+(1-m)i} - a_9 z - a_{10} z = f_3(s,i,z,w)\\
\frac{dw}{dt} &= \frac{a_{11} sw}{a_3+(1-m)s} + \frac{a_{12}\,iw}{a_5+(1-m)i} + a_9 z - a_{13} w = f_4(s,i,z,w)
\end{aligned}\right\} \quad (2)$$

with $s(0) \geq 0$, $i(0) \geq 0$, $z(0) \geq 0$ and $w(0) \geq 0$. It is observed that the number of parameters was reduced from sixteen in system (1) to fourteen in system (2). Obviously, the interaction functions of system (2) are continuous and have continuous partial derivatives on the following positive four dimensional spaces.

$R^4_+ = \{(s,i,z,w) \in R^4 : s(0) \geq 0, i(0) \geq 0, z(0) \geq 0, w(0) \geq 0\}$.

Therefore, these functions are Lipschitzian on R^4_+, and hence the existence and uniqueness of solutions for system (2) are guaranteed. Further, all the solutions for system (2) with non-negative initial conditions are uniformly bounded, as demonstrated in the following theorem.

Theorem 1 [12]

All the solutions of system (**2**) are uniformly bounded.

3. Existence and stability of equilibrium points [12]

System (2) has at most five equilibrium points, which are mentioned in the following:

◉ The equilibrium point $E_0 = (0,0,0,0)$, which is known as the vanishing point that always exists and unstable.

◉ The axial equilibrium point $E_1 = (1,0,0,0)$, which exists unconditionally.
Also, it is a locally asymptotically stable if the following conditions hold:

$$a_6 > a_1, \tag{3}$$

$$\frac{a_{11}}{a_3 + (1-m)} > (a_{13} + a_9 + a_{10}), \tag{4}$$

$$a_{13}(a_9 + a_{10}) > \frac{[a_{11}(a_9 + a_{10}) + a_9 a_7]}{a_3 + (1-m)}. \tag{5}$$

◉ The free predators' equilibrium point $E_2 = (\bar{s}, \bar{i}, 0, 0)$ which exists uniquely in the Int. R_+^2 (Interior of R_+^2) of $si-plane$, provided that:

$$a_6 < a_1, \tag{6}$$

where

$$\bar{s} = \frac{a_6}{a_1}, \tag{7}$$

$$\bar{i} = \frac{a_1 - a_6}{a_1(1+a_1)}. \tag{8}$$

And it is a locally asymptotically stable if the following conditions hold:

$$2\bar{s} + (1 + a_1)\bar{i} > 1, \tag{9}$$

$$\bar{s} < \frac{a_6}{a_1}, \tag{10}$$

$$a_{13} > \frac{a_{11}\bar{s}}{a_3 + (1-m)\bar{s}} + \frac{a_{12}\bar{i}}{a_5 + (1-m)\bar{i}}, \tag{11}$$

$$\varphi_1 > \varphi_2, \tag{12}$$

where:

$$\varphi_1 = (a_9 + a_{10})\left[a_{13} - \left(\frac{a_{11}\bar{s}}{a_3 + (1-m)\bar{s}} + \frac{a_{12}\bar{i}}{a_5 + (1-m)\bar{i}}\right)\right],$$

$$\varphi_2 = a_9\left[\frac{a_7\bar{s}}{a_3 + (1-m)\bar{s}} + \frac{a_8\bar{i}}{a_5 + (1-m)\bar{i}}\right].$$

◉ The disease-free equilibrium point $E_3 = (\tilde{s}, 0, \tilde{z}, \tilde{w})$ exists uniquely in the interior R_+^3 of $szw-$space where,

$$\tilde{w} = \frac{(1-\tilde{s})[a_3 + (1-m)\tilde{s}]}{a_2}, \tag{13}$$

$$\tilde{z} = \frac{a_7\ \tilde{s}\tilde{w}}{(a_9 + a_{10})\ [a_3 + (1-m)\ \tilde{s}]} \quad \text{and}$$

$$\tilde{s} = \frac{a_3 a_{13}(a_9 + a_{10})}{(a_9 + a_{10})[a_{11} - a_{13}(1-m)] + a_7 a_9},$$

if the following conditions hold:

$$\tilde{s} < 1, \tag{14}$$

$$a_{11} > a_{13}\ (1\text{-}m). \tag{15}$$

And it is a locally asymptotically stable if the following conditions hold:

$$\tilde{s} < \frac{a_4\tilde{w} + a_5 a_6}{a_1 a_5}, \tag{16}$$

$$2\tilde{s} + \frac{a_2 a_3}{[a_3 + (1-m)\tilde{s}\,]^2} > 1, \tag{17}$$

$$a_{13} > \frac{a_{11}\tilde{s}}{a_3 + (1-m)\tilde{s}}, \tag{18}$$

$$n_{33}\ n_{44} > n_{43}\, n_{34}, \tag{19}$$

$$n_{41}(n_{11} + n_{44}) > n_{31} n_{43}. \tag{20}$$

◉ Finally, the positive (coexistence) equilibrium point $E_4 = \left(\bar{\bar{s}}, \bar{\bar{i}}, \bar{\bar{z}}, \bar{\bar{w}}\right)$ exists in the Int. R_+^4 if the following conditions hold:

$$a_2 a_6(1-m) < a_3 a_4(1 + a_1), \tag{21}$$

$$a_{12} > a_{13}(1-m), \tag{22}$$

$$i_1 < i_2, \tag{23}$$

where:

$$i_1 = \frac{-(a_3a_4 + a_2a_5a_6)}{a_2a_6(1-m) - a_3a_4(1+a_1)}, \quad i_2 = \frac{a_5a_{13}(a_9 + a_{10})}{(a_9 + a_{10})[a_{12} - a_{13}(1-m)] + a_8a_9}.$$

The characteristic equation of $J(E_4)$ is given by:

$$\lambda^4 + H_1\lambda^3 + H_2\lambda^2 + H_3\lambda + H_4 = 0, \tag{24}$$

where:

$H_1 = -(\gamma_0 + \gamma_1)$

$H_2 = \gamma_0\gamma_1 + \gamma_2 + \gamma_3 - (\gamma_4 + \gamma_5 + \gamma_6 + \gamma_7)$

$H_3 = -[(\gamma_0(\gamma_2 - \gamma_4) + \gamma_1(\gamma_3 - \gamma_6) - m_{24}(\gamma_8 - \gamma_9 + \gamma_{10}) + \gamma_6\gamma_{11} + m_{14}(\gamma_{12} - \gamma_{13} + \gamma_{14})]$

$H_4 = (\gamma_2 - \gamma_4)(\gamma_3 - \gamma_6) + (\gamma_9 - \gamma_{10})(\gamma_{15} - \gamma_{16}) + \gamma_{17}(\gamma_4 - \gamma_{18}) + \gamma_{19}(\gamma_{13} - \gamma_{14}),$

with

$\gamma_0 = m_{11} + m_{22}$, $\gamma_1 = m_{33} + m_{44}$, $\gamma_2 = m_{33}m_{44}$, $\gamma_3 = m_{11}m_{22}$, $\gamma_4 = m_{34}m_{43} > 0$, $\gamma_5 = m_{24}m_{42} < 0$, $\gamma_6 = m_{12}m_{21} < 0$, $\gamma_7 = m_{14}m_{41} < 0$, $\gamma_8 = m_{11}m_{42}$,
$\gamma_9 = m_{32}m_{43} > 0$, $\gamma_{10} = m_{42}m_{33} < 0$, $\gamma_{11} = m_{14} + m_{41}$, $\gamma_{12} = m_{22}m_{41} < 0$,
$\gamma_{13} = m_{31}m_{42} > 0$, $\gamma_{14} = m_{41}m_{33} < 0$, $\gamma_{15} = m_{11}m_{24} < 0$, $\gamma_{16} = m_{14}m_{21} < 0$,
$\gamma_{17} = m_{12}m_{24} > 0$, $\gamma_{18} = m_{31}m_{43} > 0$, $\gamma_{19} = m_{14}m_{22} > 0$.

Now by using Routh Hurwitz criterion all the eigenvalues, which represent the roots of eq. (24), have negative real parts if and only if $H_1 > 0$, $H_3 > 0$, $H_4 > 0$ and $\Delta = (H_1H_2 - H_3)H_3 - H_1^2 H_4 > 0$. Now, $H_i > 0, i = 1,3$ and 4, provided that:

$$min\left\{W_1, W_2, \frac{a_4a_5\bar{\bar{w}} + a_6\left[a_5 + (1-m)\bar{\bar{i}}\right]^2}{a_1\left[a_5 + (1-m)\bar{\bar{i}}\right]^2}\right\} > \bar{\bar{s}} > W_3, \tag{25}$$

$$a_{13} > \frac{a_{11}\bar{\bar{s}}\left[a_5 + (1-m)\bar{\bar{i}}\right] + a_{12}\bar{\bar{i}}\left[a_3 + (1-m)\bar{\bar{s}}\right]}{\left[a_3 + (1-m)\bar{\bar{s}}\right]\left[a_5 + (1-m)\bar{\bar{i}}\right]}, \tag{26}$$

$$W_4 > a_9\left[\frac{a_7\bar{\bar{s}}}{a_3 + (1-m)\bar{\bar{s}}} + \frac{a_8\bar{\bar{i}}}{a_5 + (1-m)\bar{\bar{i}}}\right], \tag{27}$$

$$\frac{a_8\bar{\bar{i}}}{a_5 + (1-m)\bar{\bar{i}}} > \frac{a_7\left(a_3\bar{\bar{w}} - \bar{\bar{s}}\left[a_3 + (1-m)\bar{\bar{s}}\right]\right)}{\left[a_3 + (1-m)\bar{\bar{s}}\right]^2}, \tag{28}$$

where : $W_1 = \frac{a_3a_{11}\bar{\bar{w}}}{a_2\left[a_3 + (1-m)\bar{\bar{s}}\right]}$,

$$W_2 = \frac{a_4\left[[1 - 2\bar{\bar{s}} - (1+a_1)\bar{\bar{i}}][a_3 + (1-m)s] - a_2a_3\bar{\bar{w}}\right]}{a_1a_2\left[a_3 + (1-m)\bar{\bar{s}}\right]\left[a_5 + (1-m)\bar{\bar{i}}\right]},$$

$$W_3 = \frac{\left(1 - (1+a_1)\bar{\bar{i}}\right)\left[a_3 + (1-m)\bar{\bar{s}}\right]^2 - a_2a_3\bar{\bar{w}}}{2\left[a_3 + (1-m)\bar{\bar{s}}\right]^2},$$

$$W_4 = (a_9 + a_{10})\left[a_{13} - \left(\frac{a_{11}\bar{\bar{s}}}{a_3 + (1-m)\bar{\bar{s}}} + \frac{a_{12}\bar{\bar{i}}}{a_5 + (1-m)\bar{\bar{i}}}\right)\right],$$

