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Maximal Ideal Graph of Commutative Rings

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Abstract
In this paper, we introduce and study the notion of the maximal ideal graph of a commutative ring with identity. Let R be a commutative ring with identity. The maximal ideal graph of R, denoted by MG(R), is the undirected graph with vertex set, the set of non-trivial ideals of R, where two vertices I_1 and I_2 are adjacent if $I_1 \neq I_2$ and I_1+I_2 is a maximal ideal of R. We explore some of the properties and characterizations of the graph.

Keywords: The maximal ideal graph of a commutative ring R, maximal ideals and connected graphs.

بيان المثاليات الأعظمية للحلقات الابدالية

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الخلاصة
في هذا البحث نقوم بدراسة فكرة بيان المثاليات الأعظمية للحلقات الابدالية بالعنصر المحايد. لتكن R حلقة ابدالية بعنصر المحايد. البيان المثاليات الأعظمية للحلقة R يرمز له MG(R) وهو بيان غير موجه والذي مجموعة رؤوسه هي المثاليات غير تافهة للحلقة R ، حيث أن أي رأسين I_1 و I_2 متجاورين اذا كان $I_1 \neq I_2$ و I_1+I_2 هي مثالية أعظمية للحلقة R، وكذلك نستكشف بعض الخواص والمميزات لهذا النوع من البيان.

1. Introduction

The graphs assigned to a commutative ring have been studied by many mathematicians. The zero divisor graph of commutative rings was first introduced by Beck in [1]. After that, many mathematicians studied such graphs [2- 5].

Throughout this paper, R will be a commutative ring with identity. We introduce and investigate the notion of maximal ideal graph of a commutative ring R with identity, which is denoted by MG(R). It is the undirected graph with vertex set, the set of non-trivial ideals of R, where two vertices I_1 and I_2 are adjacent if $I_1 \neq I_2$ and I_1+I_2 are maximal ideals of R. First, we explore some of the properties and characterizations of these graphs. For instance, the rings R, for which the graph MG(R) is star or complete bipartite, are characterized. Next, we study the planarity as well as the connectivity of MG(R). It is shown that MG(R) is a connected graph and diam (AG) (R) $\leq$ 3.

We recall some definitions in graph theory which are needed in our work [6, 7].

The neighborhood of a vertex v in the graph G, denoted by N (v), is the set of vertices adjacent to v. The degree of a vertex v of the graph G, denoted by $\deg_G(v)$, is the number of edges incident to v. A

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graph G is a complete graph if every two of its vertices are adjacent. A complete graph of order n is denoted by K_n. A graph G is n- partite, n≥1, if it is possible to partition V(G) into n subsets V_1 ,V_2, …,V_n(called partite sets) such that every element of E(G) joins a vertex of V_i to a vertex of V_j, i≠j. A complete bipartite graph with exactly two partitions of size m and n is denoted by $K_{m, n}$. A graph G is said to be star if G= $K_{1,n}$. Two vertices u and v of a graph G are said to be connected in G if there exists a path between them. A graph G is called connected if all pairs of its vertices are connected. Let G be a connected graph. The distance between a pair of vertices u and v of G, denoted by d(u, v), is the length of the shortest u-v path in G. The diameter, eccentricity, and radius of a connected graph G are defined by diamG=Max{d(u, v): u, v ∈ V(G)}, e(v)=Max{d(u, v): for all u ∈ V(G)} and rad(G)=Min{e(v): v∈V(G)}, respectively. A vertex v of a connected graph G is a cut-vertex if the components of G-v are more than the components of G. The girth of a graph G is the length of the shortest cycle in G. A k-coloring of a graph G is a function C:V(G)→{1,2,...,k} such that C(u)≠C(v) whenever u is adjacent to v. If a k-coloring of G exists, then G is k-colorable. The chromatic number of G is defined by χ(G)=min{k; G is k-colorable}. A complete sub-graph K_n of a graph G is called a clique, and ω(G) is the clique number of G, which is the greatest integer r≥ 1 such that K_r⊆G. A graph G is called a planar graph if it can be drawn on a plane in such a way that any two of its edges either meet only at their end vertices or do not meet at all. A graph G is perfect if every induced subgraph H of G satisfies χ(H) = ω(H). A graph is a split graph if it can be partition in an independent set and a clique.

