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Supplemented and π-Projective Semimodules

Muna M. T. Altaee*, Asaad M. A. Alhossaini
College of Education for Pure Science, University of Babylon, Babylon, Iraq

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Abstract

In modules there is a relation between supplemented and π-projective semimodules. This relation was introduced, explained and investigated by many authors. This research will firstly introduce a concept of "supplement subsemimodule" analogues to the case in modules: a subsemimodule Y of a semimodule W is said to be supplement of a subsemimodule X if it is minimal with the property $X+Y=W$. A subsemimodule Y is called a supplement subsemimodule if it is a supplement of some subsemimodule of W. Then, the concept of supplemented semimodule will be defined as follows: an S-semimodule W is said to be supplemented if every subsemimodule of W is a supplement. We also review other types of supplemented semimodules. Previously, the concept of π-projective semimodule was introduced. The main goal of the present study is to explain the relation between the two concepts, supplemented semimodule and π-projective semimodules, and prove these relations by many results.

Keywords: Semimodule, supplemented semimodule, π-projective semimodule, lies above a direct summand, coclosed semimodule.

الاستكمال وشبه المقاسات الاسقاطية من النمط π

منى توفيق احمد الطائي *، اسعد محمد علي الحسيني
قسم الرياضيات, كلية التربية للعلوم الصرفة, جامعة بابل, بابل, العراق

الخلاصة

في المقاسات هناك علاقة بين المقاسات الاسقاطية من النمط π والاستكمال وقد وضحت واستعرضت هذه العلاقة وعممت من قبل العديد من الباحثين. هذا البحث اولا سوف يستعرض مفهوم استكمال شبه المقاس الجزئي كما في المقاس: يكون شبه المقاس الجزئي Y (من شبه المقاس W) استكمال لشبه المقاس الجزئي X اذا كان هو اصغري في W مع خاصية $X+Y=W$. ويكون Y شبه مقاس استكمالي اذا كان استكمال لبعض شبه المقاسات الجزئية في شبه المقاس W. ثم يستعرض مفهوم الاستكمال لشبه المقاس, حيث ان شبه المقاس يكون مستكمل اذا كان كل شبه مقاس جزئي منه هو استكمالي. ان الهدف الرئيسي لهذا البحث هو توضيح العلاقة بين مفهومي الاستكمال و شبه المقاس الاسقاطي من النمط π وذلك سيكون من خلال عدة نتائج تثبت هذه العلاقة.

1. Introduction.

By this paper, S will denotes a commutative semiring with identity $1\neq0$. W will be a semimodule over S. Previously, the concept of projective modules was introduced **[1]** and further studied **[2]**.

Also the two concepts, supplemented and π-projective modules, were introduced **[1]** and studied **[3]**.

*Email: anc012.t3@gmail.com

Then, the concept of π-projective semimodule was introduced and investigated for semimodules **[4]**. The concept of supplemented semimodule will be discussed in this paper, the relation between a supplemented and π-projective semimodule will be explained, and investigated equipped to that in modules. The organization of the research will be as follows:

- Section 2 contains the primitives related to this work.
- Section 3 will give the means of the concept of supplement subsemimodule which is: Let X and Y be subsemimodules of a semimodule W. Y is said to be supplement of X if it is minimal with the property $X+Y=W$. A subsemimodule Y is called a supplement subseminodule if it is a supplement of some subsemimodule of W. An S-semimodule W is said to be supplemented if every subsemimodule of W is a supplement. Other concepts analogues to that in the modules are also introduced as:
- In section 4, the relation between supplemented and π-projective semimodule will be studied and an access will be provided to some results related to that relation. Also, in this section a concept of coclosed semimodule will be introduced as follows: Let K be a subsemimodule of a semimodule W**,** then we say that K is coclosed in W if, for all subsemimodules H of W, $K/H \ll W/H$, implies $K=H$.

2. Preliminaries

This section will consist of definitions and other primitives related to the research.

Definition 2.1.[5] Let S be a semiring. A left S-semimodule W is a commutative monoid $(W, +, 0)$ for which we have a function $S\times W\rightarrow W$ defined by $(s, w)\mapsto s\ w$ ($s \in S$ and $w \in W$) such that for all $s, s' \in S$ and $w, w' \in W$, then the following conditions are satisfied:

a) $s(w+w')=s\ w+ sw'$
b) $(s+ s')w=s\ w+ s'\ w$
c) $s'\ (sw)= (s's)w$
d) $0w=0$

In this work, an S-semimodule will be a left unitary S-semimodule ($1w=w$ for all w in W).

Definition 2.2.[5] Let D be a nonempty subset of a left S-semimodule W, then D is said to be a subsemimodule of W if the following conditions hold:

1) $(d_1+d_2) \in D$, for all d_1 and $d_2 \in D$.
2) $sd \in D$ for all $s\in S$ and for all $d\in D$.

Definition 2.3.[6, p.154] Let W be a semimodule and D be a subsemimodule of W, then D is said to be subtractive if for all $d\in D$ and $(d+ c) \in D$ implies that $c\in D$.

Notes:

1- $\{0\}$ and W are subtractive subsemimodules of a semimodule W.
2- If every subsemimodule of any semimodule is subtractive, then the semimodule is called subtractive semimodule.

Definition 2.4. [6, p.149] A semimodule W is said to be semisubtractive, if for any $w, w'\in W$ there is always some $k\in W$ satisfying $w+ k =w'$ or $w'+ k=w$.

Definition 2.5. [5] An element w of a left S-semimodule W is cancellable if $w+ x=w+ h$ implies that $x=h$.

Definition 2.6. [5] An S-semimodule W is cancellative if every element of W is cancellable.

