DOI: 10.24996/ijs.2020.61.4.22

ISSN: 0067-2904

Some Properties of the Essential Fuzzy and Closed Fuzzy Submodules

Hassan K. Marhon*[1], Hatam Y. Khalaf[2]
[1]Ministry of Education, General Directorate for the Education of Baghdad of the **first Rusafa** , E-mail:
[2]Department of Mathematics, College of Education for pure Science, Ibn-Al-Haitham , Baghdad University, Baghdad, Iraq

Received: 30/11/ 2019 Accepted: 31/12/2019

Abstract
In this paper, we introduce and study the essential and closed fuzzy submodules of a fuzzy module X as a generalization of the notions of essential and closed submodules. We prove many basic properties of both concepts.

Keywords: Fuzzy Module, Closed Fuzzy, essential fuzzy submodule.

بعض الخواص للمقاسات الضبابية الجزئية الجوهرية والمقاسات الجزئيه الضبابيه المغلقه

حسن خليف مرهون[1], حاتم يحيى خلف[2]
[1]المديرية العامة لتربية بغداد، الرصافة الاولى, وزارة التربية, العرق
[2]قسم الرياضيات, كلية التربية للعلوم الصرفة، ابن الهيثم, جامعة بغداد, العراق

الخلاصة
في هذا البحث قدمنا مفهوم المقاسات الضبابية الجزئية الجوهرية (المغلقة) في مقاس ضبابي كتعميم لمفهوم المقاسات الجزئية الجوهرية (المغلقة). ثم اعطينا العديد من التشخيصات و الخواص الاساسية لهذين المفهومان.

Introduction

The notion of a fuzzy subset of a nonempty set S as a function from S into [0,1] was first developed by Zadeh [1]. The concept of fuzzy modules was introduced by Zahedi [2], whereas that of fuzzy submodules was introduced by Martines [3].

A non-zero proper submodule A of a module M is called an essential if $A \cap B \neq (0)$, for any non-zero submodule B of M [4, 5].

Rabi [6] fuzzified this concept to the essential fuzzy submodule of a fuzzy module X.

Goodeal [4,7] introduced and studied the concept of closed submodules, where a submodule A of an R-module M is said to be a closed submodule of M ($A \leq_c M$), if it has no proper essential extension.

In this paper, we shall give some properties of the essential fuzzy submodules. Also, we introduce the concept of the closed fuzzy submodules. We establish many properties and characterizations of these concepts.

Throughout this paper, R is a commutative ring with unity, M is an R-module, and X is a fuzzy module of an R-module M.

1.Preliminaries

In this section, we shall provide the concepts of fuzzy sets and operations on fuzzy sets, with some of their important properties which are used in the next sections of the paper.

*Email: hassanmath316@gmail.com

Definition 1.1:[1]

Let M be non-empty set and let I be the closed interval [0,1] of the real line (real number). ***A fuzzy set X*** in M (a fuzzy subset X of M) is characterized by a membership function $X: M \to I,$ which associates with each point x ∈ M its degree of membership X(x) ∈ [0,1].

Definition 1.2: [2]

Let $x_t: M \to I$ be a fuzzy set in M ,where x ∈ M, t ∈ [0,1], defined by:

$$x_t = \begin{cases} 1 & if \quad x = y \\ 0 & if \quad x \neq y \end{cases} \quad \text{for all } y \in M$$

Then x_t is called a ***fuzzy singleton.***

If x = 0 and t =1, then :

$$0_1(y) = \begin{cases} 1 & if \quad y = 0 \\ 0 & if \quad y \neq 0 \end{cases}$$

We shall call such fuzzy singleton the ***fuzzy zero singleton.***

Definition 1.3:[2]

Let A and B be two fuzzy sets in M, then :

1. A = B if and only if A(x) =B(x) ,for all x ∈ M.
2. A⊆ B if and only if A(x)≤ $B(x)$, for all x ∈ M. If A⊆ B and there exists x ∈ M such that A(x) < $B(x)$, then we write A⊂ B and A is called a proper fuzzy subset of B.
3. $x_t \subseteq A$ if and only if $x_t(y) \leq A(y)$, for all y $\in M,$ and if t > 0 then $A(x) \geq$ t. Thus, $x_t \subseteq A(x \in A_t)$, (that is, $x_t \in A$ if and only if $x_t \subseteq A$).