Throughout this work, we use J_R, I_R, m_R and M_R to denote the Jacobson radical, the set of non-trivial ideals, the set of minimal and maximal ideals of a ring R, respectively.

2. The Maximal Ideal Graph of R

In this section, we introduce the notion of the maximal ideal graph of a commutative ring with identity. We illustrate this concept by examples and remarks and give some of its properties and characterizations.

Definition2.1: Let R be a commutative ring with identity. The maximal ideal graph of R, denoted by MG(R), is the undirected graph with vertex set, the set of all non-trivial ideals of R, where two vertices I_1 and I_2 are adjacent if $I_1 \neq I_2$ and I_1+I_2 are maximal ideals of R.

We begin with the following easy result which may be needed in the sequel.

Lemma2.2:

1. Every non-maximal ideal is adjacent to at least one maximal ideal in MG(R).
2. If $P_1, P_2, \dots, P_n \in M_R$ such that $\bigcap_{i=1}^{n} P_i \notin M_R \cup \{(0)\}$, then the ideal $\bigcap_{i=1}^{n} P_i$ is adjacent to every P∈ M_R in MG(R).

Proof:

1. Let J∈V(MG(R))\ M_R. Then J⊂Q, for some Q∈ M_R. Obviously, I+Q=Q. Thus I is adjacent to Q.
2. The proof follows from the first part of the Lemma2.2.

Example1: Consider the ring z_{60}. The graph MG(z_{60}) is:

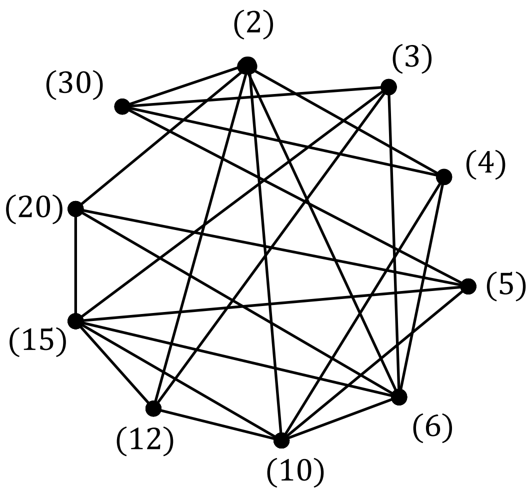

Figure 1-The graph MG(z_{60})

It is obvious from Figure-1 that every non- maximal ideal is adjacent to a maximal ideal.

Remark2.3: The co-maximal ideals of R are not adjacent in MG(R).

The next main result shows the adjacency between ideal vertices of MG(R).

Theorem2.4: Let I, J and P be three distinct vertices of MG(R) with $P\in M_R$. Then:

1. $P\in N(I)\cap N(J) \Leftrightarrow P\in N(I+J)$, where $I+J\neq P$, R.
2. $I\subset J_R \Rightarrow P\in N(I)$
3. $I\subset J \wedge J\notin M \Rightarrow I\notin N(J)$
4. $I\in N(JL) \Rightarrow I\in N(J\cap L)\cap N(J)$, for every vertices L in which $LJ\neq(0)$.

Proof:

1. Let $I+J\neq P$. If $P\in N(I)\cap N(J)$, then by Lemma2.2, I, $J\subset P$. This means that $I+J\subset P$. Thus $P\in N(I+J)$. Similarly, $P\in N(I+J)$ leads to $P\in N(I)\cap N(J)$.
2. Let $I\subset J_R$. Then $I\subset I+J_R=J_R\subseteq P$. By Lemma2.2, $P\in N(I)$.

Similarly, we can show the other parts of Theorem2.4.

Proposition2.5: If $\{I, J\}\in E(MG(R))$ with I, $J\notin M_R$, then there exists a unique $M\in M_R$ such that $M\in N(I)\cap N(J)$.