A straightforward computation shows that:

$\Delta = K_1 - K_2$, where:

$K_1 = (\gamma_0\gamma_1 - \gamma_5 - \gamma_7)(\gamma_0 + \gamma_1)[\gamma_0(\gamma_2 - \gamma_4) + \gamma_1(\gamma_3 - \gamma_6) - m_{24}(\gamma_8 - \gamma_9 + \gamma_{10}) + \gamma_6\gamma_{11} - m_{14}(\gamma_{12} - \gamma_{13} + \gamma_{14})] + \gamma_0\gamma_1(\gamma_2 - \gamma_4)^2 - m_{14}(\gamma_2 - \gamma_4)(\gamma_{12} - \gamma_{13} + \gamma_{14})[\gamma_2(\gamma_{12} - \gamma_{13} + \gamma_{14}) - \gamma_1] + \gamma_0\gamma_1(\gamma_3 - \gamma_6)^2 - m_{14}(\gamma_0 - \gamma_1)(\gamma_3 - \gamma_6)(\gamma_{12} - \gamma_{13} + \gamma_{14}),$

$K_2 = m_{24}^2(\gamma_8 - \gamma_9 + \gamma_{10})^2 + m_{14}^2(\gamma_{12} - \gamma_{13} + \gamma_{14})^2 + \gamma_6^2\gamma_{11}^2 + (\gamma_{12} - \gamma_{13} +$

$\gamma_{14})\ [2m_{14}m_{24}(\gamma_8-\gamma_9+\gamma_{10})-2m_{14}\gamma_6\gamma_{11}]-\ 2m_{24}\gamma_6\gamma_{11}(\gamma_8-\gamma_9+\gamma_{10})+2\gamma_0\gamma_1(\gamma_2-\gamma_4)(\gamma_3-\gamma_6)+(\gamma_0-\gamma_1)^2[(\gamma_9-\gamma_{10})+(\gamma_{15}+\gamma_{16})+\gamma_{17}(\gamma_{14}-\gamma_{18})+\ \gamma_{19}(\gamma_{13}-\gamma_{14})].$

Hence, Δ will be positive under conditions $(25-28)$. Therefore, all the eigenvalues of J_4 have negative real parts under the given conditions and hence E_4 is locally asymptotically stable. However, it is unstable otherwise.

4. Local bifurcation analysis

In this section, the effects of varying the parameter values on the dynamical behavior of system (2) around each equilibrium point are studied. We recall that the existence of non-hyperbolic equilibrium point of system (2) is the necessary but not sufficient condition for bifurcation to occur. Therefore, in the following theorems, an application to the Sotomayor's theorem for local bifurcation is appropriate.

Now, according to Jacobian matrix of system (2) which is given in equation (4.1)in a previous work [12], it is easy to verify that, for any nonzero vector $\dot{H}=(\dot{H}_1,\dot{H}_2,\dot{H}_3,\dot{H}_4)^T$, we have:

$$D^2F_\mu(X,\mu)(\dot{H},\dot{H})=[\,t_{ij}\,]_{4\times1}, \qquad (29)$$

where:

$$t_{11}=2\left[\frac{a_2a_3\dot{H}_1}{[a_3+(1-m)s]^2}\left(\frac{(1-m)w\dot{H}_1}{a_3+(1-m)s}-\dot{H}_4\right)-\dot{H}_1^2-(1+a_1)\dot{H}_1\dot{H}_2\right],$$

$$t_{21}=2\left[\frac{a_4a_5\dot{H}_2}{[a_5+(1-m)\mathrm{i}]^2}\left(\frac{(1-m)w\dot{H}_2}{a_5+(1-m)i}-\dot{H}_4\right)+\ a_1\dot{H}_1\dot{H}_2\right],$$

$$t_{31}=2\left[\begin{array}{c}\frac{a_3a_7\dot{H}_1}{[a_3+(1-m)s]^2}\left(\dot{H}_4-\frac{(1-m)w\dot{H}_1}{a_3+(1-m)s}\right)+\\ \frac{a_5a_8\dot{H}_2}{[a_5+(1-m)\mathrm{i}]^2}\left(\dot{H}_4-\frac{(1-m)w\dot{H}_2}{a_5+(1-m)i}\right)\end{array}\right],$$

$$t_{41}=2\left[\begin{array}{c}\frac{a_3a_{11}\dot{H}_1}{[a_3+(1-m)s]^2}\left(\dot{H}_4-\frac{(1-m)w\dot{H}_1}{a_3+(1-m)s}\right)+\\ \frac{a_5a_{12}\dot{H}_2}{[a_5+(1-m)\mathrm{i}]^2}\left(\dot{H}_4-\frac{(1-m)w\dot{H}_2}{a_5+(1-m)\mathrm{i}}\right)\end{array}\right],$$

and $$D^3F_\mu(X,\mu)(\dot{H},\dot{H},\dot{H})=[\,\ddot{t}_{ij}\,]_{4\times1}, \qquad (30)$$

where:

$$\ddot{t}_{11}=\frac{6a_2a_3(1-m)\dot{H}_1^2}{[a_3+(1-m)s]^3}\left(\dot{H}_4-\frac{(1-m)w\dot{H}_1}{a_3+(1-m)s}\right),$$

$$\ddot{t}_{21}=\frac{6a_4a_5(1-m)\dot{H}_2^2}{[a_5+(1-m)\mathrm{i}]^3}\left(\dot{H}_4-\frac{(1-m)w\dot{H}_2}{a_5+(1-m)\mathrm{i}}\right),$$

$$\ddot{t}_{31}=6\left[\begin{array}{c}\frac{a_3a_7(1-m)H_1^2}{[a_3+(1-m)s]^3}\left(\frac{(1-m)w\dot{H}_1}{a_3+(1-m)s}-\dot{H}_4\right)+\\ \frac{a_5a_8(1-m)\dot{H}_2^2}{[a_5+(1-m)\mathrm{i}]^3}\left(\frac{(1-m)w\dot{H}_2}{a_5+(1-m)\mathrm{i}}-\dot{H}_4\right)\end{array}\right],$$

$$\ddot{t}_{41}=6\left[\begin{array}{c}\frac{a_3a_{11}(1-m)\dot{H}_1^2}{[a_3+(1-m)s]^3}\left(\frac{(1-m)w\dot{H}_1}{a_3+(1-m)s}-\dot{H}_4\right)+\\ \frac{a_5a_{12}\dot{H}_2^2}{[a_5+(1-m)\mathrm{i}]^3}\left(\frac{(1-m)w\dot{H}_2}{a_5+(1-m)\mathrm{i}}-\dot{H}_4\right)\end{array}\right].$$

where $X=(s,i,z,w)^T$ and μ is any bifurcation parameter.

In the following theorems, the local bifurcation conditions near equilibrium points are established.

4.1 Local bifurcation analysis near E_1

Theorem (2): If the parameter a_6 passes through the value $\ddot{a}_6=a_1$ then system (2) at the axial equilibrium point E_1 =(1,0,0,0) possesses :

- No saddle-node bifurcation.
- Transcritical bifurcation.
- No pitchfork bifurcation.

Proof: According to the Jacobian matrix $J(E_1)$ given by eq.(4.3) [12], the system (2) at the equilibrium point $E_1 = (1,0,0,0)$ has a zero eigenvalue (say $\lambda_{1i} = 0$) at $a_6 = \ddot{a}_6$, and the Jacobian matrix J_1 with $a_6 = \ddot{a_6}$ becomes:

$\ddot{J_1} = J_1(a_6 = \ddot{a}_6) = [\dot{r}_{ij}]_{4x4}$,

where, $\dot{r}_{ij} = r_{ij}$ for all i , j =1,2,3,4 except $\dot{r}_{22} = a_1 - \ddot{a}_6 = 0$.

Now, let $\dot{H}^{[1]} = \left(\dot{H}_1^{[1]}, \dot{H}_2^{[1]}, \dot{H}_3^{[1]}, \dot{H}_4^{[1]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{1i} = 0$. Thus $\left(\ddot{J}_1 - \lambda_{1i} I\right) \dot{H}^{[1]} = 0$, which gives:

$H^{[1]} = \left(\sigma_1 \dot{H}_2^{[1]}, \dot{H}_2^{[1]}, \dot{H}_3^{[1]}, \sigma_2 \dot{H}_3^{[1]}\right)$ where $\dot{H}_2^{[1]}$ and $\dot{H}_3^{[1]}$ are any nonzero real numbers,

where:

$$\sigma_1 = -(1 + a_1), \quad \sigma_2 = \frac{(a_9 + a_{10})[a_3 + (1 - m)]}{a_2}.$$

Let $\emptyset^{[1]} = \left(\emptyset_1^{[1]}, \emptyset_2^{[1]}, \emptyset_3^{[1]}, \emptyset_4^{[1]}\right)^T$ be the eigenvector associated with the eigenvalue $\lambda_{1i} = 0$ of the matrix $[\ \ddot{J}_1]^T$. Then we have $\left(\ddot{J}_1^{\ T} - \lambda_{1i} I\right) \emptyset^{[1]} = 0$. By solving this equation $\emptyset^{[1]}$,

we obtain $\emptyset^{[1]} = \left(0, \emptyset_2^{[1]}, \sigma_3 \emptyset_4^{[1]}, \emptyset_4^{[1]}\right)^T$ where $\emptyset_2^{[1]}$ and $\emptyset_4^{[1]}$ are any nonzero real numbers,

where:

$$\sigma_3 = \frac{a_9}{a_9 + a_{10}}.$$

Now,

$$\frac{\partial f}{\partial a_6} = f_{a_6}(\mathrm{X}, a_6) = \left(\frac{\partial f_1}{\partial a_6}, \frac{\partial f_2}{\partial a_6}, \frac{\partial f_3}{\partial a_6}, \frac{\partial f_4}{\partial a_6}\right)^{\mathrm{T}} = (0, -i, 0, 0)^{\mathrm{T}}.$$

So, $\frac{\partial f}{\partial a_6}(\mathrm{E}_1, \ddot{a}_6) = (0, 0, 0, 0)^T$ and hence $(\emptyset)^T \frac{\partial f}{\partial a_6}(E_1, \ddot{a}_6) = 0$.