Definition 2.7. [7] An S-semimodule W is said to be a direct sum of subsemimodules $W_1, W_2, \ldots, W_k$ of W, if each $w\in W$ can be written uniquely as $w=w_1+w_2+\ldots+w_k$ where $w_i \in W_i$, $1\leq i \leq k$. It is denoted by $W=W_1\oplus W_2\oplus\ldots\oplus W_k$. Each W_i is called a direct summand of W.

Remark 2.8.[8] Let W be a subtractive cancellative semisubtractive S-semimodule, then $\boldsymbol{W} = W_1 \oplus W_2$ if and only if $W = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$.

Definition 2.9.[7] If S is a semiring and W, N are left S-semimodules, then a map $q:W\rightarrow N$ is called a homomorphism of S-semimodules, if :

(i) $q(w + w') = q(w) + q(w')$
(ii) $q(s\ w) = sq(w)$, for all w , $w'\in W$ and $s\in S$.

The set of S-homomorphisms of W into N is denoted by Hom (W, N). A homomorphism q is called an epimorphism if its onto, it is called a monomorphism if q is one-one, and it is isomorphism if q is one-one and onto.

Remarks 2.10 [9].

For a homomorphism of S-semimodules q: $W\rightarrow N$ we define

(i) $ker\ (\ q\ ) = \{\ w\in W \mid\ q(w) = 0\ \}$

(ii) $q(W) = \{q(w) | w \in W\}$.
(iii) $Im(q) = \{n \in N \mid n + q(w) = q(w')$ for some $w, w' \in W\}$

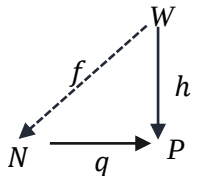
As previously described **[9],** $ker(q)$ is a subtractive subsemimodule of W, $Im(q)$ is a subtractive subsemimodule of W and $q(W)$ is a subsemimodule of W. In the module theory, $q(W) = Im(q)$, whereas in the semimodule theory this is not true in general. It is clear that $q(W) \subseteq Im(q)$, where the equality is satisfied if $q(W)$ is subtractive subsemimodule of K.

According to the same study **[9]**, End(W) means the set of all S endomorphisms of W. Using standard arguments, it can be shown that for each S-semimodule W, End(W) is a semiring.

Definition 2.11.[9] A subsemimodule K of W is called small (superfluous) if for any subsemimodule K' of W, $K+K' = W$ implies $K' = W$ and denoted by ($K \ll W$).

Definition 2.12.[9] A semimodule W is said to be hollow if every proper subsemimodule H of W is small.

Definition 2.13.[10, p.7] A left S-semimodule W is said to be projective if for every epimorphism $q: N \to P$ and for every homomorphism $h: W \to P$ there is a homomorphism $f: W \to N$ such that the diagram ($qf=h$) commutes.



Definition 2.14. [10, p.5] The sequence $L \xrightarrow{\lambda} N \xrightarrow{\phi} B$ is called an exact sequence if $\ker\phi = \mathrm{Im}\lambda$.

Definition 2.15.[10, p.27] A proper short exact sequence $0 \to L \xrightarrow{\lambda} N \xrightarrow{\phi} B \to 0$ is called split or split exact split if there is a homomorphism $\chi: B \to N$ such that $\phi\chi = I_B$.

Note: A monomorphism ϕ is split if it has a left invers and an epimorphism λ is split if it has a right inverse.

Definition 2.16.[11] If H is a subsemimodule of a semimodule W, then W/H is called quotient (factor) semimodule of W by H, defined by $W/H = \{[w] \mid w \in W\}$.

Definition 2.17.[4] A semimodule W is indecomposable if the direct summands of it are only {0} and itself .

Definition 2.18.[4] An S-semimodule W is said to be π-projective if for every two subsemimodules A and B of W, with $A+B=W$, there exist f and g $\in$End(W) such that $f + g = 1_W$, $f(W) \subseteq A$, and g(W) $\subseteq B$.

Definition 2.19 Amply supplemented: A semimodule W is called amply supplemented if for any two subsemimodules H, D of W such that $H+D=W$, there is a supplement H' of H such that $H' \subseteq D$.

Definition 2.20 Weakly supplemented: Let W be an S-semimodule and let X be a subsemimodule of W if there is a subsemimodule Y of W such that $X+Y=W$, $X \cap Y \ll W$ ($\ll$ denotes a small sub semimodule), then we can say that Y is a weak supplement in W. The semimodule W is weakly supplemented if every subsemimodule of W has a weak supplement.

Definition 2.21 Mutual supplements: Let A, B be subsemimodules of a semimodule W, then A, B are said to be mutual supplements if they are supplements of each other.

Definition 2.21 Lies above a direct summand: a subsemimodule A of a semimodule W lies above a direct summand if there exists a decomposition $W = K \oplus K'$, with $K \subseteq A$ and $K \cap A \ll K'$.

In this work, S- semimodule will be cancellative, subtractive and semisubtractive.

3. Supplemented semimodules:

The following concept was introduce **[1]** and then studied **[12]**.We will introduce this concept for semimodules and give some results related to this concept.

Definition 3.1. Let A and B be subsemimodules of a semimodule W, B is said to be supplement of A if it is minimal with the property $A+B=W$. A subsemimodule B is called a supplement subseminodule if it is a supplement of some subsemimodule of W **[12, p.25].**

Definition 3.2. An S-semimodule W is said to be supplemented if every subsemimodule of W is a supplement **[12, p.26].**

Example 3.3. (i) $3\mathbb{Z}_{12}$ is a supplement of $4\mathbb{Z}_{12}$ in $\mathbb{Z}_{12}$ as $\mathbb{N}$-semimodule.

(ii) $\{\bar{0}, \bar{2}\}$ is not supplement of $\mathbb{Z}_4$ in $\mathbb{Z}_4$ as $\mathbb{N}$-semimodule.

(iii) $2\mathbb{Z}_8$ is not supplement of $4\mathbb{Z}_8$ in $\mathbb{Z}_8$ as $\mathbb{N}$-semimodule.