Proof: Suppose that $M_1, M_2\in M_R$ and each of I and J are adjacent to both M_1 and M_2 in MG(R). Then by Lemma2.2, I, $J\subset M_1\cap M_2$. Since $I+J\in M_R$, then $M_1=I+J=M_2$.

The next result shows that the degree of maximal ideals determines the finiteness of MG(R).

Proposition2.6: Let R be Artinian. If $\deg I<\infty$, for every $I\in M_R$, then MG(R) is a finite graph.

Proof: Since R is Artinian ([8], Theorem8.7), then R is isomorphic to $R_1\times R_2\times \cdots \times R_n$, where (R_i, P_i) is a local Artinian ring. The maximally of I gives that $I=R_1\times R_2\times \cdots \times R_{i-1}\times P_i\times R_{i+1}\times \cdots \times R_n$, where $1\leq i\leq n$. Since degI is finite, then I_{R_i} is finite. Thus MG(R) is a finite graph.

The next result gives the conditions on MG(R) for which R is a local ring.

Poposition2.7: If $MG(R)\cong K_n$ or $MG(R)\cong K(n, 1)$, where $n\in Z^+$, then R is local.

Proof: If $MG(R)\cong K_n$, then by Remark2.4, R is local. Let MG(R) be a star with center I. If MG(R) consists of only one edge, then it refers to completeness case. Assume that $|MG(R)|\geq 3$. If $I\notin M_R$, then by Lemma2.2, $V(MG(R))\setminus\{I\}=M_R$. Thus $I=J_R\neq(0)$. Now, suppose that P, $S\in M_R$ with $P\neq S$. Obviously, $(0)\neq PS\notin M_R$. Thus $PS=I=J_R$. This contradicts that $|MG(R)|\geq 3$. Therefore, $I\in M_R$. Again by Lemma2.2, $M_R=\{I\}$. Thus the proof is completed.

The converse of Proposition2.7 will be true if V(MG(R)) is a totally ordered set. We illustrate it in the following result.

Proposition2.8: If V(MG(R)) is a totally ordered set, then MG(R) is a star.

Proof: Since V(MG(R)) is a totally ordered set, then MG(R) contains a vertex I which is adjacent to each other vertex. If $J\neq I$ and $P\neq I$ are two distinct vertices of MG(R), then either $P\subset J$ or $J\subset P$. For both cases, J and K are not adjacent vertices. Thus MG(R) is a star with center I.

Corollary2.9: For any prime number p, the graph $MG(z_{p^n})$ is star.

Proof: It follows from Proposition2.8.

Now, we give the condition for which MG(R) be a complete bipartite, as follows.

Theorem2.10: Let $J_R\notin\{(0)\}\cup M_R$. Then $MG(R)\cong K_{m,n}$; m, $n\in Z^+$ if and only if $I_R - M_R \subseteq J_R$.

Proof: Let $I_R - M_R \subseteq J_R$. Choose $V_1=M_R$ and $V_2=\{I\in V(MG(R)): I\subseteq J_R\}$. From Lemma2.2, every two vertices in V_1 are independent with respect to the graph MG(R). Since $|MG(R)|\neq 1$, then $J_R\notin M_R$. Thus $I+J\notin M_R$ for every I, $J\in V_2$. This means that every two vertices in V_2 are independent with respect to the graph MG(R). On the other hand, Theorem2.4 mentions that every $I\in V_1$ is adjacent to each $J\in V_2$. This ends the proof.

Conversely, if MG(R) is a complete bipartite with partite sets W_1 and W_2, we can prove that $W_i=M_R$ and $W_j=\{I\in V(MG(R)): I\subseteq J_R\}$, for i, j=1, 2 with $i\neq j$. This completes the proof.

Corollary2.11: Let $J_R\notin M_R\cup\{(0)\}$. If MG(R) is not a complete bipartite, then MG(R) is a 3-partite graph.