Next, we give some operations on fuzzy sets:.

Definition 1.4:[2]

Let A and B be two fuzzy sets in M, then:

1.$(A \cup B)$(x) = max$\{A(x), B(x)\}$, for all x ∈ M.
2.$(A \cap B)(x) = min\{A(x), B(x)\}$, for all x ∈ M.

A∪ B and $A \cap B$ are fuzzy sets in M.

In general, if $\{A_\alpha, \alpha \in \Lambda\}$ is a family of fuzzy sets in M, then:

$$\left(\bigcap_{\alpha \in \Lambda} A_\alpha\right)(x) = inf\{A_\alpha(x), \alpha \in \Lambda\}, \text{for all } x \in S.$$

$$\left(\bigcup_{\alpha \in \Lambda} A_\alpha\right)(x) = \sup\{A_\alpha(x), \alpha \in \Lambda\}, \text{for all } x \in S.$$

Now, we give the definition of the level subset, which is a set between a fuzzy set and an ordinary set.

Definition 1.5:[3]

Let A be a fuzzy in M, for all t∈ [0,1], then the set $A_t = \{x \in M, A(x) \geq t\}$ is called ***level subset of X.***

Note that, A_t is a subset of M in the ordinary sense.

The following are some properties of the level subset.

Remark 1.6:[1]

Let A and B be two fuzzy subsets of a set M and let t ∈ [0,1], then:

1. $(A \cap B)_t = A_t \cap B_t$.
2. $(A \cup B)_t = A_t \cup B_t$.
3. A = B if and only if $A_t = B_t$.

Definition 1.7:[1]

Let A be a fuzzy set in M, then A is called ***empty fuzzy*** set, denoted by $\emptyset$, if and only if A(x) = 0, for all x ∈ M.

Definition 1.8: [2]

Let M be an R-module. A fuzzy set X of M is called fuzzy module of M if:

1. X(x−y) ≥ min $\{X(x), X(y)\}$, for all x, y ∈ M.
2. X(rx) ≥ X(x), for all x ∈M, r ∈ R.
3. X(0) = 1 (0 is the zero element of M).

Definition 1.9: [3]
Let X and A be two fuzzy modules of an R-module M. A is called a fuzzy submodule of X if A ⊆ X.
Proposition 1.10:[8]
Let A be a fuzzy set of an R-module M. Then the level subset A_t, t ∈[0,1] is a submodule of M if and only if A is a fuzzy submodule of X, where X is a fuzzy module of an R-module M.
Now, we shall give some properties of the fuzzy submodule, which are used in the next section.
Definition 1.11:[2]
Let A and B be two fuzzy subsets of an R-module M. Then (A + B)(x) = sup $\{\min\{A\,(a), B(b), x = a + b\}\}$ a, b ∈ M, for all x ∈ M. A+B, is a fuzzy subset of M.
Proposition 1.12:[2]
Let A and B be two fuzzy submodule of a fuzzy module X, then A+B is a fuzzy submodule of X.
Remark 1.13:[9]
If X is a fuzzy module of an R-module M and x_t ⊆X, then for all fuzzy singleton r_k of R , $r_k x_t$= $(rx)_\ell$ ℓ =min$\{k\,, t\}$.
Definition 1.14:[6]
Let X and Y be two fuzzy modules of an R-module M_1 and M_2, respectively . Define X ⊕Y :$M_1 \oplus M_2 \to [0,1]$ by:
(X⊕Y)(a, b) = min $\{X(a), Y(b)$, for all (a, b) ∈ $M_1 \oplus M_2\}$.
X⊕Y is called a fuzzy ***external direct sum*** of X and Y .
If A and B are fuzzy submodules of X, Y respectively, then
A⊕B :$M_1 \oplus M_2 \longrightarrow [0,1]$ is defined by :
(A⊕$B)(a, b) =$ min $\{A(a), B(b)$, for all (a, b) ∈ $M_1 \oplus M_2\}$
Note that, if X = A + B and A ∩ B=0, then X is called ***internal direct sum*** of A and B which is denoted by A⊕B. Moreover, A and B are called ***direct summand of X.***
Definition 1.15:[9]
Let A be a fuzzy module in M, then we define:
1. $A^* = \{x \in M: A(x) > 0\}$ is called support of A, also
   $A^* = \cup A_t$, t ∈ (0,1].
2. $A_* = A_{A(0_M)} = \{x \in M: A(x) = 1 = A(0_M)\}$.
Definition 1.16:[10]
A fuzzy module X of an R-module M is called simple if X = 0_1.
Definition 1.17:[6]
Let A be a fuzzy submodule of a fuzzy module X of an R-module M, then A is called an essential fuzzy submodule (briefly A$\leq_e X$), if A ∩ B ≠ 0_1, for any non-trivial fuzzy submodule B of X. Equivalently, a fuzzy submodule A of X is called essential if A ∩ U = 0_1, implies that U = 0_1, for all fuzzy submodule U of X.
2. Properties of Essential Fuzzy Submodules
In this section we shall give some properties of essential fuzzy submodule which were introduced in a previous study [6].
Remark 2.1:
If X is a fuzzy module of an R-module M and A is an essential fuzzy submodule of X, then A_t is not necessarily an essential of X_t. For example:
Example:
Let M =Z_6 be a Z-module. Define X: M ⟶ [0,1], A : M ⟶ [0,1] by:

$$X(a) = \begin{cases} 1 & if\ a = 0 \\ \frac{1}{2} & if\ a = 2,4 \\ 0 & otherwise \end{cases} \quad , \quad A(a) = \begin{cases} 1 & if\ a = 0 \\ \frac{1}{3} & if\ a = 2,4 \\ 0 & otherwise \end{cases}$$

It is clear that X is fuzzy module of Z_6. A is fuzzy submodule of X and A ≠ 0_1. Suppose that there exists a fuzzy submodule of X such that A ∩ B = 0_1. Hence, for all a ∈ Z_6, a ≠ 0 (A∩B)(a) = 0. But if a ∈ (2)- (0)

(A∩B)(a) = min $\{A(a), \text{then } B(a)\}$ = min $\left\{\frac{1}{3}, B(a)\right\}$. Hence, 0 = min $\left\{\frac{1}{3}, B(a)\right\}$, which implies that B(a) = 0 .
If a $\notin$ (2), then B(a) = 0, since B(a)$\leq X(a)$, for all a $\in Z_6$ and X(a) $\neq 0\ \forall\ a \in$ (2).
If a = 0, then B(0) = 1= X(0). Thus B = 0_1, then A is essential fuzzy submodule of X. On the other hand, $A_{\frac{1}{2}} = 0$, $X_{\frac{1}{2}}$ =(2) , hence $A_{\frac{1}{2}}$ is not essential in $X_{\frac{1}{2}}$.

Proposition 2.2

Let X be a fuzzy module of an R-module M and let A be a fuzzy submodule of X. If A_t is an essential submodule of $X_t\ \forall\ t \in [0,1]$, then A is an essential fuzzy submodule of X.

Proof:

Suppose that there exists a fuzzy submodule B of X such that A $\cap B = 0_1$. Hence, $(A \cap B)_t$=$(0_1)_t$, $\forall\ t \in$ [0,1]. It follows that $A_t \cap B_t = (0)$, $\forall\ t \in (0,1]$. But A_t is an essential submodule of X_t, hence $B_t = (0)$, $\forall t \in (0,1]$. Thus, B =0_1; that is A is an essential fuzzy submodule of X .

The following condition remark will be needed in some properties in our work.

Remark: Let X be a fuzzy module of an R-module M and A, B are non-trivial fuzzy submodules of X, if $A_* \subseteq B_*$, implies that A ⊆ B.

Proposition 2.3:

Let X be a fuzzy module of an R-module M such that X satisfies the previous remark and A be a fuzzy submodule of X, then A is an essential fuzzy submodule of X if and only if A_* is an essential submodule in X_*.