Definition 3.4. Let A, B be subsemimodules of a semimodule W, then A, B are said to be mutual supplements if they are supplements of each other **[12, p.26].**

Example 3.5 (i) $5\mathbb{Z}_{10}$, $2\mathbb{Z}_{10}$ are mutual supplements in $\mathbb{Z}_{10}$ as $\mathbb{N}$-semimodule.

(ii) As an $\mathbb{N}$-semimodule, $\mathbb{Z}_8$ is a supplement of $4\mathbb{Z}_8$ in $\mathbb{Z}_8$, while $4\mathbb{Z}_8$ is not a supplement of $\mathbb{Z}_8$, so $\mathbb{Z}_8$ and $4\mathbb{Z}_8$ are not mutual supplements in $\mathbb{Z}_8$.

Remark 3.6. Let A and B be subsemimodules of a semimodule W, then X is said to be supplement of B if and only if $W=X+Y$ with $X\cap Y$ is small in X **[1, p.348].**

Proof :($\Longrightarrow$) If X is a supplement of Y and $C\subset X$ with $(X\cap Y)+C=X$, then we have $W=X+Y=(X\cap Y)+C+Y=C+Y$ $(X\cap Y\subset Y)$ then $X=C$ by the minimality of X, $X\cap Y\ll X$.

($\Longleftarrow$) Let $W=Y+X$ and $Y\cap X\ll X$. Let $D\subset X$ with $Y+D=W$, we have $X=W\cap X=(Y+D)\cap X=(Y\cap X)+D$, since $(Y\cap X)\ll X\rightarrow X=D$ is minimal at the desired rate. ▩

Example 3.7.(i) $\mathbb{Z}_6$, $\mathbb{Z}_{10}$ and $\mathbb{Z}_{15}$ as $\mathbb{N}$-semimodules are supplemented semimodules.

(ii) $\mathbb{Z}_{12}$, as $\mathbb{N}$-semimodule is not supplemented semimodule, because $6\mathbb{Z}_{12}$ is not a supplement of any subsemimodule of $\mathbb{Z}_{12}$. ($\mathbb{Z}_{12}=6\mathbb{Z}_{12}+\mathbb{Z}_{12}$, $6\mathbb{Z}_{12}\cap\mathbb{Z}_{12}$ is not small in $6\mathbb{Z}_{12}$).

Definition 3.8. A semimodule W is called amply supplemented if for any two subsemimodules H, D of W such that $H+D=W$, there is a supplement H' of H such that $H'\subseteq D$ **[1, p.359].**

Example 3.9. (i) As $\mathbb{N}$-semimodules, $\mathbb{Z}_9$, $\mathbb{Z}_{25}$ and $\mathbb{Z}_{49}$ are amply supplemented, since they are hollow.

(ii) As a $\mathbb{Z}$-semimodule, $\mathbb{Z}$ is not amply supplemented.

Definition 3.10. Let W be an S-semimodule and let X be a subsemimodule of W if there exists a subsemimodule Y of W such that $X+Y=W$, $X\cap Y\ll W$, then we say that Y is a weak supplement in W. W is weakly supplemented if every subsemimodule of W has a weak supplement **[12, p.27].**

In the coming results, we need to define the concept "lies above a direct summand" for semimodules, where this concept was given previously for modules **[1, p.357]**.

Definition 3.11. A subsemimodule A of a semimodule W lies above a direct summand if there exists a decomposition $W=K\oplus K'$, with $K\subseteq A$ and $K'\cap A\ll K'$.

Example 3.12. As an $\mathbb{N}$-semimodule, $3\mathbb{Z}_{12}\oplus 4\mathbb{Z}_{12}=\mathbb{Z}_{12}$, $4\mathbb{Z}_{12}\subseteq 2\mathbb{Z}_{12}$ and $3\mathbb{Z}_{12}\cap 2\mathbb{Z}_{12}\ll 3\mathbb{Z}_{12}$, hence $2\mathbb{Z}_{12}$ lies above a direct summand.

The following lemma is needed for the next results.

Lemma 3.13. Let C, D and K be subsemimodules of a U- semimodule W then:

1) If $C\ll D$ and $D\subseteq K$, then $C\ll K$.
2) If $C\ll W$ and $q:W\rightarrow Y$ is a homomorphism, then $q(C)\ll q(W)$.
3) If $C\ll W$, $C\subseteq D$, and D is a direct summand of W, then $C\ll D$.
4) If $C\ll D$ and $K\subseteq C$, then $K\ll D$.
5) If $C\ll W$ and $D\ll W$, then $C+D\ll W$.

Proof: Similar to the proof in the case of modules **[13, 5.1.3].** ▩

Lemma 3.14. If W is a supplemented semimodule and $W\cong W'$, then W' is supplemented.

Proof: Let W and W' be semimodules such that $W\cong W'$, then $W\xrightarrow{\beta}W'$, $\beta(X)=X'$, $X\leqslant W$ and $X'\leqslant W'$. Since W is supplemented, then there exists a subsemimodule Y such that $X+Y=W$ and $X\cap Y\ll Y$ then $\beta(X)+\beta(Y)=W'$ implies $X'+\beta(Y)=W'$ and $\beta(Y)\cap X'\ll\beta(Y)$, by Lemma (**3.13**). Thus W' is supplemented.

In **[1, p.362], [1, 41.16], [1, 41.11], [1, p.355], [1, 41.11],** and **[1, 41.12]** respectively, the next results were proved for modules. We will prove them for semimodules.

Lemma 3.15. Every factor semimodule of a supplemented semimodule is also supplemented.