Therefore, according to Sotomayor's theorem, the saddle-node bifurcation cannot occur**.** While the first condition of transcritical bifurcation is satisfied. Now, since

$$Df_{a_6}(E_1, \ddot{a}_6)\dot{H}^{[1]} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sigma_1 \dot{H}_2^{[1]} \\ \dot{H}_2^{[1]} \\ \dot{H}_3^{[1]} \\ \sigma_2 \dot{H}_3^{[1]} \end{pmatrix} = \begin{pmatrix} 0 \\ -\dot{H}_2^{[1]} \\ 0 \\ 0 \end{pmatrix},$$

$$\left(\emptyset^{[1]}\right)^T \left[Df_{a_6}(E_1, \ddot{a}_6)\dot{H}^{[1]}\right] = -\dot{H}_2^{[1]} \emptyset_2^{[1]} \neq 0.$$

Now, by substituting $\dot{H}^{[1]}$ in (29)**,** we get**:**

$$D^2 f(E_1, \ddot{a}_6)\left(\dot{H}^{[1]}, \dot{H}^{[1]}\right) = \begin{pmatrix} -2\sigma_4 \dot{H}_2^{[1]}[a_2 \sigma_4 \dot{H}_3^{[1]} + \dot{H}_2^{[1]}(\sigma_1 + 1 + a_1)] \\ 2\dot{H}_2^{[1]}[\frac{-a_4 \sigma_2 \dot{H}_3^{[1]}}{a_5} + \sigma_1 a_1 \dot{H}_2^{[1]}] \\ 2\sigma_2 \dot{H}_2^{[1]} \dot{H}_3^{[1]}[a_7 \sigma_4 \sigma_1 \dot{H}_3^{[1]} + \frac{a_8}{a_5}] \\ 2\sigma_2 \dot{H}_2^{[1]} \dot{H}_3^{[1]}[a_{11} \sigma_4 \dot{H}_3^{[1]} + \frac{a_{12}}{a_5}] \end{pmatrix},$$

where: $\sigma_4 = \frac{a_3}{[a_3 + (1 - m)]^2}$.

Hence, it is obtained that:

$$\left(\emptyset^{[1]}\right)^T D^2 f(E_1, \ddot{a}_6)\left(\dot{H}^{[1]}, \dot{H}^{[1]}\right) =$$

$$2\dot{H}_2^{[1]}\left(\emptyset_2^{[1]}[\sigma_1 a_1 \dot{H}_2^{[1]} - \frac{a_4 \sigma_2 \dot{H}_3^{[1]}}{a_5}] + \sigma_2 \dot{H}_3^{[1]} \emptyset_4^{[1]}[\sigma_4 \sigma_1 \dot{H}_3^{[1]}(a_7 + a_{11}) + \frac{(a_8 + a_{12})}{a_5}]\right) \neq 0$$

Thus, according to Sotomayor's theorem, system (2) has transcrirtical bifurcation but does not experience a pitchfork bifurcation at E_1 with the parameter $a_6 = \ddot{a}_6$ where $\ddot{a}_6 = a_1$.

4.2 Local bifurcation analysis near E_2

Theorem (3): Suppose that the following conditions are satisfied:

$$\delta \neq 0, \tag{31}$$

$\delta = -\aleph_5\aleph_1[a_2\beta_1 + \aleph_1 + (1+a_1)\aleph_2] - \aleph_2\aleph_4[a_4\beta_2 - a_1\aleph_1] + \aleph_6[a_7\aleph_1\beta_1 + a_8\aleph_2\beta_2] + a_{11}\aleph_1\beta_1 + a_{12}\aleph_2\beta_2$,

$\aleph_1 = -\frac{\dot{u}_{24}}{\dot{u}_{21}}$, $\aleph_2 = \frac{\dot{u}_{11}\dot{u}_{24} - \dot{u}_{14}\dot{u}_{21}}{\dot{u}_{21}\dot{u}_{12}}$, $\aleph_3 = -\frac{\dot{u}_{34}}{\dot{u}_{43}}$, $\aleph_4 = \frac{-\dot{u}_{11}(\dot{u}_{43}\dot{u}_{34} - \dot{u}_{44}\dot{u}_{33})}{\dot{u}_{24}(\dot{u}_{21}\dot{u}_{12} - \dot{u}_{24}\dot{u}_{11})}$,

$\aleph_5 = \frac{-\dot{u}_{21}\aleph_2}{\dot{u}_{11}}$, $\aleph_6 = -\frac{\dot{u}_{43}}{\dot{u}_{33}}$, $\beta_1 = \frac{a_3}{(a_3+(1-m)\bar{s})^2}$, $\beta_2 = \frac{a_5}{(a_5+(1-m)\bar{i})^2}$,

Then system (2) at the equilibrium point $E_2 = (\bar{s}, \bar{i}, 0,0)$ with the parameter

$$\ddot{a}_{13} = \frac{a_{11}\bar{s}}{a_3 + (1-m)\bar{s}} + \frac{a_{12}\bar{i}}{a_5 + (1-m)\bar{i}}$$

possesses**:**

- No saddle-node bifurcation.
- Transcritical bifurcation.
- No pitchfork bifurcation.

Proof: According to the Jacobian matrix J_2 given by eq.(4.4) [12], system (2) at the equilibrium point $E_2 = (\bar{s}, \bar{i}, 0,0)$ has zero eigenvalue (say $\lambda_{2w} = 0$) at $a_{13} = \ddot{a}_{13}$**,** and the Jacobian matrix J_2 with $a_{13} = \ddot{a}_{13}$ becomes:

$\ddot{J}_2 = J_2(a_{13} = \ddot{a}_{13}) = [\dot{u}_{ij}]_{4x4}$,

where, $\dot{u}_{ij} = u_{ij}$ for all i, j =1, 2, 3, 4 except $\dot{u}_{44} = \frac{a_{11}\bar{s}}{a_3+(1-m)\bar{s}} + \frac{a_{12}\bar{i}}{a_5+(1-m)\bar{i}} - \ddot{a}_{13} = 0$.

Let $\dot{H}^{[2]} = (\dot{H}_1^{[2]}, \dot{H}_2^{[2]}, \dot{H}_3^{[2]}, \dot{H}_4^{[2]})^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{2w} = 0$. Thus $(\ddot{J}_2 - \lambda_{2w}I)\dot{H}^{[2]} = 0$, which gives:

$\dot{H}_1^{[2]} = \aleph_1\dot{H}_4^{[2]}$, $\dot{H}_2^{[2]} = \aleph_2\dot{H}_4^{[2]}$ and $\dot{H}_3^{[2]} = \aleph_3\dot{H}_4^{[2]}$, where $\dot{H}_4^{[2]}$ is any nonzero real number, with $\aleph_1$ and $\aleph_2$ which are mentioned in the state of the theorem.

Let $\emptyset^{[1]} = (\emptyset_1^{[2]}, \emptyset_2^{[2]}, \emptyset_3^{[2]}, \emptyset_4^{[2]})^T$ be the eigenvector associated with the eigenvalue $\lambda_{2w} = 0$ of the matrix $\ddot{J}_2$.

Then we have $(\ddot{J}_2 - \lambda_{2w}I)\,\emptyset^{[2]} = 0$. By solving this equation for $\emptyset^{[2]}$, we obtain $\emptyset^{[2]} = (\aleph_5\emptyset_4^{[2]}, \aleph_4\emptyset_4^{[2]}, \aleph_6\emptyset_2^{[2]}, \emptyset_4^{[2]})^T$, where $\emptyset_2^{[2]}$ are any nonzero real numbers, with $\aleph_4, \aleph_5$ and $\aleph_6$ *w*hich are mentioned in the state of the theorem.

Now, consider that:

$$\frac{\partial f}{\partial a_{11}}(X, a_{13}) = f_{a_{13}}(X, a_{13}) = \left(\frac{\partial f_1}{\partial a_{13}}, \frac{\partial f_2}{\partial a_{13}}, \frac{\partial f_3}{\partial a_{13}}, \frac{\partial f_4}{\partial a_{13}}\right)^T = (0,0,0,-w)^T$$

So, $$\frac{\partial f}{\partial a_{13}}(E_2, a_{13}) = (0,0,0,0)^T,$$

and hence $(\emptyset^{[2]})^T f_{a_{13}}(E_2, \ddot{a}_{13}) = 0$.

Therefore, according to Sotomayor's theorem, the saddle-node bifurcation cannot occur**.** While the first condition of transcritical bifurcation is satisfied. Now, since

$$Df_{a_{13}}(E_2, \ddot{a}_{13})\dot{H}^{[2]} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \aleph_1\dot{H}_4^{[2]} \\ \aleph_2\dot{H}_4^{[2]} \\ \aleph_3\dot{H}_4^{[2]} \\ \dot{H}_4^{[2]} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\dot{H}_4^{[2]} \end{pmatrix}$$

$(\emptyset^{[2]})^T[Df_{a_{13}}(E_2, \ddot{a}_{13})H^{[2]}] = -\emptyset_4^{[2]}\dot{H}_4^{[2]} \neq 0$.

Moreover, by substituting $H^{[2]}$ in (29), we get:

$$D^2f(E_2,\ddot{a}_{13})(\dot{H}^{[2]},\dot{H}^{[2]}) = \begin{bmatrix} -2\aleph_1\left(\dot{H}_4^{[2]}\right)^2[a_2\beta_1+\aleph_1+(1+a_1)\aleph_2] \\ -2\aleph_2\left(\dot{H}_4^{[2]}\right)^2[a_4\beta_2-a_1\aleph_1] \\ 2\left(\dot{H}_4^{[2]}\right)^2[a_7\aleph_1\beta_1+a_8\aleph_2\,\beta_2] \\ 2\left(\dot{H}_4^{[2]}\right)^2[a_{11}\aleph_1\beta_1+a_{12}\aleph_2\beta_1] \end{bmatrix}.$$

Hence, it is obtained that:

$$\left(\emptyset^{[2]}\right)^T D^2f(E_2,\ddot{a}_{13})(\dot{H}^{[2]},\dot{H}^{[2]}) = 2\delta\dot{H}_2^{[2]}\emptyset^{[2]},$$

where δ are mentioned in the state of the theorem.

So, by condition (31), we obtain that:

$$\left(\emptyset^{[2]}\right)^T D^2f(E_2,\ddot{a}_{13})(\dot{H}^{[2]},\dot{H}^{[2]}) \neq 0.$$

Thus, according to Sotomayor's theorem, system (2) has a transcritical bifurcation at the equilibrium point $E_2=\left(\bar{s},\bar{i},0,0\right)$ with the parameter $\ddot{a}_{13}=\frac{a_{11}\bar{s}}{a_3+(1-m)\bar{s}}+\frac{a_{12}\bar{i}}{a_5+(1-m)\bar{i}}$.