Proof:

(⇒) Let A_* be an essential submodule of A_*. To prove that A is essential in X.
Suppose that B is a fuzzy submodule of X, B$\neq 0_1$ and A $\cap B = 0_1$, then $(A \cap B)_t = (0)$, by remark(1.6)(3). So $(A \cap B)_* = (0)$, implies that $A_* \cap B_* = (0)$, so B_* =(0) , since A_* is an essential in X_*. Then $B_* = (0_1)_*$, then by the previous remark , $A_* \subseteq B_*$ implies that A$\subseteq B$. Therefore, $0_1 = B \cap A = B$. Thus, B = 0_1 and A is essential in X.

(⇐) If A is essential in X, then to show that A_* is essential in X_*, let N be a submodule of X_* and $A_* \cap N = (0)$. Let B(x) = $\begin{cases} 1 & if\ \ x \in N \\ 0 & otherwise \end{cases}$.
It is clear that B is a fuzzy submodule of X and $B_* = N$, then $A_* \cap N = A_* \cap B_* = (A \cap B)_*$. Thus, $(A \cap B)_* = (0) = (0_1)_*$. Then by the previous remark, A $\cap B = 0_1$. Since A is an essential fuzzy submodule of X, then $(A \cap B)_t = (0)$, so $(A \cap B)_* = (0)$ implies that $A_* \cap B_* = (0)$, by remark (1.6)(1) so B_* =(0), since A_* is an essential in X_*. Thus $B_* = (0_1)_*$ and so by the previous remark , B = 0_1 and so A is an essential in X.

Remark 2.4:

Every non-trivial fuzzy submodule of X is essential fuzzy itself.

Proof:

Let A be non-trivial fuzzy submodule of X and B be fuzzy submodule of A, where B $\neq 0_1$, then A ∩ B = B $\neq 0_1$. Therefore, A is an essential fuzzy submodule of A.

Proposition 2.5:

Let X be a fuzzy module of an R-module M and A_1,A_2,B_1,B_2 be fuzzy submodules of X, where A_1 is an essential fuzzy in $B_1$and A_2 is an essential in B_2, then $A_1 \cap A_2$ is essential in $B_1 \cap B_2$.

Proof:

Let C be any fuzzy submodule of X such that $0_1 \neq C \subseteq B_1 \cap B_2$. Since A_2 is essential in B_2, then $A_2 \cap C \neq 0_1$. Therefore, $A_2 \cap C$ is non-trivial fuzzy submodule of X such that $A_2 \cap C \subseteq B_1$. As A_1 is essential in B_1, hence $A_1 \cap (A_2 \cap C) \neq 0_1$, then $(A_1 \cap A_2) \cap C \neq 0_1$. Therefore, $A_1 \cap A_2$ is essential in $B_1 \cap B_2$.

Proposition 2.6:

Let X be a fuzzy module of an R-module M and A_1, A_2 be fuzzy submodules of X . If A_1 is essential in X and A_2 is essential in X, then $A_1 \cap A_2$ is essential in X.

Proof:

It is clear by remark(2.4) and proposition(2.5).

Proposition 2.7:

Let X be a fuzzy module of an R-module M and A ≤ B ≤ X. If A is an essential fuzzy in B and B is an essential fuzzy in X, then A is an essential fuzzy submodule in X.

Proof:

Let A be an essential in B, B be an essential in X, and C be any non-trivial fuzzy submodule of X. Since B is an essential fuzzy submodule in X, then we have C ∩ B $\neq 0_1$, and then since A is essential in B, we have (C ∩ B) ∩ A $\neq 0_1$; that is C ∩ A $\neq 0_1$. Thus, A is an essential fuzzy submodule in X.

Recall that a submodule A of an R-module C. A relative complement for A in C is any submodule B of C which is maximal with respect to the property A $\cap B = 0$, [4].

We fuzzify this concept as follows:

Definition 2.8:

Let X be a fuzzy module of an R-module M and A be a fuzzy submodule of X. A relative complement for A in X is any fuzzy submodule B of X which is maximal with respect to the property A $\cap B = 0_1$.

Proposition 2.9:

Let A be a fuzzy submodule of X of an R-module M, such that X satisfies the previous remark. If B is a relative complement for A, where B is any fuzzy submodule of X, then A$\oplus B$ is essential in X.