Proof: Since MG(R) is not a complete bipartite, then by Theorem2.10, $I\nsubseteq J_R$, for some $I\in MG(R)\setminus M_R$. We set $V_1=M_R$, $V_2==\{I\in V(MG(R)): I\subseteq J_R\}$ and $V_3=V(MG(R))\setminus(V_1\cup V_2)$. It is not difficult to show that every two vertices in V_i are independent, for i=1, 2, 3. Thus MG(R) is a 3-partite graph.

Example2: The graph $MG(z_{36})$ is a 3-partite graph, as the following figure shows:

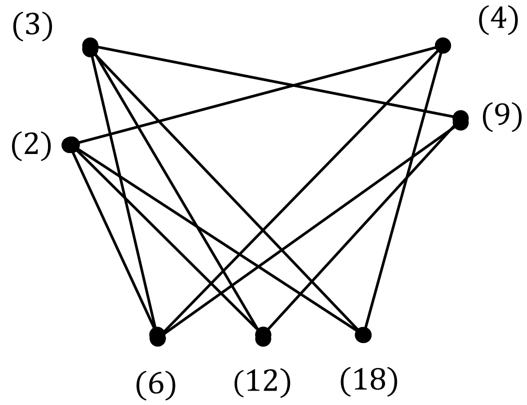

Figure 2- The graph MG(z_{36})

Now we are at the position of the following main result.

Theorem2.12: Let$m_R \neq \emptyset$. If V (MG(R)) =$m_R \cup M_R$, then:

1. The graph MG(R) is split.
2. The graph MG(R) is perfect.
3. The clique number of MG(R) is ω(MG(R)) = max {$|m_R|$,$|m_R| + 1$}.

Proof:

1. Let A be the induced subgraph of MG(R) bym_R. Let S, T$\in m_R$ with S$\neq$T. Obviously, S+T$\neq$R. If we assume that S+T$\in m_R$, then S=S+T=T, which is a contradiction. Therefore, S+T$\in$MG(R). Thus A is a complete graph. From Remark2.3, the vertices in M_R are independent. Hence MG(R) is a split graph.

2. Let C:I_1, I_2, $\cdots$ I_{2n+1}, I_1 be an induced cycle in MG(R) with n$\geq$2. If C does not contain any maximal ideal vertex, then by the first part of Theorem2.12, {I_1, I_3}$\in$E(MG(R)), which is a contradiction. Let $I_1\in M_R$. Obviously, I_{2n+1}, $I_2 \notin M_R$. Then they are adjacent in MG(R), which is a contradiction. Now, assume that C′ is an induced odd cycle in $\overline{MG(R)}$ of length n≥ 5. Then C′ contains at least P, Q$\in M_R$ with P$\neq$Q such that they are not adjacent in C′. From Lemma2.2, P and Q are adjacent in $\overline{MG(R)}$. This contradics Lemma2.2. Hence, by the strong perfect graph theorem in [9], MG(R) is a perfect graph.

3. The proof follows from the first part of Theorem2.12.[10]

Example3: Consider the ring z_{12}. The following graph shows that MG(z_{12}) is a split and perfect graph. Also ω(MG(z_{12}))= $|m_R| + 1$=3.

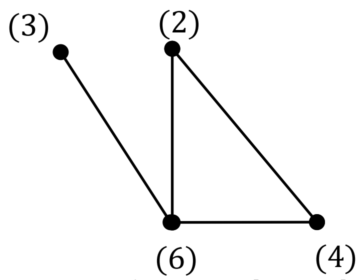

Figure 3- The graph MG(z_{12})

In the next result, we find the girth of MG(R).

Theorem2.13: Let $J_R \neq$(0). The girth g(MG(R)) is either 3, 4, or ∞.