4.3 Local bifurcation analysis near $E_3(\tilde{s},0,\tilde{z},\tilde{w})$

Theorem (4): Suppose that the following conditions are satisfied:

$$2\tilde{s}+\frac{a_2a_3\tilde{w}}{[a_3+(1-m)\tilde{s}]^2}<1, \tag{32}$$

$$\tilde{s}<\frac{1}{2}, \tag{33}$$

$$\ddot{\pounds}\neq 0, \tag{34}$$

where:

$$\ddot{I}_1=\frac{-\ddot{n}_{31}}{\ddot{n}_{33}},\quad \ddot{I}_2=\frac{\ddot{n}_{14}\ddot{n}_{31}}{\ddot{n}_{41}\ddot{n}_{34}-\ddot{n}_{31}\ddot{n}_{44}},\quad \ddot{I}_3=\frac{\ddot{n}_{42}\ddot{I}_2}{\ddot{n}_{31}},\quad \ddot{I}_4=\frac{-(\ddot{n}_{12}+\ddot{n}_{32}\ddot{I}_4+\ddot{n}_{42}\ddot{I}_2)}{\ddot{n}_{22}},$$

$$\mu_1=\frac{a_3(1-m)\tilde{w}}{[a_3+(1-m)\tilde{s}]^3},\quad \ddot{\pounds}=\mu_1[a_2-(a_7\ddot{I}_2+\ddot{I}_1a_{11})].$$

Then system (2) at the equilibrium point $E_3=(\tilde{s},0,\tilde{z},\tilde{w})$ with the parameter value: $\ddot{a}_2=\frac{(1-2\tilde{s})[a_3+(1-m)\tilde{s}]^2}{a_3\tilde{w}}$ has a saddle − node bifurcation, but neither a transcritical nor a pitchfork bifurcation can occur at E_3.

Proof: By using the characteristic equation given by eq. (4.5) [12], system (2) at the equilibrium point E_3 has zero eigenvalue (say $\lambda_{3s}=0$) at $a_2=\ddot{a}_2$ and the Jacobian matrix J_3 with parameter $a_2=\ddot{a}_2$ becomes:

$$\ddot{J}_3=J_3(a_2=\ddot{a}_2)=\left[\ddot{n}_{ij}\right]_{4\times 4},\quad \text{where,}$$

$\ddot{n}_{ij}=n_{ij}$ for all i, j = 1,2,3,4 except $\ddot{n}_{11}=1-\tilde{s}-\frac{\ddot{a}_2\,a_3\tilde{w}}{[a_3+(1-m)\tilde{s}]^2}=0$ and $\ddot{n}_{14}=\frac{\ddot{a}_2\,\tilde{s}}{a_3+(1-m)\tilde{s}}$.

Note that, $\ddot{a}_2>0$, which is provided by condition (33).

Let $\dot{H}^{[3]}=\left(\dot{H}_1^{[3]},\dot{H}_2^{[3]},\dot{H}_3^{[3]},\dot{H}_4^{[3]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{3s}=0$. Thus $\left(\ddot{J}_3-\lambda_{3s}I\right)\dot{H}^{[3]}=0$, which gives:

$\dot{H}^{[3]}=\left(\dot{H}_1^{[3]},0,\ddot{I}_1\,\dot{H}_1^{[3]},0\right)^T$, where $\dot{H}_1^{[3]}$ is any nonzero number with $\ddot{I}_1$, which is mentioned in the state of the theorem.

Let $\emptyset^{[3]}=\left(\emptyset_1^{[3]},\emptyset_2^{[3]},\emptyset_3^{[3]},\emptyset_4^{[3]}\right)^T$ be the eigenvector associated with the eigenvalue $\lambda_{3s}=0$ of the matrix $\ddot{J}_3$. Then we have $\left(\ddot{J}_3^{\,T}-\lambda_{3s}I\right)\emptyset^{[3]}=0$.

By solving this equation for $\emptyset^{[3]}$, we obtain:

$\emptyset^{[3]}=\left(\emptyset_1^{[3]},\ddot{I}_4\emptyset_1^{[3]},\ddot{I}_3\,\emptyset_1^{[3]},\ddot{I}_2\emptyset_1^{[3]}\right)^T$ where $\emptyset_4^{[3]}$ is any nonzero number with $\ddot{I}_2$, $\ddot{I}_3$ and $\ddot{I}_4$, which is mentioned in the state of the theorem.

Now,

$$\frac{\partial f}{\partial a_2}=f_{a_2}(X,a_2)=\left(\frac{\partial f_1}{\partial a_2},\frac{\partial f_2}{\partial a_2},\frac{\partial f_3}{\partial a_2},\frac{\partial f_4}{\partial a_2}\right)^T=\left(-\frac{a_3\mathrm{w}+s[a_3+(1-m)s]}{[a_3+(1-m)s]^2}s\,,0\,,0\,,0\right)^T,$$

So, $f_{a_2}(E_3\,,\,\ddot{a}_2)=\left(-\frac{a_3\tilde{\mathrm{w}}+\tilde{s}[a_3+(1-m)\tilde{s}]}{[a_3+(1-m)\tilde{s}]^2}\tilde{s},0,0,0\right)^T$,

and hence $\left(\emptyset^{[3]}\right)^T f_{a_2}(E_3\,,\,\ddot{a}_2)=-\emptyset_1^{[3]}\frac{a_3\tilde{\mathrm{w}}+\tilde{s}[a_3+(1-m)\tilde{s}]}{[a_3+(1-m)\tilde{s}]^2}\tilde{s}\neq 0.$

Therefore, according to Sotomayor's theorem, neither a transcritical nor a pitchfork bifurcation can occur at E_3, while the first condition of a saddle-node bifurcation is satisfied.

Moreover, by substituting $H^{[3]}$ in (29), we get:

$$D^2f(E_3,\,\ddot{a}_2)\left(\dot{H}^{[3]},\dot{H}^{[3]}\right)=\left[t_{ij}\right]_{4x1}=\begin{bmatrix}2a_2\mu_1(\dot{H}_1^{[3]})^2\\0\\-2\,(\dot{H}_1^{[4]})^2a_7\mu_1\\-2\,(\dot{H}_1^{[4]})^2a_{11}\mu_1\end{bmatrix}.$$

Hence, it is obtained that:

$\left(\emptyset^{[3]}\right)^T D^2f(E_3,\,\ddot{a}_2)\left(\dot{H}^{[3]},\dot{H}^{[3]}\right)=2\emptyset_1^{[3]}\left(\dot{H}_1^{[3]}\right)^2\ddot{\pounds}.$

So, according to the condition (34) we obtain that:

$\left(\emptyset^{[3]}\right)^T D^2f(E_4,\ddot{a_2})\left(\dot{H}^{[3]},\dot{H}^{[3]}\right)\neq 0.$

Thus, by using Sotomayor's theorem, system (2) has a saddle-node bifurcation at $E_3=(\tilde{s}\ ,0\,,\tilde{z}\,,\widetilde{w})$ at the parameter : $\ddot{a}_2=\frac{(1-2\tilde{s})[a_3+(1-m)\tilde{s}]^2}{a_3\widetilde{w}}$.

4.4 Local bifurcation analysis near E_4

Theorem (5): Suppose that the following conditions are satisfied:

$$\bar{\bar{s}}>\frac{\left(1-(1-a_1)\bar{\bar{i}}\right)\left[a_3+(1-\mathrm{m})\bar{\bar{s}}\right]^2-a_2a_3\bar{\bar{\mathrm{w}}}}{2\left[a_3+(1-\mathrm{m})\bar{\bar{s}}\right]^2},\tag{35}$$

$$\bar{\bar{w}}>\frac{a_6\left[a_3+(1-\mathrm{m})\bar{\bar{s}}\right]^2}{a_4a_5},\tag{36}$$

$$Đ_1\neq 1,\tag{37a}$$

$$Đ_5\left(\bar{\bar{s}}+Đ_2\bar{\bar{i}}\right)\neq\bar{\bar{i}},\tag{37b}$$

$$£\neq 0,\tag{38a}$$

$$\dot{£}\neq 0,\tag{38b}$$

where:

$$Đ_1=\frac{\ddot{m}_{21}\ddot{m}_{12}}{\ddot{m}_{11}\ddot{m}_{24}-\ddot{m}_{14}\ddot{m}_{12}},\quad Đ_2=\frac{-\ddot{m}_{14}}{\ddot{m}_{12}}Đ_1,\quad Đ_3=\frac{-(\ddot{m}_{32}+\ddot{m}_{31}\,Đ_2+\ddot{m}_{34}\,\,Đ_1}{\ddot{m}_{33}},$$

$$Đ_4=\frac{\ddot{m}_{12}\ddot{m}_{21}\ddot{m}_{33}}{\ddot{m}_{12}(\ddot{m}_{43}\ddot{m}_{31}-\ddot{m}_{33}\ddot{m}_{21})-\ddot{m}_{11}(\ddot{m}_{43}\ddot{m}_{32}-\ddot{m}_{42}\ddot{m}_{33})},$$

$$Đ_5=\frac{(\ddot{m}_{43}\ddot{m}_{32}-\ddot{m}_{21}\ddot{m}_{33})Đ_4}{\ddot{m}_{33}},\qquad Đ_6=\frac{-\ddot{m}_{43}Đ_4}{\ddot{m}_{33}},$$

$$\ddot{Y}_1=\frac{a_3Đ_2}{[a_3+(1-m)\bar{\bar{s}}\,]^2}\left[\frac{(1-m)\bar{\bar{w}}}{a_3+(1-m)\bar{\bar{s}}}-Đ_2\right],\ \ddot{Y}_2=\frac{a_5}{\left[a_5+(1-m)\bar{\bar{i}}\,\right]^2}\left[\frac{(1-m)\bar{\bar{w}}}{a_5+(1-m)\bar{\bar{i}}}-Đ_1\right],$$

$$\ddot{Y}_3=\frac{a_3(1-m)(Đ_2)^2}{[a_3+(1-m)\bar{\bar{s}}\,]^3}\left[Đ_1-\frac{(1-m)\bar{\bar{w}}}{a_3+(1-m)\bar{\bar{s}}}\right],\ddot{Y}_4=\frac{a_5(1-m)}{\left[a_5+(1-m)\bar{\bar{i}}\,\right]^3}\left[Đ_1-\frac{(1-m)\bar{\bar{w}}}{a_5+(1-m)\bar{\bar{i}}}\right],$$

$£=Đ_5Đ_2[a_2\ddot{Y}_1+Đ_2-(1+a_1)]+a_4\ddot{Y}_2+a_1Đ_1-Đ_6[a_7\ddot{Y}_1+a_8\ddot{Y}_2]-Đ_4[a_{11}\ddot{Y}_1+a_{12}\ddot{Y}_2],$

$\dot{£}=a_2Đ_5\ddot{Y}_3+a_4\ddot{Y}_4-[Đ_6(a_7\ddot{Y}_3+a_8\ddot{Y}_4)+Đ_4(a_{11}\ddot{Y}_3+a_{12}\ddot{Y}_4)].$

Then system (2) at the equilibrium point $E_4=\left(\bar{\bar{s}}\ ,\bar{\bar{i}}\ ,\bar{\bar{z}}\ ,\bar{\bar{w}}\right)$ with the parameter value:

$$\ddot{a}_1=\frac{a_4a_5\bar{\bar{w}}-a_6[a_5+(1-m)\bar{\bar{i}}\ ]^2}{\bar{\bar{s}}[a_5+(1-m)\bar{\bar{i}}\ ]^2},$$

has a saddle − node bifurcation, a transcritical and a pitchfork bifurcation at E_4.