Proof:

Since B is a relative complement of A, we can prove that B_* is a relative complement of A_*. As B ∩ A $= 0_1$, then (B ∩ A)$_* = (0_1)_*$, so $B_* \cap A_*$ =(0). Suppose that N is a submodule of X_* and B_* is a submodule of N such that N ∩ A_* =(0). Let

C: M ⟶ [0,1], define by:

$$C(x) = \begin{cases} 1 & \text{if} \quad x \in N \\ 0 & \text{otherwise} \end{cases}.$$

It is clear that C is a fuzzy submodule of X and $C_* = N$. Thus $C_* \cap A_*$= (0) and so $(C \cap A)_* = (0_1)_*$. By the previous remark, C ∩ A = 0_1. But B_* is a submodule of C_*= N, implies that B is a fuzzy submodule of C (by the previous remark). Thus C = B, since B is a relative complement of A. It follows that $C_* = B_*$ (see remark (1.6)(3)); that is N = B_* and B_* is a relative complement of A_*. Hence, $A_* \oplus B_*$ is an essential submodule in X_* [4,proposition(1.3)]. So $(A\oplus B)_*$ is an essential submodule in X_*. Therefore, A ⊕ B is an essential fuzzy submodule of X by proposition (2.4).

3.Properties of Closed Fuzzy Submodules

In this section, we introduce the notion of the closed fuzzy submodule of a fuzzy module as a generalization of (ordinary) notion closed submodule, where a submodule A of an R-module M is said to be closed submodule of M (briefly A $\leq_c$ M), if A has no proper essential extension ; that is if A$\leq_e B \leq M$, then A = B [4], [7]. We shall give some properties of this concept.

Definition 3.1:

Let A be a fuzzy submodule of X of an R-module M, then A is called closed fuzzy submodule of X (shortly A$\leq_c X$), if A has no proper essential extension; that is A$\leq_e B \leq X$, then A = B.

Proposition 3.2:

Let X be a fuzzy module of an R-module M which satisfies the previous remark, and A is a fuzzy submodule of X , then A is closed fuzzy submodule of X if and only if A_* is closed submodule in X_*.

Proof:

(⇒) Suppose that $A_* \leq_e N \leq X_*$. We have to show that $A_* = N$. Let

$$B(x) = \begin{cases} 1 & if \quad x \in N \\ 0 & otherewise \end{cases}.$$

It is clear that B is a fuzzy submodule of X and $B_* = N$, $A_* \subseteq B_* = N$. So by the previous remark, A$\subseteq B$. But $A_* \leq_e N = B$, then by proposition(2.4), A$\leq_e B$. Therefore, A = B, since A is a closed fuzzy submodule in X, so $A_* = B_* = N$, then $A_* = N$.

(⇐) If A_* is a closed submodule in X_*. To show that A is a closed fuzzy submodule in X , assume that A$\leq_e B \leq X$. We must prove that A = B. Since A $\leq_e B$, then $A_* \leq_e B_*$ by proposition (2.4). But A_* is a closed submodule in X_*, so that $A_* = B_*$, then, by the previous remark, A = B .

Remarks and Examples 3.3:

1. For every fuzzy module X. 0_1 is closed fuzzy submodule of X and X is closed fuzzy submodule of X .

2. Every direct summand of a fuzzy module X is a closed fuzzy submodule of X.

Proof:

Let A be a direct summand of X , then there exists fuzzy submodule of X such that X = A$\oplus B$, then A + B = X and A$\cap B = 0_1$. Suppose that A is an essential fuzzy submodule of C , where C is a fuzzy submodule of X . To show that A = C,
we claim that B$\cap C = 0_1$. Since $(A \cap B) \cap C = 0_1$, then A$\cap (B \cap C) = 0_1$. If B$\cap C \neq 0_1$, then A$\cap (B \cap C) \neq 0_1$, since B$\cap C \leq C$ and A is essential in C, which is a contradiction. Then B$\cap C = 0_1$. Therefore, B + C = B $\oplus$ C.
Now, X = A$\oplus B$, since A $\leq C$. Then X = A$\oplus B \leq C + B = C\oplus B$, so A$\oplus B = C\oplus B$, since $(A\oplus B) = (C\oplus B)$.
$(A\oplus B)_t = (C\oplus B)_t$, then by a previous study [6,lemma(2.3.3)], $A_t\oplus B_t = C_t\oplus B_t$ $\forall$ $t \in$ [0,1], which implies that $A_t = C_t$. Thus, A = C by remark(1.6)(3) . Therefore, A is closed fuzzy submodule of X.
3. Let M =Z_6 be a Z-module. Let X :M $\longrightarrow$ [0,1] , defined by
X(a) = 1 ,$\forall$ $a \in Z_6$, and let A : M $\longrightarrow$ [0,1], B : M $\longrightarrow$ [0,1], define by:

A(x) =$\begin{cases} 1 & if \ x \in (\bar{2}) \\ 0 & othrewise \end{cases}$, B(x) = $\begin{cases} 1 & if \ x \in (\bar{3}) \\ 0 & otherewise \end{cases}$

It is clear that A and B are fuzzy submodules of X and A$\oplus B = X$, hence A and B are closed fuzzy submodule by remark and example(3.3)(2).
4. Every fuzzy submodule of semi-simple fuzzy module is closed fuzzy module, where a fuzzy module X of an R-module M is called semi-simple if X is a sum of simple fuzzy submodules of X [10, p.66].

Proof:

It is clear.
5. If A is closed fuzzy submodule of X of an R-module M and A$\subseteq B \subseteq X$, then A is a closed fuzzy in B.

Proof:

Assume that A is essential in D, where D is a fuzzy submodule of B .It is clear that D is a fuzzy submodule of X . Hence, A = D, since A is a closed fuzzy in X . Thus A is a closed fuzzy in B.

Theorem 3.4:

Let $\{A_\propto\}$ and $\{X_\propto\}$ be collections of a fuzzy modules of an R-module M, such that $A_\propto$ is a closed fuzzy submodule of $X_\propto$, for each $\propto$. Then $\oplus A_\propto$ is closed fuzzy in $\oplus X_\propto$. $\propto\in\wedge$ is any index set .

Proof:

Suppose that $\oplus A_\propto$ is essential in B , where B is a fuzzy submodule of $\oplus X_\propto$ for any $\propto$i $\in \wedge$, $X_{\propto i}$ is essential in $X_{\propto i}$. Hence by proposition (2.5) , $A_{\propto i}$ = $\oplus A_\propto \cap X_{\propto i}$ is an essential B$\cap X_{\propto i} \subseteq X_{\propto i}$. Hence, $A_{\propto i} = B \cap X_{\propto i}$, since $A_{\propto i}$ is closed fuzzy in $X_{\propto i}$ by the assumption. But $\oplus$ $A_\propto \subseteq B$, hence B$\cap A_{\propto i} = A_{\propto i}$ $\forall\propto$i $\in\wedge$. Since B$\cap (\oplus A_{\propto i} \cap X_{\propto i}) = B \cap X_{\propto i} = A_{\propto i}$. It follows that B$\subseteq \oplus_{\propto i} A_{\propto i}$, because $x_1 \in B \leq \oplus X_{\propto i}$, $(x_1)_{\propto i}$. The $\propto$i- component of x_1 is in $X_{\propto i}$. Thus $(x_1)_{\propto i} \in B \cap X_{\propto i}$ = $A_{\propto i}$ and this implies that $(x_1)_{\propto i} \in A_{\propto i}$ for any $\propto$i $\in \wedge$ .Thus B$\leq \oplus_{\propto i} A_{\propto i}$; that is B =$\oplus A_\propto$ and $A_\propto$ is closed fuzzy in $\oplus X_\propto$.

Theorem 3.5:

Let B be a fuzzy submodule of X of an R-module M which satisfies the previous remark, such that $(X/A)_* = X_*/A_*$. Then the following statements are equivalent .
1. B is a closed fuzzy submodule in X.
2. B is a relative complement, for some fuzzy A$\subseteq X$.
3. If A is any relative complement fuzzy of B in X , then B is relative complement fuzzy of A in X .
4. If B$\leq K \leq_e X$, then K/B $\leq_e$ X/B.