Proof: If MG(R) contains an edge {S, T} with S, T$\notin M_R$, then S, T$\neq$S+T$\in M_R$. Thus S+T is adjacent to both S and T. This means that C: S, T, S+T, S is a cycle in MG(R). In this case, g(MG(R))=3. Suppose that for every {I, J}$\in$E(MG(R)), either I$\in M_R$ or J$\in M_R$. If MG(R) does not possess any cycle, then g(MG(R))=∞. Now, suppose that C_n: I_1, I_2, …, I_n, I_1 is a cycle in MG(R) of length n. Since the maximal ideals are not adjacent in MG(R), the vertices of C are alternatively maximal and non-maximal ideals. Consequently, $J_R \notin M_R$. Let $I_1\in M_R$. From Lemma2.2, J_R is adjacent to each of I_1, I_3

and I_5. If $I_2=J_R$, then C': I_2, I_3, I_4, I_5, I_2 is a cycle in MG(R). If $J_R \neq I_2$, then C'': J_R, I_1, I_2, I_3, J_R is a cycle in MG(R). From both cases, we have shown that g(MG(R)) is either 3 or 4.

The next result shows the upper bound of clique number of MG(R).

Proposition2.14: The clique of MG(R) contains in an its induced subgraph by $\{I\in V(MG(R)): I\subseteq P\}$, for exactly one $P\in M_R$.

Proof: Let G be the clique of MG(R). Since the co-maximal ideals are not adjacent in MG(R), G has at most one maximal ideal. The adjacency of every two vertices of G and Proposition2.5 illustrates that there exists exactly one $P\in M_R$ for which G is a subgraph of the graph induced by $\{I\in V(MG(R)): I\subseteq P\}$.

3. The Planarity of MG(R)

First, we find the clique number of MG(R).

Proposition3.1: If the subgraph induced by $\{I\in V(MG(R)): I\subseteq P\}$is planar, for every $P\in M_R$, then ω(MG(R)) is either 2 or 3 or 4.

Proof: The proof follows from Proposition2.14 and Koratowsky theorem [6].

In the next result, we show that MG(R) is a planar graph under a certain condition on vertex set of MG(R).

Theorem3.2: If $V(MG(R))=m_R \cup M_R$ is finite and $|m_R| \leq 3$, then the graph MG(R) is planar.

Proof: To show that MG(R) is planar, we refer to Koratowsky theorem. Since $|m_R| \leq 3$, then any subgraph of MG(R) induced by five vertices is not complete. This means that MG(R) does not contain any complete subgraph K_5. If we assume that MG(R) contains a $K_{3,3}$ with partite sets $V_1=\{I_1, I_2, I_3\}$ and $V_2=\{J_1, J_2, J_3\}$, then by Lemma2.2 either $V_1\subseteq M_R$ or $V_2\subseteq M_R$. Assume that $V_1\subseteq M_R$. Then $V_2\subseteq m_R$. From Proposition2.5, any two of L, M and N are independent. This contradicts that every minimal ideal are adjacent in MG(R). Therefore, MG(R) is a planar graph.

The next result demonstrates that the planarity of MG(R) limits the order of M_R.

Proposition3.3: Let $J_R \neq (0)$. If MG(R) is planar graph, then $|M_R| \leq 4$.

Proof: Let MG(R) be a planar graph. Assume by contrary that MG(R) has at least five distinct maximal ideals, say M, N, P, Q and S. Obviously, any one of the vertices MNP, MNPQ and MNPQS are non-zero ideals and adjacent to each of ideals M, N and P in MG(R). Thus MG(R) contains a complete bipartite graph $K_{3,3}$. This contradicts the Koratowsky theorem. Therefore $|M_R| \leq 4$.

Before closing this section, we give the following main result.

Theorem3.4: Let $R\cong R_1\times R_2\times \cdots \times R_n$, with $R_1, R_2, \cdots, R_n$ are distinct fields. Then MG(R) is planar graph if and only if $n\leq 4$.

Proof: Let MG(R) be a planar graph. Assume that n>4. Obviously, $(0)\times R_2\times \cdots \times R_n \in M_R$ and the sum of every two of ideals $(0)\times R_2\times \cdots \times R_n$, $(0)\times(0)\times R_3\times \cdots \times R_n$, $(0)\times R_2\times(0)\times R_4\times \cdots \times R_n$, $(0)\times R_2\times R_3\times(0)\times R_5\times \cdots \times R_n$, $(0)\times R_2\times R_3\times R_4\times(0)\times \cdots \times R_n$ is equal to $(0)\times R_2\times \cdots \times R_n$. Then MG(R) contains a complete sub-graph of order 5. This contradicts the planarity of MG(R). Therefore, $n\leq 4$.