Proof: The characteristic equation given by eq.(4.6) [12] of system (2) having zero eigenvalue (say $\lambda_{4i}=0$) if and only if $H_4=0$, and then E_4 becomes a non-hyperbolic equilibrium point. Clearly, the Jacobian matrix of system (2) at the equilibrium point E_4 with parameter $a_1=\ddot{a}_1$ becomes:

$$\ddot{J}_4=J_4(\,a_1=\ \ddot{a}_1)=\left[\,\ddot{m}_{ij}\,\right]_{4\times4},$$

where, $\ddot{m}_{ij}=m_{ij}$ for all i , j =1,2,3,4 except m_{ij} which is given by:

$$\ddot{m}_{11}=1-2\bar{\bar{s}}-(1+\ \ddot{a}_1)\bar{\bar{i}}-\frac{a_2\ a_3\bar{\bar{w}}}{\left[a_3+(1-m)\bar{\bar{s}}\right]^2},\ \ddot{m}_{12}=-(1+\ \ddot{a}_1)\bar{\bar{s}}$$

and $\ddot{m}_{21}=\ \ddot{a}_1\bar{\bar{i}}$. Note that, $\ddot{a}_1>0$ provided that condition (36).

Let $\dot{H}^{[4]}=\left(\dot{H}_1^{[4]},\dot{H}_2^{[4]},\dot{H}_3^{[4]},\dot{H}_4^{[4]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{4i}=0$. Thus $\left(\ddot{J}_4-\lambda_{4i}I\right)\dot{H}^{[4]}=0$, which gives:

$\dot{H}^{[4]}=\left(Đ_2\dot{H}_2^{[4]},\dot{H}_2^{[4]},Đ_3\dot{H}_2^{[4]},Đ_1\dot{H}_2^{[4]}\right)^T$, where $\dot{H}_2^{[4]}$ is any nonzero number with $Đ_1,Đ_2$ and $Đ_3$ which is mentioned in the text of the theorem.

Let $\emptyset^{[4]}=\left(\emptyset^{[4]},\emptyset^{[4]},\emptyset^{[4]},\emptyset^{[4]}\right)^T$ be the eigenvector associated with the eigenvalue $\lambda_{4i}=0$ of the matrix $\ddot{J}_4$. Then we have $\left(\ddot{J}_4{}^T-\lambda_{4i}I\right)\emptyset^{[4]}=0$.

By solving this equation for $\emptyset^{[4]}$, we obtain:

$\emptyset^{[4]}=\left(Đ_5\emptyset_2^{[4]},\emptyset_2^{[4]},Đ_6\emptyset_2^{[4]},Đ_4\emptyset_2^{[4]}\right)^T$ where $\emptyset_2^{[4]}$ is any nonzero number with $Đ_4,Đ_5$ and $Đ_6$ which is mentioned in the state of the theorem.

Now,

$$\frac{\partial f}{\partial a_1}=f_{a_1}(X\,,a_1)=\left(\frac{\partial f_1}{\partial a_1},\frac{\partial f_2}{\partial a_1},\frac{\partial f_3}{\partial a_1},\frac{\partial f_4}{\partial a_1}\right)^T=(-si\,,si\,,0\,,0)^T,$$

So, $f_{a_1}(E_4\,,\,\ddot{a}_1)=\left(-\bar{\bar{s}}\bar{\bar{i}}\,,\bar{\bar{s}}\bar{\bar{i}}\,,0,0\right)^T$ and hence it is obtained that:

$$\left(\emptyset^{[4]}\right)^Tf_{a_1}(E_4\,,\,\ddot{a}_1)=\bar{\bar{s}}\bar{\bar{i}}\emptyset_2^{[4]}(1-Đ_5).$$

So, according to the condition $(37a)$ we obtain that:

$$\left(\emptyset^{[4]}\right)^Tf_{a_1}(E_4\,,\,\ddot{a}_1)\neq0.$$

Therefore, according to Sotomayor's theorem 4, neither a transcritical nor a pitchfork bifurcation 4 can occur at E_4 , while the first condition of a saddle-node 4 bifurcation is satisfied. Moreover, by substituting $H^{[4]}$ in (29), we get:

$$D^2f(E_4\,,\,\ddot{a}_1)\left(\dot{H}^{[4]},\dot{H}^{[4]}\right)=\begin{bmatrix}2(\dot{H}_2^{[4]})^2Đ_2[a_2\ddot{Y}_1+Đ_2-(1+a_1)]\\2\,(\dot{H}_2^{[4]})^2[a_4\ddot{Y}_2+a_1Đ_1]\\-2\,(\dot{H}_1^{[4]})^2[a_7\ddot{Y}_1+a_8\ddot{Y}_2]\\-2\,(\dot{H}_1^{[4]})^2[a_{11}\ddot{Y}_1+a_{12}\ddot{Y}_2]\end{bmatrix}.$$

Hence, it is obtained that:

$$\left(\emptyset^{[4]}\right)^TD^2f(E_4\,,\,\ddot{a}_1)\left(\dot{H}^{[4]},\dot{H}^{[4]}\right)=2\emptyset_2^{[4]}\left(\dot{H}_2^{[4]}\right)^2\ddot{O}_1\,£.$$

So, according to the condition $(38a)$ we obtain that:

$$\left(\emptyset^{[4]}\right)^TD^2f(E_4,\ddot{a}_1)\left(\dot{H}^{[4]},\dot{H}^{[4]}\right)\neq0$$

Thus, by using Sotomayor's theorem, system (2) has a saddle-node bifurcation at $E_4 = \left(\bar{\bar{s}}, \bar{\bar{i}}, \bar{\bar{z}}, \bar{\bar{w}}\right)$ at $\ddot{a}_1$.

Now, if the condition (37a) is not satisfied, we obtain that:

$$\left(\emptyset^{[4]}\right)^T f_{a_1}(E_4, \ddot{a}_1) = 0.$$

Therefore, according to Sotomayor 's theorem, the saddle - node bifurcation cannot occur **.** While the first condition of transcritical bifurcation is satisfied . Now, since

$$Df_{a_1}(E_4, \ddot{a}_1)\dot{H}^{[4]} = \begin{pmatrix} -\bar{\bar{i}} & -\bar{\bar{s}} & 0 & 0 \\ \bar{\bar{i}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} Đ_2\dot{H}_2^{[4]} \\ \dot{H}_2^{[4]} \\ Đ_3\dot{H}_2^{[4]} \\ Đ_1\dot{H}_2^{[4]} \end{pmatrix} = \begin{pmatrix} -\dot{H}_2^{[4]}(\bar{\bar{s}} + Đ_2\bar{\bar{i}}) \\ \bar{\bar{i}}\dot{H}_2^{[4]} \\ 0 \\ 0 \end{pmatrix},$$

$$\left(\emptyset^{[4]}\right)^T\left[Df_{a_1}(E_4, \ddot{a}_1)\dot{H}^{[4]}\right] = -\dot{H}_2^{[4]}\emptyset_2^{[4]}\,[Đ_5\left(\bar{\bar{s}} + Đ_2\bar{\bar{i}}\right) - \bar{\bar{i}}\,].$$

So, according to the condition (37b) we obtain that:

$$\left(\emptyset^{[4]}\right)^T Df(E_4, \ddot{a}_1)\left(\dot{H}^{[4]}, \dot{H}^{[4]}\right) \neq 0.$$

Now, by substituting $\dot{H}^{[4]}$ in (29), we get**:**

$$D^2f(E_4, \ddot{a}_1)\left(\dot{H}^{[4]}, \dot{H}^{[4]}\right) = \begin{bmatrix} 2(\dot{H}_2^{[4]})^2Đ_2[a_2Ÿ_1 + Đ_2 - (1 + a_1)] \\ 2(\dot{H}_2^{[4]})^2[a_4Ÿ_2 + a_1Đ_1] \\ -2(\dot{H}_1^{[4]})^2[a_7Ÿ_1 + a_8Ÿ_2] \\ -2(\dot{H}_1^{[4]})^2[a_{11}Ÿ_1 + a_{12}Ÿ_2] \end{bmatrix}.$$

Hence, it is obtained that:

$$\left(\emptyset^{[4]}\right)^T D^2f(E_4, \ddot{a}_1)\left(\dot{H}^{[4]}, \dot{H}^{[4]}\right) = 2\emptyset_2^{[4]}\left(\dot{H}_2^{[4]}\right)^2 £.$$

So, according to the condition $(38a)$ we obtain that:

$$\left(\emptyset^{[4]}\right)^T D^2f(E_4, \ddot{a}_1)\left(\dot{H}^{[4]}, \dot{H}^{[4]}\right) \neq 0.$$

Now, if the condition (38a) is not satisfied, by 4substituting $\dot{H}^{[4]}$ in (30)**,** we obtain that:

$$D^3f(E_4, \ddot{a}_1)\left(\dot{H}^{[4]}, \dot{H}^{[4]}, \dot{H}^{[4]}\right) = \begin{bmatrix} 6(\dot{H}_2^{[4]})^3a_2Ÿ_3 \\ 6(\dot{H}_2^{[4]})^3a_4Ÿ_4 \\ -6(\dot{H}_1^{[4]})^3[a_7Ÿ_3 + a_8Ÿ_4] \\ -6(\dot{H}_1^{[4]})^3[a_{11}Ÿ_3 + a_{12}Ÿ_4] \end{bmatrix},$$

$$\left(\emptyset^{[4]}\right)^T D^3f(E_4, \ddot{a}_1)\left(\dot{H}^{[4]}, \dot{H}^{[4]}, \dot{H}^{[4]}\right) = 6\emptyset^{[4]}(\dot{H}_2^{[4]})^3\dot{£}.$$

So, according to the condition $(38b)$ we obtain that:

$$\left(\emptyset^{[4]}\right)^T D^3f(E_4, \ddot{a}_1)\left(\dot{H}^{[4]}, \dot{H}^{[4]}, \dot{H}^{[4]}\right) \neq 0.$$

Thus, by using Sotomayor's theorem, system (2) has a transcritical bifurcation and pitch fork bifurcation at $E_4 = \left(\bar{\bar{s}}, \bar{\bar{i}}, \bar{\bar{z}}, \bar{\bar{w}}\right)$ by the conditions (37a) and (38a), respectively, which are not satisfied at $\ddot{a}_1$.