Proof: Let W be a supplemented semimodule and let H be a proper subsemimodule of W, to show that $\frac{W}{H}$ is supplemented semimodule. Let $\frac{K}{H}$ be a subsemimodule of $\frac{W}{H}$, then K is a subsemimodule of W containing H and since W is supplemented, then there exists a subsemimodule L of W such that $K+L=W$ and $K\cap L\ll L$. Then $\frac{K}{H}+\frac{L+H}{H}=\frac{W}{H}$ and $\frac{K}{H}\cap\frac{L+H}{H}=\frac{K\cap(L+H)}{H}=\frac{(K\cap L)+H}{H}$ which is small in $\frac{L+H}{H}$ since it is the image of K∩L under the natural epimorphism from W onto $\frac{W}{H}$ (Lemma **(3.13)**). Hence $\frac{W}{H}$ is supplemented semimodule.

Corollary 3.16. A direct summand of a supplemented semimodule is supplemented.

Proof: Let A be a direct summand of a supplemented semimodule W, then there exists B subsemimodule of W such that $A \oplus B = W$, then $A \cong W/B$ (since when $W=A+B$, then $A \cong A/W$and $B \cong W/A$). By Lemma**(3.15)** W/B is supplemented semimodule and by Lemma**(3.14)** A is supplemented.

Lemma 3.17. Let A be a sub semimodule of a semimodule W, then the following are equivalent:

(1) A lies above a direct summand of W.

(2) A has a supplement H in W such that $A \cap H$ is a direct summand in A.

Proof :($1 \Longrightarrow 2$) Assume that A lies above a direct summand of W, then $W=K \oplus H$ with $K \subseteq A$ and $A \cap H \ll H$. $A=W \cap A=(K \oplus H) \cap A=K \oplus (H \cap A) \rightarrow H \cap A$ is a direct summand of A. We claim that H is a supplement of A, and to verify this: $A+H=K+(H \cap A)+H=W$. Since $A \cap H \ll H$, hence H is supplement of A.

($2 \Longrightarrow 1$) Let H be a supplement of A in W such that $A \cap H$ is a direct summand in A, then $W=A+H$ with $A \cap H \ll H$. There exists a subsemimodule B of A such that $A=B \oplus (A \cap H)$. $W=A+H=B \oplus (A \cap H)+H=B+H$ and $B \cap H=B \cap (A \cap H)=0$, hence $W=B \oplus H$, then A lies above a direct summand in W. ▓

Lemma 3.18. If every subsemimodule A of a semimodule W is of the form $A=K+H$ with K supplemented and $H \ll W$, then W is amply supplemented.

Proof: Let $W=B+A$, then by assumption $A=K+H$, K supplemented and $H \ll W$. Now, $W=B+A=B+K+H$ since $H \ll W \rightarrow W=B+K$. Since $B \cap K \subseteq K$, then $K=B \cap K+D$ with $(B \cap K) \cap D=B \cap D \ll D$ and $W=B+(B \cap K)+D=B+D$. It is clear that $D \subseteq A$. ▓

Proposition 3.19. Let L be a subsemimodule of an S-semimodule W, then the followings are equivalent:

1) L lies above a direct summand of W.

2) There are two idempotent $f, g \in End(W)$, with $f+g=1_W$ such that $f(W) \subseteq L$ and $g(L) \ll g(W)$.

(3) There is a direct summand U of W with $U \subseteq L$, $L=U+K$ and $K \ll W$.

Proof: ($1 \Longrightarrow 2$**)** Assume that $W=E_1 \oplus E_2$ such that $E_1 \subseteq L$ and $L \cap E_2 \ll E_2$. Let $f=\pi_1$ and $g=\pi_2$, $\pi_i: E_1 \oplus E_2 \rightarrow E_i$ be the natural projections. It is clear that both f and g are idempotent endomorphisms of W, $f(W)=E_1 \subseteq L$. On the other hand $f+g=1_W$, so $L=f(L)+g(L)$, hence $L \cap E_2=[f(L)+g(L)] \cap E_2=f(L) \cap E_2+g(L) \cap E_2=g(L)$, since $g(L) \subseteq E_2$ and $f(L) \cap E_2=0$, therefore $g(L)=L \cap E_2 \ll E_2=g(W)$.

($2 \Longrightarrow 3$) By (2) there exists $f, g \in End(W)$ idempotent such that $f(W) \subseteq L$, $g(L) \ll g(W)$ and $f+g=1_W$, then we have $y \in f(W) \cap g(w)$ implies that $y=f(w)=g(w')$, but $w=g(w)+g(w') \rightarrow g(w)=g^2(w')+g^2(w) \rightarrow g(w)=g(w')+g(w) \rightarrow g(w')=0 \rightarrow y=0$, that is, $f(W) \cap g(W)=0$. $f+g=1_W$, $W=f(W) \oplus g(W)$. Put $U=f(W)$, then $L=L \cap W=L \cap [U \oplus g(W)]=U+L \cap g(W)$ (since $f(W) \subseteq L$), so, $L \cap g(W) \subseteq g(L) \ll g(W)$, hence $K=L \cap g(W) \ll W$.

($3 \Longrightarrow 1$**)** By (3), $W=U \oplus U'$ with $U \subseteq L$, $L=U+K$ and $K \ll W$. We claim that U' is a supplement of L in W. Note that $W=U+U' \subseteq L+U'$, thus $W=L+U'$. If there exists $U'' \subseteq U'$, $W=L+U''$, then $W=U+K+U''=U+U''$(since $K \ll W$) which implies that $U''=U'$. This proves the claim.

Proposition 3.20. Let W be an S-semimodule, then the followings are equivalent:

(1) W is amply supplemented and every supplement subsemimodule is a direct summand.

(2) Every subsemimodule of W lies above a direct summand of W.