Proof:

(1)$\Rightarrow$(2) : If B is a closed fuzzy submodule of X. By proposition (3.2), B_* is closed submodule in X_*. Hence, B_* is a relative complement of some N, where N is a submodule of X_*. Let A : M $\longrightarrow$ [0,1], define by:

A(x) = $\begin{cases} 1 & \text{if } x \in N \\ 0 & \text{otherwise} \end{cases}$

It is clear that A is a fuzzy submodule of X and $A_* = N$, so B_* is a relative complement of A_*. Hence, $A_* \cap B_* = (0)$ and so B ∩ A = 0_1. Suppose that C is a fuzzy submodule of X and B is a fuzzy submodule of C such that C ∩ A = 0_1, so $C_* \cap A_*$
(0). But B_* is a submodule of C_* and B_* is a relative complement of A_*, so $B_* = C_*$. Thus B = C (by the previous remark) and B is a relative complement of A.
(2)⇒(1) : If B is a relative complement of A. To prove that B is a closed fuzzy submodule in X, suppose that $B \leq_e C \leq X$. Then $B \cap A \leq_e C \cap A$, by proposition (2.6). Hence, $0_1 \leq_e C \cap A$ and so C ∩ A = 0_1. But B is a fuzzy submodule of C and B is a relative complement of A, hence C = B. Thus, B is closed fuzzy submodule in X.
(2)⇒(3) : If B is a relative complement of A, then B_* is a relative complement of A_* by proposition (2.10). Then A_* is a relative complement of B_* [4 ,proposition(1.4)] and so A is a relative complement of B.
(3)⇒(2): It is clear.
(1)⇒(4): Let $B \leq K \leq_e X$. Then $B_* \leq K_*$ by proposition (1.10) and $K_* \leq_e X_*$ by proposition (2.4). Therefore, $B_* \leq K_* \leq_e X_*$. But B is a closed fuzzy submodule in X, implies that B_* is a closed submodule in X_* by proposition (3.2). Hence $K_* / B_* \leq_e X_* / B_*$ [4 ,proposition(1.4)], that is $(K/B)_* \leq_e (X/B)_*$ and $K/B \leq_e X / B$ by proposition (2.4) .
(4)⇒(3): Given that $B \leq X$, A is relative complement of B in X, and $B \leq K \leq_e X$, then $K/B \leq_e X/B$. We have to show that B is a relative complement of A. Since $A \cap B = 0_1$, then B can be enlarged to complement B' of A. By the modular law:

$$B_*' \cap (A_* \oplus B_*) = (B_*' \cap A_*) + B_*.$$
$$= (0) + B_* = B_* .$$

This implies that $\left(\frac{B_*'}{B_*}\right) \cap \frac{A_* \oplus B_*}{B_*} = \{B_*\}$. That is $\left(\frac{B'}{B}\right)_* \cap \left(\frac{A \oplus B}{B}\right)_* = \{B_*\}$. Also $A \oplus B \leq_e X$, so $B \leq A \oplus B \leq_e X$, therefore by the assumption $A \oplus B/B \leq_e X/B$. This implies that $(A \oplus B/B)_* \leq_e (X / B)_*$ by proposition(2.4). So that $\left(\frac{B'}{B}\right)_* \cap \left(\frac{A \oplus B}{B}\right)_* = \{B_*\}$, implies that $\left(\frac{B'}{B}\right)_* = \{B_*\}$.
Let $x \notin B_*$, then $\left(\frac{B'}{B}\right)(x + B_*) = 0$, implies that $\sup\{B'(x + y) | y \in B_*\} = 0$
Then $B'(x + 0) = B'(x) = 0$, hence $x \notin B_*$, implies that $B_*' \subseteq B_*$. Hence, $B_*' = B_*$. Then, by the previous remark, $B' = B$.

Proposition 3.6:
Let X be a fuzzy module of an R-module M and let A ≤ B ≤ X. If A is closed fuzzy in B and is B closed fuzzy in X, then A is closed fuzzy in X.
Proof:
Since A is closed fuzzy in B and B is closed fuzzy in X, therefore A_* is closed in B_* and B_* is closed in X_*, by proposition (3.2). This implies that A_* is closed in X_* [4 , proposition(1.5)]. Thus, A is closed fuzzy in X by proposition (3.2).