5. Hopf bifurcation analysis

In this section, the possibility of the occurrence of a Hopf bifurcation near the positive equilibrium point of the system (2) is investigated, as shown in the following theorem.

Theorem 5*:* Suppose that the local conditions $(21-23)$ with the following conditions are satisfied:

$$Å_1 > Å_2, \tag{39}$$
$$Å_3 > Å_4, \tag{40}$$
$$Å_5 > Å_6, \tag{41}$$
$$Å_7 < Å_8, \tag{42}$$

$$\frac{4H_1\text{H} - H_1^3}{4} < H_3, \tag{43}$$

$$\gamma_0 < \min\left\{ \frac{{\gamma_0}^2+{m_{33}}^2}{(\gamma_3-\gamma_6)}, (\gamma_3-\gamma_6) \right\}, \tag{44}$$

where:

$\text{Å}_1 = [\gamma_0 + \gamma_3 - \gamma_6][{\gamma_0}^2 - \gamma_7 - 2\gamma_0] - \gamma_0 m_{24}(\gamma_8 - \gamma_9 + \gamma_{10}) + 3\text{Ë}[{\gamma_0}^2 + \gamma_0(\gamma_3 - \gamma_6)] + 3\text{Ë}m_{33} - \gamma_0 m_{14}(\gamma_{12} - \gamma_{13} + \gamma_{14}) + \gamma_0\gamma_6\gamma_{11} + \gamma_{17}\gamma_{18} + \gamma_{10} - \gamma_{16}$

$\text{Å}_2 = -\gamma_{15} - \gamma_{14}\gamma_{17} - \gamma_{19}(\gamma_{13} - \gamma_{14}) - {\gamma_0}^2\gamma_4 - \gamma_9,$

$\text{Å}_3 = [2m_{33}(\gamma_3 - \gamma_6)2\text{Ë}\gamma_0 + 2\text{Ë}(\gamma_3 - \gamma_6) + m_{24}(\gamma_8 - \gamma_9 + \gamma_{10}) - \gamma_0\gamma_4 - \gamma_6\gamma_{11}] + \gamma_0{\gamma_4}^2 + \gamma_0(\gamma_3 - \gamma_6)^2 + 2\gamma_0[(\gamma_9 - \gamma_{10}) + \gamma_{19}(\gamma_{13} - \gamma_{14})] + 2m_{33}[-\gamma_0 m_{24}(\gamma_8 - \gamma_9 + \gamma_{10}) + {\gamma_0}^3\gamma_4\gamma_6\gamma_{11}\text{-}(\gamma_{15} + \gamma_{16})] + {\gamma_0}^2(3\text{Ë}^2 m_{33} + {m_{33}}^3) - [{\gamma_0}^2\gamma_4 + (\gamma_9 - \gamma_{10})] - 2(\gamma_{14} - \gamma_{18})m_{33}\gamma_{17} - 2\text{Ë}\,\gamma_{19}(\gamma_{13} - \gamma_{14}),$

$\text{Å}_4 = (\gamma_0\gamma_5 + \gamma_1\gamma_5 + \gamma_0\gamma_7)[(\gamma_3 - \gamma_6) + \gamma_0 m_{33}] + m_{14}(\gamma_{12} - \gamma_{13} + \gamma_{14})({\gamma_0}^2 - \gamma_7) + 2\gamma_0\gamma_{17}(\gamma_{14} - \gamma_{18}) + m_{33}[(\gamma_{12} - \gamma_{13} + \gamma_{14})^2 m_{14}\gamma_4 - m_{14}(\gamma_{12} - \gamma_{13} + \gamma_{14})] - m_{14}(\gamma_{12} - \gamma_{13} + \gamma_{14})[\gamma_4 + 2\text{Ë}\gamma_0 - 2m_{33}{\gamma_0}^3\gamma_4] + (\gamma_3 - \gamma_6)[m_{14}(\gamma_{12} - \gamma_{13} + \gamma_{14}) + 2\gamma_0\gamma_4] - 2m_{33}(\gamma_9 - \gamma_{10}) + \gamma_0(\gamma_3 - \gamma_6)(3\text{Ë}^2 m_{33} + {m_{33}}^3) + 3\text{Ë}^2{m_{33}}^2 + 2\text{Ë}[-2\gamma_0(\gamma_4 + \gamma_3 - \gamma_6) - \gamma_0 m_{24}(\gamma_8 - \gamma_9 + \gamma_{10}) + \gamma_0\gamma_6\gamma_{11}] + (\gamma_{15} + \gamma_{16})(2\gamma_0 - 2\text{Ë}) - 2\text{Ë}\,\gamma_{17}(\gamma_{14} - \gamma_{18}) - 2\gamma_{19}(\gamma_{13} - \gamma_{14})m_{33},$

$\text{Å}_5 = [(\gamma_0 + \gamma_1)\gamma_5 + \gamma_0\gamma_7)][m_{24}(\gamma_8 - \gamma_9 + \gamma_{10}) + \text{Ë}(\gamma_3 - \gamma_6)({\gamma_0}^2 - \gamma_7) - \gamma_6\gamma_{11} + \gamma_0 m_{33} + m_{14}(\gamma_{12} - \gamma_{13} + \gamma_{14}) + \gamma_0\gamma_4] + \gamma_0(\gamma_3 - \gamma_6)[{m_{33}}^3 + m_{33}\left(2 + \text{Ë}^3 - 2\gamma_4\right) - 1 - m_{14}(\gamma_{12} - \gamma_{13} + \gamma_{14} - 1)] + \gamma_{19}(\gamma_{13} - \gamma_{14})\left(1 - \text{Ë}^3 m_{33} + 2\gamma_0 m_{33} - 2\text{Ë}m_{33}\right) + m_{33}(2\gamma_0 - 2\text{Ë} - \text{Ë}^3)(\gamma_9 - \gamma_{10}) - m_{24}m_{33}({\gamma_0}^2 - \gamma_7)(\gamma_8 - \gamma_9 + \gamma_{10}) + (\gamma_3 - \gamma_6)^2(\gamma_0 m_{33} + 2\text{Ë}m_{33}\gamma_0) + \text{Ë}\gamma_6\gamma_{11}(2m_{14} - {\gamma_0}^2 + \gamma_7) + \text{Ë}^3{m_{33}}^2 + \text{Ë}m_{33}(\text{Ë}^2 - 2\gamma_0)[\gamma_0\gamma_4 + \gamma_6\gamma_{11} - {\gamma_4}^2] + \text{Ë}m_{33}(\text{Ë}^2 - 2\gamma_0)[\gamma_0\gamma_4 + \gamma_6\gamma_{11} - {\gamma_4}^2] + {m_{33}}^2[({\gamma_0}^2 - \gamma_7)(\gamma_3 - \gamma_6) - \gamma_0 m_{24}(\gamma_8 - \gamma_9 + \gamma_{10})(\gamma_0 - \text{Ë} + 2\text{Ë}m_{14}) - (\gamma_{15} + \gamma_{16}) + \text{Ë}m_{14}m_{33}\left(\gamma_4 - \text{Ë}^2\gamma_0 - {\gamma_0}^2 + \gamma_7\right)(\gamma_{12} - \gamma_{13} + \gamma_{14}) - \gamma_{17}(\gamma_{14} - \gamma_{18})(\text{Ë} + {m_{33}}^2).$

and

$$\text{Ë} = \frac{a_{11}\overset{=}{s}}{a_3 + (1-m)\overset{=}{s}} + \frac{a_{12}\overset{=}{i}}{a_5 + (1-m)\overset{=}{i}}.$$

Then at the parameter value $a_{13} = \overset{*}{a}_{13}$, system (2) has a Hopf bifurcation near the point E_4.

Proof*:* Consider the characteristic equation of system (2) at E_4 which is given by eq. (24), then by using the Hopf bifurcation theorem, for n=4, we need to find a parameter, say $\left(\overset{*}{a}_{13}\right)$, to verify that the necessary and sufficient conditions for the Hopf bifurcation are satisfied, that is:

$A_i\left(\overset{*}{a}_{13}\right) > 0;\ i = 1,3,4,\ \Delta_1\left(\overset{*}{a}_{13}\right) > 0,\ H_1^3\left(\overset{*}{a}_{13}\right) - 4\,\Delta_1\left(\overset{*}{a}_{13}\right) > 0$ and $\Delta_2\left(\overset{*}{a}_{13}\right) = 0,$

where $A_i;\ i = 1,3,4$ represents the coefficients of the characteristic eq. (24).

Straightforward computation gives that:

$A_i\left(\overset{*}{a}_{13}\right) > 0;\ i = 1,3,4$ and $\Delta_1\left(\overset{*}{a}_{13}\right) > 0$ under the local conditions (25-28),

while $A_1^3\left(\overset{*}{a}_{13}\right) - 4\,\Delta_1\left(\overset{*}{a}_{13}\right) > 0$, provided that the condition (43) holds.