Proof:($1 \Longrightarrow 2$) Let W be an amply supplemented and every supplement subsemimodule is a direct summand and let A be a subsemimodule of W. Let K be a supplement of A, then $W=A+K$ and $A \cap K \ll K$. Since W is amply supplemented thus A contains a supplement of K, say H, i.e. $H+K=W$ and $H \cap K \ll H$, since H is a direct summand of W, hence $W=H \oplus D$ where D is a suitable subsemimodule of W, since $H \cap D=0$, then D is a supplement of H, by Proposition **(3.19)(**$3 \rightarrow 1$**)** D is a supplement of $H+A \cap K$, but $H+A \cap K=(H+K) \cap A=W \cap A=A$. This implies that $A \cap D \ll D$ which means that A lies above a direct summand.

($2 \Longrightarrow 1$) Assume that every subsemimodule of W lies above a direct summand of W. Let H be a subsemimodule of W, then $W=A \oplus B$, with $A \subseteq H$ and $H \cap B \ll B$. But $W=H+B$, implies B is a supplement of H in W, and then W is supplemented. Let D be any subsemimodule of W. Then by Proposition **(3.19)** part **(1→3)**, there is a direct summand K of W with $K \subseteq D$, $D=K+E$ and $E \ll W$. Since W is supplemented, then by corollary (**3.16**) K is also supplemented and by Lemma **(3.18)** it is amply supplemented. Now, suppose that C is a supplement of a subsemimodule D of W, since D

lies above a direct summand of W by Lemma **(3.17),** $D\cap C$ is a direct summand of D and again by Lemma **(3.17)(**2→1),$W=K\oplus C$ for some $K \subseteq W$. Hence C is a direct summand of W. ▓

4. Supplemented and π-projective semimodules

The class of supplemented π-projective modules was studied by many references. In this section, this class will be converted to semimodules.

The next proposition is a generalization for the proposition (41.15) which was previously introduced [1].

Proposition 4.1. Let A and B be mutual supplements in a π-projective semimodule W, then $W=A\oplus B$.

Proof: Assume that A and B are mutual supplements in W, then $W=A+B$ such that $A\cap B\ll A$ and $A\cap B \ll$B. It is enough to prove that $A\cap B=0$. Consider the map $h:A\coprod B\rightarrow W$ defined by $h(a, b)=a+b$, it is a split epimorphism [12], so ker $h=\{(a, b)\in A\coprod B: h(a, b)=0\} =\{(a, b)\in A\coprod B: a+b = 0\}$ is a direct summand of $A\coprod B$, hence there is a submodule K such that $A\coprod B=$ ker $h\oplus K$. On the other hand, $(a, b) \in$ ker h implies $a, b \in A\cap B$, then ker $h\subset (A\cap B)\coprod (A\cap B) \ll A\coprod B$, hence ker $h\ll A\coprod B$ and so ker $h=\{(0, 0)\}$, that is, h is an injective map. Now, $a \in A\cap B$ implies $h(a, 0)=h(0, a)= a$, so $(a, 0)= (0, a)$ which implies that $a=0$. Therefore, $A\cap B=0$. ▓

The next proposition is a generalization for the proposition (2.3.1) in which was described earlier [12].

Proposition4.2. For a semimodule W, the following assertions are equivalent:

1. W is supplemented and π-projective.
2. (a) W amply supplemented; and
   (b) The intersection of mutual supplements is zero.
3. (a) Every subsemimodule of W lies above a direct summand ; and
   (b) If C and D are direct summands of W with $W=C+D$, then $C\cap D$ is also direct summand in W.
4. For every subsemimodules C and D of W, with $W=C+D$, there are idempotents h and $q\in$ End(W) such that $h+ q=1_W$, $h(W)\subseteq C$,$q(W)\subseteq D$, and $q(C)\ll q(W)$.

Proof: (1⟹ 2) (a) Let $W=C+D$ and A be a supplement of C in W. Since W is π-projective , so there exist homomorphisms h and q such that $h(W)\subseteq D$, $q(W)\subseteq$C and $h+ q=1_W$. We claim that $h(C)\subseteq C$, $W=C+h(A)$ and $h(C\cap A)=C\cap h(A)$. Now to verify this claim, let $c\in C$, then $q(c)\in$C and $c=h(c)+ q(c)\in C$ ($h+ q=1_W$), since W is subtractive, then $h(c)\in C$, hence $h(C)\subseteq C$. Since A is supplement of C in W , then $W=C+A$ implies $h(W)=h(C)+h(A)\subseteq C+h(A)$ and since W is π-projective, then $W=h(W)+q(W)\subseteq C+C+ h(A)=C+ h(A)$, thus W=$C+ h(A)$. Now to prove that $h(C\cap A)=C\cap h(A)$, let $w\in C\cap h(A)$, implies that $w\in C$ and $w\in h(A)$, that is $w=h(a)$, for some a in A, but $a= h(a)+ q(a)$ and $h(a) \in$C, $q(a) \in$C, so $a \in$C, then $w\in h(C\cap A)$, then $(C\cap h(A))\subseteq h(C\cap A)$. Clearly, $h(C\cap A)\subseteq h(C)\cap h(A))\subseteq (C\cap h(A))$. Since A is supplement of C in W, then $C\cap A\ll A$ and by Lemma**(3.13)** $h(C\cap A)\ll h(A)$, since $h(C\cap A)= C\cap h(A)$, then $C\cap h(A) \ll h(A)$, that is, $h(A)$ is a supplement of C in W and $h(A)\subseteq D$. Hence W is amply supplemented.

(b) Directly from Proposition **(4.1).**

(2⟹3) (a) It is enough by Proposition **(3.17)** to prove that every supplement subsemimodule of W is a direct summand (since W is amply supplemented). Let D be a supplement subsemimodule of W, then D is a supplement of some subsemimodule C of W, implies that $W=C+D$ and $C\cap D\ll D$. But W is amply supplemented, thus C contains a supplement C' of D in W, then $W= C'+D$ and $C'\cap D\ll C'$. Since $C'\cap D\subseteq C\cap D\ll D$, hence $C'\cap D\ll D$, therefore C' and D are mutual supplements, then by 2(b), $C'\cap D=0$, thus $W=C'\oplus D$.