Proposition 3.7:
Let X be a fuzzy module of an R-module M and let A be a fuzzy submodule of X , then A is a direct summand of X if and only if A_t is a direct summand of X_t $\forall\, t \in [0,1]$.
Proof:
(⇒) If A is a direct summand of X , then $A \oplus B = X$ for some fuzzy submodule of X , hence $(A \oplus B)_t = X_t$, implies that $A_t \oplus B_t = X_t$ [6 ,lemma 2.3.3]. Then A_t is a direct summand of X_t.
(⇐) If $A_t \oplus N = X_t$ for some N submodule of X_t. Define

$$B(x) = \begin{cases} t & if \;\; x \in N \\ 0 & otherwise \end{cases}$$

It is clear that B is a fuzzy submodule of X and $B_t = N$, then B_t is a submodule of X_t by proposition (1.10). But A_t is a direct summand of X_t, then $A_t \oplus B_t = X_t$, hence $(A \oplus B)_t = X_t$; that is $A \oplus B = X$ by remark(1.6)(3).

Remark 3.8:
The intersection of two closed fuzzy submodules needs not be a closed fuzzy submodule in general, as the following example shows:
Example

Let M be the Z-module $Z \oplus Z_2$. Let X : M ⟶ [0,1], defined by :
X(a ,b) = 1 , for all (a, b) $\in Z \oplus Z_2$
Let A : M⟶ [0,1] , B : M⟶ [0,1] , defined by :

A(a ,b) = $\begin{cases} 1 \text{ if } (a,b) \in (1,\overline{0})Z \cong Z \\ 0 \quad \text{otherwise} \end{cases}$, B(a, b) = $\begin{cases} 1 \text{ if } (a,b) \in (1,\overline{1})Z \\ 0 \quad \text{otherwise} \end{cases}$.

It is clear that A and B are fuzzy submodules of X and that $A_* = (1,\overline{0})Z \cong Z$,$B_* = (1,\overline{1})Z$. Since A_* is a direct summand of $X_* = M$, then A is direct summand of X, by proposition (3.7). Also B_* is direct summand of X_*, then B is direct summand of X . Therefore, A and B are closed fuzzy submodules of X, by remark and example (3.3)(2). But
$(A\cap B)(a,b) = \min\{A(a), B(b)\}\ \forall\ (a\ ,b) \in Z \oplus Z_2$

$(A\cap B)(a,b) = \begin{cases} 1 \text{ if } (a,b) \in (2,\overline{0})Z \\ 0 \quad \text{otherwise} \end{cases}$

$(A \cap B)_* = (2,\overline{0})Z$, which is not a closed submodule in X_*, since $(2,\overline{0})Z \leq_e (1,\overline{0})Z \leq X$, but $(2,\overline{0}) \neq (1,\overline{0})Z$. Therefore, $A \cap B$ is not a closed fuzzy submodule of X, by proposition (3.2).

References

1. Zadeh, L.A. **1965**. Fuzzy Sets. *Information and Control*, **8**: 338-353.
2. Zahedi, M. M. **1992**. On L-Fuzzy Residual Quotient Module and P. Primary Submodule. *Fuzzy Sets and Systems*, **51**: 333-344.
3. Martinez, L. **1996**. Fuzzy Module Over Fuzzy Rings in Connection with Fuzzy Ideals of Rings. *J. Fuzzy Math*., **4**: 843-857.
4. Goodearl, K. R. **1976**. *Ring Theory*, *Nonsingular Rings and Modules*. Marcel Dekker, New York.
5. Inaam, M. A. H, Farhan, D. Sh. and Shuker, N. Al. **2019**. Essentially Second Modules. *Iraqi Journal of Science*, **60**(6): 1374- 1380.
6. Rabi, H.J. **2001**. Prime Fuzzy Submodules and Prime Fuzzy Modules. M. Sc. Thesis, Univ. of Baghdad.
7. Inaam, M. A. and Tha'ar, Y. G. **2016**. On Closed Rickart Modules. *Iraqi Journal of Science*, **57**(4B): 2764- 2753.
8. Mukhejee T.K., Sen M. K. and Roy D. **1996**. On Submodule and their Radicals. *J. Fuzzy Math*., **4**: 549-558.
9. Mashinchi, M. and Zahedi, M. M. **1992**. On L-Fuzzy Primary Submodule. *Fuzzy Sets and Systems*, **49**: 231-236.
10. Maysoun, A.H. **2002**. F-regular Fuzzy Modules. M. Sc. Thesis, Univ. of Baghdad.