On the other hand, it is observed that $\Delta_2 = 0$ gives that:

$$H_3\,(H_1 H_2 - H_3) - H_1^2\,H_4 = 0.$$

By a straight forward computation, we get:

$$M_1\,{\overset{*}{a}_{13}}^3 + M_2{\overset{*}{a}_{13}}^2 + M_3\overset{*}{a}_{13} + M_4 = 0, \tag{45}$$

where:

$M_1 = -m_{33}[{\gamma_0}^2 + {m_{33}}^2 + \gamma_0(\gamma_3 - \gamma_6)]$,
$M_2 = Å_1 + Å_2$,
$M_3 = Å_3 + Å_4$,
$M_4 = Å_5 + Å_6$,
where $Å_i$; $i = 1,2,3,4,5,6$ which are mentioned in the text of the theorem.
Clearly, $M_i > 0$, $i = 2,3,4$ provided that in addition to the conditions $(25 - 28)$, the conditions $(39 - 42)$ holds.
Note that, by using Descartes rule of sign, eq.(45) has a unique positive root $\overset{*}{a}_{13}$.
Now, at $a_{13} = \overset{*}{a}_{13}$ the characteristic equation given by eq. (24). can be written as:

$\left(\lambda^2 + \frac{H_3}{H_1}\right)\left(\lambda^2 + H_1\lambda + \frac{\Delta_1}{H_1}\right) = 0$, which has four roots:

$$\lambda_{1,2} = \pm i\sqrt{\frac{H_3}{H_1}} \quad \text{and} \quad \lambda_{3,4} = \frac{1}{2}\left(-H_1 \pm \sqrt{H_1^2 - 4\frac{\Delta_1}{H_1}}\right).$$

Clearly, at $a_{13} = \overset{*}{a}_{13}$ there are two pure imaginary eigenvalues (λ_1 and λ_2) and two eigenvalues which are real and negative. Now for all values of a_{13} in the neighborhood of $\overset{*}{a}_{13}$, the roots are in general of the following form:

$$\lambda_1 = \varepsilon_1 + i\varepsilon_2 \quad ,\lambda_2 = \varepsilon_1 - i\varepsilon_2 \ , \quad \lambda_{3,4} = \frac{1}{2}\left(-H_1 \pm \sqrt{H_1^2 - 4\frac{\Delta_1}{H_1}}\right).$$

Clearly, $Re\big(\lambda_N(a_{13})\big)\big|_{a_{13}=\overset{*}{a}_{13}} = \varepsilon_1\left(\overset{*}{a}_{13}\right) = 0$, $N = 1,2$, which implies that the first condition of the necessary and sufficient conditions for Hopf bifurcation are satisfied at $a_{13} = \overset{*}{a}_{13}$. Now, according to the verification of the transversality condition, we must prove that:

$$\dot{\Theta}\left(\overset{*}{a}_{13}\right)\dot{\Psi}\left(\overset{*}{a}_{13}\right) + \dot{\Gamma}\left(\overset{*}{a}_{13}\right)\dot{\Phi}\left(\overset{*}{a}_{13}\right) \neq 0,$$

where $\dot{\Theta}, \dot{\Psi}, \dot{\Gamma}$ and $\dot{\Phi}$ are given in lemma (1) of a previous work [11]. Note that for $a_{13} = \overset{*}{a}_{13}$ we have $\varepsilon_1(\overset{*}{a}_{13}) = 0$ and $\varepsilon_2(\overset{*}{a}_{13}) = \sqrt{\frac{H_3}{H_1}}$, thus it gives the following simplifications:

$\dot{\Psi}\left(\overset{*}{a}_{13}\right) = -2\, H_3\left(\overset{*}{a}_{13}\right)$,

$\dot{\Phi}\left(\overset{*}{a}_{13}\right) = 2\,\frac{\varepsilon_2\left(\overset{*}{a}_{13}\right)}{H_1}(H_1H_2 - 2\,H_3)$,

$\dot{\Theta}\left(\overset{*}{a}_{13}\right) = H_4'\left(\overset{*}{a}_{13}\right) - \frac{H_3}{H_1}H_2'\left(\overset{*}{a}_{13}\right)$,

$\dot{\Gamma}\left(\overset{*}{a}_{13}\right) = \varepsilon_2\left(\overset{*}{a}_{13}\right)\left(H_3'\left(\overset{*}{a}_{13}\right) - \frac{H_3}{H_1}H_1'\left(\overset{*}{a}_{13}\right)\right)$,

where:

$H_1' = \frac{dH_1}{da_{13}}\big|_{a_{13}=\overset{*}{a}_{13}} = 1$,

$H_2' = \frac{dH_2}{da_{13}}\big|_{a_{13}=\overset{*}{a}_{13}} = -(m_{11} + m_{22} + m_{33})$,

$H_3' = \frac{dH_3}{da_{13}}\big|_{a_{13}=\overset{*}{a}_{13}} = m_{11} + m_{22} + m_{11}m_{22}$,

$H_4' = \frac{dH_4}{da_{13}}\big|_{a_{13}=\overset{*}{a}_{13}} = -m_{33}(\gamma_3 - \gamma_6)$.

Then , we get that: $\dot{\Theta}\left(\overset{*}{a}_{13}\right)\dot{\Psi}\left(\overset{*}{a}_{13}\right) + \dot{\Gamma}\left(\overset{*}{a}_{13}\right)\dot{\Phi}\left(\overset{*}{a}_{13}\right) = Å_7 + Å_8 \neq 0$, where:

$Å_7 = 2\, H_3 \left(\frac{H_3}{H_1} + m_{33}(\gamma_3 - \gamma_6)\right),$

$Å_8 = 2\, \frac{{\varepsilon_2}^2}{H_1}(H_1 H_2 - \, 2\, H_3)\left(m_{11} + m_{22} + m_{11}m_{22} + \frac{H_3}{H_1}(m_{11} + m_{22} + m_{33})\right).$

Now, according to condition (45), we have:

$$\dot{\boldsymbol{\Theta}}\left(\overset{*}{a}_{13}\right)\, \dot{\boldsymbol{\Psi}}\left(\overset{*}{a}_{13}\right) + \dot{\boldsymbol{\Gamma}}\, \left(\overset{*}{a}_{13}\right)\, \dot{\boldsymbol{\Phi}}\left(\overset{*}{a}_{13}\right) \neq \mathbf{0}\,.$$

So, we obtain that the Hopf bifurcation occurs around the equilibrium point E_4 at the parameter $a_{13} = \overset{*}{a}_{13}$.

6. Numerical simulation

In this section, the dynamical behavior of system (2) is studied numerically for a set of parameters and different sets of initial points. The objectives of this part are:

1- Investigating the effect of varying the value of each parameter on the dynamical behavior of system (2).

2- Confirming the obtained analytical results.

It is observed that, for the following set of hypothetical parameters that satisfies stability conditions of the positive equilibrium point, system (2) has a globally asymptotically stable positive equilibrium point, as shown in Figure− (1) $(a - d)$.

$$\boldsymbol{a_1 = 2, a_2 = 0.3, m = 0.7,\ a_3 = 0.5,\ a_4 = 0.2,}$$
$$\boldsymbol{a_5 = 0.5, a_6 = 0.01,\ a_7 = 0.1,\ a_8 = 0.1\,, a_9 = 0.5,}$$
$$\boldsymbol{a_{10} = 0.01,\ \ a_{11} = 0.1,\ a_{12} = 0.1,\ a_{13} = 0.1}$$

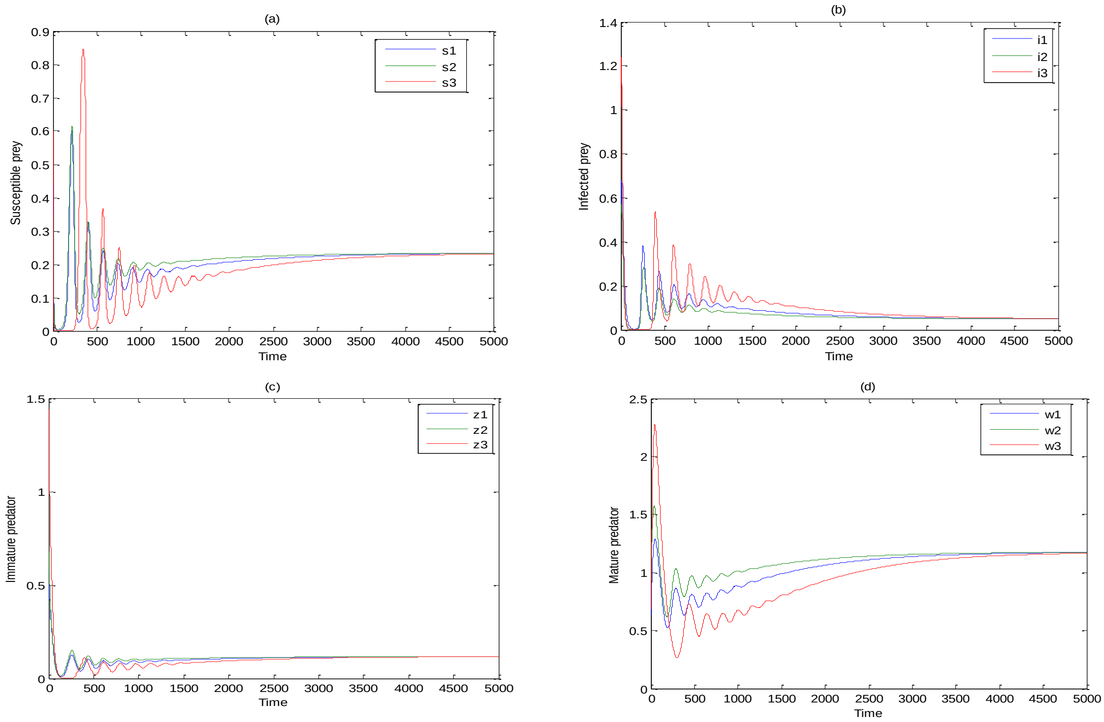

Figure 1-The time series of the solution of system (2) starting from the three different initial points (0.4, 0.5, 0.6, 0.6), (0.4,0.5 , 0.7 , 0.9) and (0.8, 0.9, 1.5, 0.5)
for the data given by eq. (6.1). (a) the trajectories of s as a function of time, (b) the trajectories of i as a function of time, (c) trajectories of z as a function of time, (d) the trajectories of w as a function of time.

Clearly, Figure – (1) shows that system (2) is globally asymptotically stable as the solution of system (2) approaches asymptotically to the positive equilibrium point $E_4 = (0.233, 0.050, 0.116, 1.173)$, starting from three different initial points, which confirms our obtained analytical results.

Now, in order to discuss the effects of the parameters' values of system (2) on the dynamical behavior of the system, the system is solved numerically for the data given in eq. (6.1) with varying one parameter at each time, The obtained results are given in Table-1, while more details are provided elsewhere [4].