(b) Suppose that C and D are direct summands of W with $W=C+D$. W is amply supplemented, then there exists a supplement D' of C in W such that $D'\subseteq D$ and C' a supplement of D' in W such that $C'\subseteq C$. Therefore, $C\cap D'\ll D'$ and $C'\cap D'\ll C'$. Now, $C'\cap D'\subseteq C\cap D'$ and since $C\cap D'\ll D'$, then $C'\cap D'\ll D'$, that is, C' and D' are mutual supplements, and by 2(b) $C'\cap D'=0$, hence $W=C'\oplus D'$. $W\cap C=C=C'\oplus(C\cap D')$ and C is a direct summand in W, then $W=C\oplus H=C'\oplus C\cap D'\oplus H$, where H is a suitable subsemimodule of W, implies that $C\cap D'$ is a direct summand of W, since $C\cap D'\ll D'$, then $C\cap D'\ll W$, then $C\cap D'=0$. Therefore, $W=C\oplus D'$, where $D'\subseteq D$. Similarly, $W=C'\oplus D$, where $C'\subseteq C$. Since $W=C+D$, then $C=W\cap C=(C'\oplus D)\cap C=C'\oplus(D\cap C)$, also $D=W\cap D=(C\oplus D')\cap D=(C\cap D)\oplus D'$, hence $W=C'\oplus D'\oplus C\cap D$, that is, $(C\cap D)$ is a direct summand of W.

(3⟹4) Let every subsemimodule of W lies above a direct summand, by Proposition(3.20) W is amply supplemented and every supplement is a direct summand in W. Let $W=C+D$, since W is amply

supplemented, then there is a supplement $D'$of C in W such that $D'\subseteq D$. This means that $W=C+D'$ and $C\cap D'$ is small in D'. Also, there is a supplement $C'\subseteq C$ of D' in W, then $W=C'+D'$ and $C'\cap D'$ is small in C', which means that C' and D' are mutual supplements. Thus $C'\cap D'=0$, then $W=C'\oplus D'$. Now, let $h:W\rightarrow C'$ and $q:W\rightarrow D'$ are the natural projections, so $h+q=1_W$, then $h(W)\subseteq C'$, then $h(W)\subseteq C$ and $q(W)\subseteq D'\subseteq D$, also $q(C)\subseteq q(W)\subseteq D'$ and $q(C)\subseteq C$, then $q(C)\subseteq C\cap D'$. Now, to prove that $C\cap D'\subseteq q(C)$, let $a\in C\cap D'$, then $a\in D'$ implies $a=q(a)$ and $a\in C$ implies $q(a)\in q(C)$, hence $a\in q(C)$, so $C\cap D'\subseteq q(C)$, then $C\cap D'=q(C)$. But $C\cap D'\ll D'$, then $q(C)\ll D'$. By Proposition **(3.19),** $q(C)\ll q(W)$.

(4$\Longrightarrow$1) Let C and D be subsemimodules of W such that $C+D=W$. By (4) there exist idempotent h and $q\in$End(W) with $h(W)\subseteq C$ and $q(W)\subseteq D$ such that $h+q=1_W$. Thus, W is π-projective. Now to show that W is supplemented. Let A be subsemimodule of W, then we can write $W=A+W$ and by (4) there exist idempotent f and $g\in$End(W) such that $g+f=1_W$ $f(W)\subseteq A$, $g(W)\subseteq W$ and $g(A)\ll g(W)$. Claim that $g(W)$ is a supplement of A. To verify this claim :1) to satisfy that $W=g(W)+A$, let $w\in W$, since $f+g=1_W$, then $w=f(w)+g(w)$, but $g(w)\in g(W)$ and $f(w)\in A$ implies that $W\subseteq g(W)+A$ and it is clear that $g(W)+A\subseteq W$, hence $W=g(W)+A$. Now we must show that $g(W)\cap A\ll g(W)$, here we must prove that $g(W)\cap A\ll g(A)$. Let $c\in g(W)\cap A$, then $c=g(w)\in A$, for some $w\in W$, since $f(W)\subseteq A$, then $f(w)\in A$ and since $f(w)+g(w)=w$, then $w\in A$, so that $c\in g(A)$, thus $g(W)\cap A\subseteq g(A)$, but $g(A)\ll g(W)$, therefore $g(W)\cap A\ll g(W)$, hence $g(W)$ is a supplement of A, thus W is supplemented semimodule. ▩

The next corollary is a generalization for the corollary (2.3.2) in[12].

Corollary 4.3. Let W be π-projective supplemented semimodule, then for any two subsemimodules A and B of W such that $W=A+B$, there exist $A'\subseteq A$ and $B'\subseteq B$ (A' and B' are subsemimodules of W) such that $W=A'\oplus B'$.

Proof: Let A and B be subsemimodules of a π-projective supplemented semimodule W with $W=A+B$, then by Proposition**(4.2),** there exist idempotent h and $g\in$End(W), $h+g=1_W$, $h(W)\subseteq A$ and $g(W)\subseteq B$. Since $h+g=1_W$, then $W=h(W)\oplus g(W)$, take $A'=h(W)$ and $B'=g(W)$, thus $W=A'\oplus B'$.

The next definition is a generalization for the definition (2.3.3) in[12].

Definition 4.4 Let W be an S-semimodule and let A be subsemimodule of W, then if for every subsemimodule B of W with $W=A+B$, there exists a weak supplement B' of A with $B'\subseteq B$. Then we say that A has an ample weak supplement in W. If every subsemimodul of W has an ample weak supplement in W, then W is said to be ample weak supplemented.

The next proposition is a generalization for the proposition (2.3.4) [12].

Proposition 4.5. If W is weakly supplemented and π-projective semimodule, then W is amply weak supplemented.