Conversely, let $n\leq 4$. Clearly, $E(MG(R))=\emptyset$, when $n\in\{1, 2\}$. Now, suppose that n=3. Then V(MG(R)) consists of $I_1=R_1\times(0)\times(0)$, $I_2=(0)\times R_2\times(0)$, $I_3=(0)\times(0)\times R_3$ $I_4=R_1\times R_2\times(0)$, $I_5=R_1\times(0)\times R_3$ and $I_6=(0)\times R_2\times R_3$, and the graph MG(R) is:

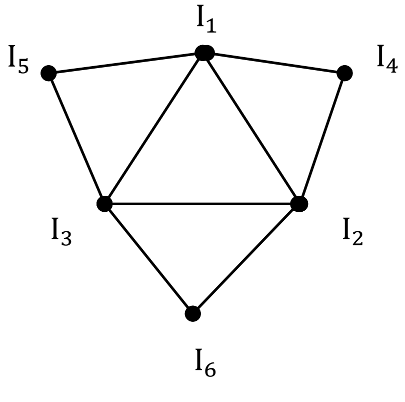

Figure 4- The graph $MG(R_1\times R_2\times R_3)$

Obviously, MG(R) is planar graph, when n=3.

Now, suppose that n=4. The maximal ideal vertices of MG(R) are $(0)\times R_2\times R_3\times R4$, $R1\times (0)\times R_3\times R_4$, $R_1\times R_2\times (0)\times R_4$ and $R_1\times R_2\times R_3\times (0)$, and the other vertices are $(0)\times (0)\times R_3\times R_4$, $(0)\times R_2\times (0)\times R_4$, $(0)\times R_2\times R_3\times (0)$, $R_1\times (0)\times (0)\times R_4$, $R_1\times R_2\times (0)\times (0)$, $R_1\times (0)\times R_3\times (0)$, $R_1\times (0)\times (0)\times (0)$, $(0)\times R_2\times (0)\times (0)$, $(0)\times (0)\times R_3\times (0)$, $(0)\times (0)\times (0)\times R_4$. This graph does not contain K_5. Also, for every three distinct vertices I, J and K of MG(R), there exists at most two vertices adjacent to each of I, J and K. Thus MG(R) does not contain K(3, 3). In this case, MG(R) is a planar graph.

4. The Connectivity of MG(R)

We start this section with the following result.

Theorem4.1: Let R be a finite non-local ring with MG(R)) is a non-empty graph. Then every two vertices are disconnected if and only if $R=R_1\times R_2$, where R_1 and R_2 are fields.

Proof: If $R=R_1\times R_2$ with R_1 and R_2 are fields, then $V(MG(R))=\{(0)\times R_2, R_1\times (0)\}$. Obviously, $(0)\times R_2$ and $R_1\times (0)$ are not adjacent in MG(R).

Conversely, suppose that every two vertices are disconnected. Since R is a finite non-local ring, then $R\cong R_1\times R_2\times \cdots \times R_n$, where (R_i, P_i) is a local ring for every i=1, 2, …, n and n≥2. If $P_1\neq(0)$, then $(P_1\times R_2\times \cdots \times R_n)+(P_1\times P_2\times \cdots \times R_n)\in M_R$, which is a contradiction. Hence $P_1=(0)$. Similarly, $P_2=P_3=\cdots=P_n=(0)$. Thus $R_1, R_2, ..., R_n$ are fields. If n≥3, then $(0)\times R_2\times \cdots \times R_n$ and $(0)\times (0)\times R_3\times \cdots \times R_n$ are adjacent in MG(R), which is a contradiction. Therefore, n=2.

In the next main result, we investigate the connectivity of MG(R).

Theorem4.2: If every two distinct maximal ideals of R have a non-zero intersection, then MG(R) is connected with diamMG(R)≤3.