Table 1-NUMERICAL BEHAVIORS OF SYSTEM (2) FOR THE DATA GIVEN IN (6.1)WITH VARYING ONE PARAMETER AT EACH TIME

Range of parameter	Numerical behavior of system (2)	Bifurcation point
$0.1 \leq a_1 < 1.9$ $1.9 \leq a_1 < 2.2$	Approaches to the infected prey free equilibrium point E_3 Approaches to the positive equilibrium point E_4	$a_1 = 1.9$
$0.1 \leq a_2 < 0.29$ $0.29 \leq a_2 < 1$	Approaches to the infected prey free equilibrium point E_3 Approaches to the positive equilibrium point E_4	$a_2 = 0.29$
$0.25 < a_3 \leq 0.4$ $0.4 < a_3 \leq 1$	Approaches to the infected prey free equilibrium point E_3 Approaches to the positive equilibrium point E_4	$a_3 = 0.4$
$0.1 \leq a_4 < 0.22$ $0.22 \leq a_4 < 1$	Approaches to the positive equilibrium point E_4 Approaches to the infected prey free equilibrium point E_3	$a_4 = 0.22$
$0.1 \leq a_5 < 0.45$ $0.45 \leq a_5 < 0.6$	Approaches to the infected prey free equilibrium point E_3 Approaches to the positive equilibrium point E_4	$a_5 = 0.45$
$0 \leq a_6 < 0.3$ $0.3 \leq a_6 < 1$	Approaches to the positive equilibrium point E_4 Approaches to the infected prey free equilibrium point E_3	$a_7 = 0.11$
$0.0001 \leq a_7 < 0.11$ $0.11 \leq a_7 < 0.15$	Approaches to the positive equilibrium point E_4 Approaches to the infected prey free equilibrium point E_3	$a_7 = 0.11$
$0.01 \leq a_{13} < 0.97$ $0.97 \leq a_{13} < 0.13$	Approaches to the infected prey free equilibrium point E_3 Approaches to the positive equilibrium point E_4	$a_{13} = 0.97$

The effects of varying the predation rate on susceptible prey a_2 in the range of $0.1 \leq a_2 < 0.29$ while keeping the other parameters as the data given in eq.(6.1), causes extinction in the infected prey and the system will approach to the infected prey free equilibrium point E_3 , as shown in Figure-(2) a, for a typical value of $a_2 = 0.2$. In the range of $0.29 \leq a_2 < 1$, it is observed that the solution of system (2) approaches asymptotically to the positive equilibrium point E_4, as shown in Figure- (2) b for a typical value of $a_2 = 0.35$.

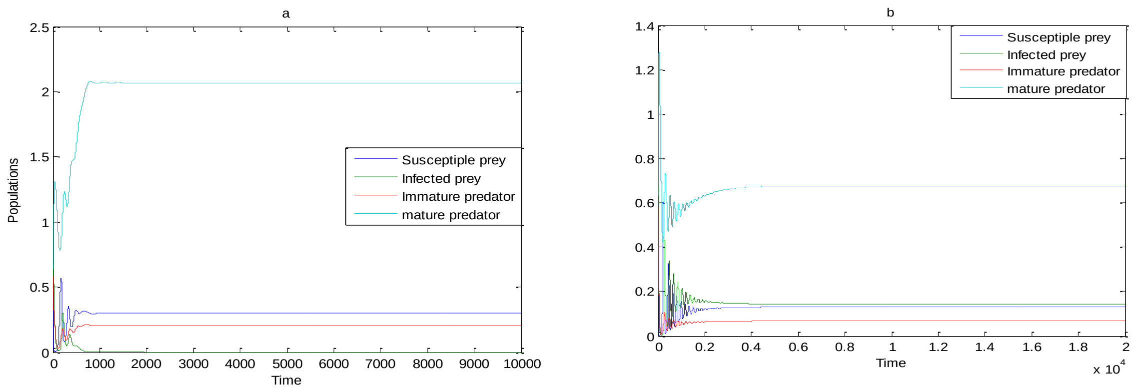

Figure 2-(a): Time series of the solution of system (2) which approaches asymptotically to the infected prey free equilibrium point $E_3 = (0.298, 0, 0.205, 2.070)$ in the positive quadrant of szw – space, for the data given in $eq.(6.1)$ with $a_2 = 0.2$. *(b):* Time series of the solution of system (2) approaches asymptotically to the *positive* equilibrium point $E_4 = (0130, 0.144, 0.067, 0.677)$ in the int. 0f R_+^4 , For the data given in eq. (6.1) with $a_2 = 0.35$.

By varying of the parameter a_7 ,which represents the conversion rate from the susceptible prey to the immature predator in the rang $0.0001 \leq a_7 < 0.11$, and keeping the rest of parameters values as in the data given in eq. (6.1), the solution of system (2) still approaches asymptotically to the positive equilibrium point E_4, as shown in Figure-(3) (a), for a typical value of a_7 = 0.08. However, by increasing this parameter further to $0.11 \leq a_7 \leq 0.15$, it is observed that system (2) still approaches the infected prey free equilibrium point E_3, as shown in Figure (3) (b), for a typical value of a_7 = 0.12.

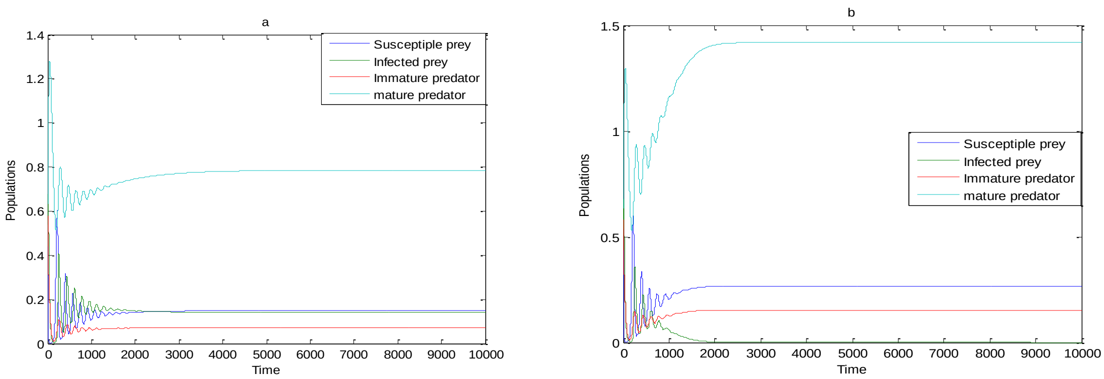

***Figure* 3** – (a) Time series of the solution of system (2) for the data given by (6.1) with $a_7 = 0.08$, which approaches to $E_4 = (0.150, 0.139, 0.074, 0.785)$. (b) Time series of the solution of system (2) for the data given by eq. (6.1) with $a_7 = 0.12$, which approaches to $E_3 = (0.267, 0, 0.153, 1.460)$.

7. Conclusions and discussion

In this paper, we proposed and analyzed an eco-epidemiological mathematical model consisting of a prey-predator model involving an SI infectious disease in a prey-stage structured predator species with a prey refuge . Further, in this model, we used Holling type II of functional responses for the predation of susceptible and infected preys which are outside refuge, as well as a linear incidence rate for describing the transition of disease. Our aim is to study the role of infectious diseases on the dynamics. Also, system (2) was solved numerically for different sets of initial points and different sets of parameters, starting with the hypothetical set of data given by eq. System (6.1). The following observations were obtained.

- System (2) has only one type of attractor in Int. R_+^4 approaches to a globally stable point.
- For the set of hypothetical parameters' values given in eq. (6.1), system (2) approaches asymptotically to a globally stable positive point $E_4 = (0.234, 0.049, 0.117, 1.178)$.

- As the infection rate of prey a_1 increases to 1.9, with keeping the rest of parameters as in eq. (6.1), the solution of system (2) approaches to the infected free equilibrium point E_3. However if $1.9 \le a_1 < 2.2$, then the infected prey will grow again and then the trajectory is transferred from the infected prey free equilibrium point to the positive equilibrium point E_4. Thus, the parameter $a_1 = 1.9$ is a bifurcation point.
- As the attack rate of the mature predator on the susceptible prey a_2 increases to 0.29, with keeping the rest of parameters as in eq. (6.1), the solution of system (2) approaches the infected free equilibrium point E_3. However, if $0.29 \le a_2 < 1$, then the infected prey will grow again and then the trajectory is transferred from the infected prey free equilibrium point to the positive equilibrium point E_4. Thus, the parameter $a_2 = 0.29$ is a bifurcation point.
- As the half saturation rate of the predator upon the susceptible prey increases to 0.4, with keeping the rest of parameters as in eq.(6.1), the solution of system (2) approaches to the disease free equilibrium point $E_3 = (\tilde{s}, 0, \tilde{z}, \tilde{w})$. In the rang $0.4 < a_3 \le 1$, the trajectory is transferred from the disease free equilibrium point E_3 to the positive equilibrium point E_4. Thus, a_3 =0.4 is a bifurcation point.
- As the attack rate of the mature predator on the Infected prey a_4 increases to 0.22, with keeping the rest of parameters as in eq. (6.1), the solution of system (2) approaches to the positive equilibrium point E_4. In the rang $0.22 \le a_4 < 1$, the trajectory is transferred from the positive equilibrium point E_4 to the disease free equilibrium point E_3. Thus, a_4 =0.22 is a bifurcation point **.**
- As the half saturation rate of the predator upon the Infected prey increases to 0.45, with keeping the rest of parameters as in eq.(6.1), the solution of system (2) approaches to the disease free equilibrium point $E_3 = (\tilde{s}, 0, \tilde{z}, \tilde{w})$. In the rang $0.45 \le a_5 < 0.6$ the trajectory is transferred from the disease free equilibrium point E_3 to the positive equilibrium point E_4. Thus, a_5 =0.45 is a bifurcation point.
- Moreover, increasing the parameter a_6, which represents the death rate of the infected prey due to the disease, to 0.3, and keeping the rest of parameter values as data given in eq. (6.1), the solution of system (2) approaches to the positive equilibrium point E_4. But, for $0.3 < a_6 < 1$, the trajectory is transferred from the positive equilibrium point E_4 to the infected prey free equilibrium point E_3 *and*, thus, the parameter a_6 =0.3 is bifurcation point.
- Now, increasing the conversion rate of food from susceptible prey to immature predator a_7 to 0.11, with keeping the rest of parameters as in eq. (6.1), the solution of system (2) approaches to the positive equilibrium point E_4. For $0.11 < a_7 < 0.15$, the trajectory is transferred from the infected prey free equilibrium point E_3 and, thus, the parameter a_7 =0.11 is a bifurcation point.
- The parameters $a_8, a_9, a_{10}, a_{11}, a_{12}$ and m, which represent the rate of food from infected prey to immature predator , the growth rate of immature predator onto mature predator, the natural death rate of immature predator, the conversion rate of food from susceptible prey onto mature predator, the conversion rate of food from infected prey onto mature predator,
and the number of prey inside the refuge parameter respectively, are not bifurcation points of system (2), with keeping the rest of parameters as in eq. (6.1).
- As the natural death rate of mature predators a_{13} increases to 0.097, with keeping the rest of parameters as in eq. (6.1), the solution of system (2) approaches to the infected prey free equilibrium point E_3. In the range $0.097 \le a_{13} < 0.15$, the trajectory is transferred from the infected prey free equilibrium point E_3 to the positive equilibrium point E_4 and, thus, $a_{13} = 0.097$ is a bifurcation point.

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