Proof: Let A be a subsemimodule of W and $W=A+B$ for some subsemimodule B of W. Since W is weakly supplemented, then there exists a weak supplement D of A in W, thus $W=A+D$ and $A\cap D\ll W$. Since W is π-projective, then there exist h and $q\in$ End (W) such that $h+q=1_W$, $h(W)\subseteq B$ and $q(W)\subseteq A$. It is clear that $h(A)\subseteq A$ and $q(B)\subseteq B$. Claim that $h(D)$ is a weak supplement of A in W. We know that $W=h(W)+q(W)\subseteq h(A+D)+A=h(A)+h(D)+A=A+h(D)$, so, $W=A+h(D)$. Since $A\cap h(D)\subseteq h(A\cap D)\ll W$, then $A\cap h(D)\ll W$, hence $h(D)$ is a weak supplement of A . Since $h(D)\subseteq B$, then A has an ample weak supplement in W. Hence W is amply weak supplemented. ▩

The next definition is a generalization for the definition (2.3.6) in[12].

Definition 4.6. Let K be a subsemimodule of a semimodule W**,** then we say that K coclosed in W if $K/H\ll W/H$, thus $K=H$, for all subsemimodules H of W contained in K.

Example 4.7. Let $H=\{\overline{0},\overline{5},\overline{10}\}$ be the subsemimodule of $\mathbb{Z}$-semimodule $W=\mathbb{Z}_{15}$, then H is coclosed in W, because the only subsemimodule N contained in H such that $H/N\ll W/N$ is H.

Note: We know that every direct summand of any module is coclosed**[12. p.63].**

Proposition 4.8. Every coclosed subsemimodule in a π-projective supplemented semimodule is a direct summand.

Proof: Let K be a coclosed subsemimodule of a semimodule W where W is π-projective supplemented. There exists a supplement subsemimodule H of K in W, that is $W=K+H$ and $K\cap H\ll H$. Claim that $K\cap H\ll K$. Let $K\cap H+K'=K$ such that K' is a subsemimodule of K. Let $p:W\rightarrow W/K'$ be the natural epimorphism, since $K\cap H\ll H$, then $K\cap H\ll W$. Thus $p(K\cap H)\ll W/K'$. Since $p(K\cap H)=\frac{K\cap H+K'}{K'}$, then $K/K'\ll W/K'$. But K is coclosed in W, therefore $K=K'$, hence $K\cap H\ll K$, implies that K and H are

mutual supplements. But, since W is π-projective semimodule, by Proposition **(4.1)** $K\cap H=0$. Thus K is a direct summand in W. ▓

The next proposition is a generalization for the proposition (41.14) [1].

Proposition 4.9. Every direct summand of π-projective semimodule is π-projective.

Proof: Suppose that W is a π-projective semimodule. Let K be a direct summand of W, then $W=K\oplus D$, where D is a suitable subsemimodule of W. Let A and B be subsemimodules of K such that $K=A+B$, then $W=A+B+D$ and since W is a π-projective, then there exist α and β such that $\alpha+\beta=1_{W,}$ $\alpha(W)\subseteq A$ and $\beta(W)\subseteq B+D$, where α and $\beta\in$End(W). Let $p:W\rightarrow K$ be the projection map of W onto K, let h, $q\in$End(K) be such that $h=p\alpha$ and $q=p\beta$. To show that $h(K)\subseteq A$, $q(K)\subseteq B$ and $h+q=1_K$. For $k\in K$, $h(k)=p(\alpha(k))=\alpha(k)\in A$. While $\beta(k)=b+d\in B+D$ and $p(d)=0$, $p(b)=b$, so $q(k)=b$. That is, $h(K)\subseteq A$ and $q(K)\subseteq B$, finally, $h+q=p\alpha+p\beta=p(\alpha+\beta)=p1_W=1_K$. Hence K is π-projective. ▓

By Proposition**(4.8)** and Proposition**(4.9)** we can obtain the following corollary.

Corollary 4.10. Let W be a π-projective supplemented semimodule, and let K be a coclosed subsemimodule in W, then K is a π-projective semimodule.

The next proposition is a generalization for the proposition (41.16) [1] which was previously proved for modules [10, p.64-67].

Proposition 4.11. Let W be a subtractive supplemented π-projective S-semimodule and E=End(W) , then:

1. Every direct summand of W is supplemented and π-projective and every supplement subsemimodule of W is a direct summand.
2. Let h and q be idempotent in E and C is a direct summand in W. If $q(C)\ll q(W)$, then $C\cap q(W)=0$ and $C\oplus q(W)$ is a direct summand in W.
3. For any $0\neq w\in W$, there is a decomposition $W=W_1\oplus W_2$ such that W_2 hollow and $w\notin W_1$.

Proof:1) To prove that every supplement subsemimodule of W is a direct summand, suppose that K is a subsemimodule, there exists a subsemimodule D of W such that $K+D=W$ and $K\cap D\ll K$. Since W is a supplemented π-projective semimodule, by Proposition **(4.2)** W is amply supplemented, hence there exists $D'\subseteq D$ such that D′ is a supplement of K in W, thus $W=K+D'$ and $K\cap D'\ll D'$. But $(K\cap D')\subseteq(K\cap D)\ll K$, then K and D' are mutual supplements and by Proposition **(4.1)** $K\cap D'=0$, then $W=K\oplus D'$.