Proof: Let K, L∈V(MG(R)) with K≠L. If {K, L}∈E(MG(R)), then they are connected. Suppose that {K, L}∉E(MG(R)). Then either K+L=R or K+L⊂P, for some $P\in M_R$. If K+L⊂P, then by Lemma2.2, P_2: K, P, L is a path in MG(R). If K+L=R, then at least one of K and L is a maximal ideal and neither K⊂L nor L ⊂K. Assume that $K\in M_R$. If $L\in M_R$, again by Lemma2.2, P_2': K, K∩L, L is a path in MG(R). Let $L\notin M_R$. Then there exists $M\in M_R$ such that L is adjacent to P. If P=K, then K is adjacent to L. Let K≠P. Then P_3: K, K∩P, P, L is a path in MG(R). From each case, we have shown that K and L are connected and d(K, L)≤3. Thus MG(R) is connected with diamMG(R)≤3.

Observe that the graph MG(R) may not be connected, when two distinct maximal ideals of R have a zero intersection.

Example4: Consider the ring z_6. Obviously, the following graph is disconnected.

Figure 5- The graph MG **(z_6)**

Next, we turn to the following result.

Proposition4.3: If R is a principal ideal ring in which every two distinct maximal ideals of R have a non-zero intersection, then diamMG(R) ≤2.

Proof: From Theorem4.2, d (P, Q) ≤2, for every P, Q∈V (MG(R)) with P≠Q, except for the possibility that P+Q=R and $\{P, Q\}\not\subseteq M_R$. Now, suppose that P+Q=R and $P\in M_R$ but $Q\notin M_R$. Then there exits $T\in M_R$ such that Q is adjacent to T. Since R is a principal ideal ring, then Q+ (T∩P) = (Q+T) ∩(Q+P) = T∩ R =T. Thus Q is adjacent to T∩P. Since P is also adjacent to T∩P, then d (P, Q) ≤2. Finally, diamMG(R) ≤2.

The next result discovers the characterizations of the cut-vertices of MG(R).

Theorem4.4: Suppose that every two distinct maximal ideals of R have a non-zero intersection. If L is a cut-vertex of MG(R), then L=P∩Q, for some $P, Q\in M_R$.

Proof: If $L\in M_R$, then by setting M=N=L, the proof will be completed. Now, suppose that $L\notin M_R$. Let J and K be two vertices in different components of MG(R)-L. We have three cases:

Case1: If $J, K\in M_R$, then J∩K∈N (J) ∩N (K). Since L is a cut-vertex of MG(R), then L= J∩K.

Case2: If J∈ M_R and K∉ M_R, then K∈N(S), for some S∈ M_R. Since J∩S is adjacent to J and S, then L= J∩S.
Case3: If J, K∉ M_R, then P∈N (J) and Q∈N (K), for some P, Q∈ M_R such that P and Q are adjacent to J and K, respectively. Since I is a cut-vertex, then P≠Q. By the same way of Case2, we obtain that L=P∩Q.

In the next main result, we find the radius of MG(R).

Theorem4.6: Let $J_R \neq (0)$. If $|M_R| \geq 2$, then rad (MG(R))= 2.

Proof: From Lemma2.2, d (J_R, K) =1, for every K∈ M_R. Since every vertex I∉ M_R is adjacent to a vertex inM_R, then d (J_R, I) ≤2. Assume that P, Q∈ M_R with P≠Q. If PQ is adjacent toJ_R, then J_R+PQ=P, for some P∈ M_R. Since J_R+PQ⊆P, Q, then P=P=Q. This contradicts that P≠Q. Therefore, PQ is not adjacent toJ_R. Thus the eccentrisity of J_R is e (J_R) =2. If there exists I∈V (MG(R)) with e (I) =1, then I is adjacent to each vertex J∈ M_R. Clearly, I∉ M_R. Since PQ is not adjacent toJ_R, for every M, Q∈ M_R with P≠Q, then neither I=J_R nor I=PQ. Thus I⊈J_R. Hence MG(R) contains a P∈ M_R which is adjacent to I. This contradicts that e(I)=1. Therefore J_R has the minimum eccentricity over all vertices of MG(R). So, rad(MG(R))=e(J_R)= 2.

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