2) Let h, q be idempotent in E and C is a direct summand in W. Since W is supplemented and π-projective, then by Proposition **(4.2)** every subsemimodule of W lies above a direct summand. Since W is π-projective, then $W=h(W)\oplus q(W)$ by 1) $h(W)$ is supplemented π-projective semimodule. But $h(C)\subseteq W$, so there is a decomposition $h(W)=K\oplus D$ such that $K\subseteq h(C)$ and $h(C)\cap D\ll D$. For the projection $p:K\oplus D\oplus q(W)\rightarrow D$, $p(C)=h(C)\cap D$. To verify this, let $c\in C$, then $q(c)\in q(W)$ implies that $p(q(c))=0$. Then $p(c)=p(h(c)+q(c))=p(h(c))=h(c)$, therefore $p(C)\subseteq p(h(C))=h(C)$, so $p(C)\subseteq h(C)\cap D$. Since $h(W)=K\oplus D$ and $K\subseteq h(C)$, then $h(C)=K\oplus(h(C)\cap D)$. Hence $p(h(C))=p(K\oplus(h(C)\cap D))=h(C)\cap D$ which is small in D. Claim that $p+q=1_{D\oplus q(W)}$. To verify this claim, let $(d+q(w))\in D\oplus q(W)$, then $(p+q)(d+q(w))=d+q(d)+q(w)$. We need to prove that $q(d)=0$. Since $q(d)\subseteq q(W)$, $d\in D\subseteq h(W)$, since W is π-projective, then $h(d)+q(d)=d$ and since W subtractive, then $q(d)\in h(W)$, therefore $q(d)\in q(W)\cap h(W)$, but $h(W)\cap q(W)=0$, then $q(d)=0$. Thus $q+h=1_{D\oplus q(W)}$. Also, $C\cap(D\oplus q(W))\subseteq p(C)+q(C)$. To prove this, let $x=d+q(w)=p(x)+q(x)\in p(C)+q(C)$, since $p(C)=h(C)\cap C\ll D\ll W$ and $q(C)\subseteq q(W)\ll W$, thus $p(C)+q(C)\ll W$. Since $K\subset h(C)$, then $W=h(C)\oplus(D\oplus q(W))$ and for $c\in$C, $h(c)+q(c)=c\in C+q(W)$, then $h(C)\subseteq C+q(W)$, thus $W=C+(D\oplus q(W))$, then by Proposition **(4.2)** $C\cap(D\oplus q(W))$ is a direct summand of W.

3) Let $0\neq w\in W$ and $\Gamma=\{B: B$ is a direct summand of $W\}$. It is clear that $\Gamma\neq\phi$ where $0\in\Gamma$. By Zorn's lemma, Γ has a maximal element W_1. Suppose that $W=W_1\oplus W_2$ for a suitable semimodule W_2 of W. Now to show that W_2 is hollow, assume that W_2 is not hollow, then there exist H and D proper subsemimodules of W_2 such that $W_2=H+D$. By (1), W_2 is supplemented and π-projective. By Proposition**(4.2)** (1⟹2(a)), W_2 is amply supplemented, thus there exists a supplement D_1 of H in W such that $D_1\subseteq D$, that is $W_2=D_1+H$ and $D_1\cap H\ll D_1$. Also, there exists a supplement H_1 of D in W_2 such $H_1\subseteq H$, then $W_2=D+H_1$ and $D\cap H_1\ll H_1$, therefore D_1 and H_1 are mutual supplements. By Proposition **(4.1),** $D_1\cap H_1=0$, hence $W_2=D_1\oplus H_1$, thus $W=W_1\oplus D_1\oplus H_1$. Since W_1 is maximal, then either $D_1=0$ or $H_1=0$, which is a contradiction since $D_1$and H_1 are mutual supplements, implies that W_2 is hollow. ▓

The next definition is a generalization for the definition (p.16) [12].

Definition 4.12. Let W be a semimodule**,** then W is said to be direct projective if for every direct summand H of W, every epimorphism from W to H splits .

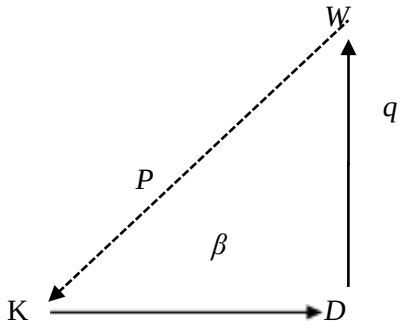
The next lemma is a generalization for the lemma (1.1.7) [12].

Lemma 4.13. If W is a direct projective semimodule, then:

1) If K and D are direct summands of W, then every epimorphism $K \to D$ splits.

2) If K and D are direct summands with $K+D=W$, then $K \cap D$ is a direct summand.

Proof: 1) Let K and D be direct summands of a direct projective semimodules W, $\beta: K \to D$ be an epimorphism, and let $p: W \to K$ be a projection map , βp is an epimorphism and splits (because W is a direct projective), implies that there exists a homomorphism $q: D \to W$ such that $\beta(pq)=1_D$.Thus β splits.



2)) Let K and D be direct summands of W with $W=K+D$, hence $W=D \oplus Y$ where Y is a suitable subsemimodule of W, so $Y \simeq \frac{W}{D} = \frac{K+D}{D}$, and by the second isomorphism theorem $\frac{K+D}{D} \simeq \frac{K}{K \cap D}$, thus $Y \simeq \frac{K}{K \cap D}$. Since K and Y are direct summands of W, then by (1) the homomrphism $fp: K \to Y$ splits where $p: K \to \frac{K}{K \cap D}$ and $f: \frac{K}{K \cap D} \to Y$ are the isomorphisms. But fp is an epimorphism, hence $\ker(fp)=K \cap D$ is a direct summand of K, hence is a direct summand of W. ▓

The next proposition is a generalization for the proposition (2.3.10) [12].

Proposition 4.14 Let W be a supplemented direct projective S-semimodule, then the followings are equivalent:

1) Every subsemimodule of W lies above a direct summand.

2) W is a π-projective semimodule.

Proof: (1$\Longrightarrow$2) Let K and D be direct summands of W with $W=K+D$. By Lemma**(4.13)** 2) $K \cap D$ is a direct summand of W. Then by Proposition**(4.2)** W is π-projective.

(2$\Longrightarrow$1) Clear. ▓

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