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Quasi J-Regular Modules

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Abstract
Throughout this note, R is commutative ring with identity and M is a unitary R-module. In this paper, we introduce the concept of quasi J-pure submodules as a generalization of quasi–pure submodules and give some of its basic properties. Using this concept, we define the class of quasi J-regular modules, where an R-module M is called quasi J-regular module if every submodule of M is quasi J-pure. Many results about this concept are proved.

Keywords: J-pure submodules, quasi J-pure submodules, J-regular modules and quasi J-regular modules.

المقاسات شبه المنتظمة من النمط –J

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الخلاصة
ليكن M مقاسا ايمن على حلقة R ذات محايد. في هذا البحث قدمنا مفهوم المقاسات الجزئية شبه النقية من النمط –J كتعميم لمفهوم المقاسات الجزئية شبه النقية. وباستخدام المفـهوم هذا نعرف المقاسات شبه المنتظمة من النمط –J إذ يقال ان المقاس M على الحلقة R بأنه شبه منتظم من النمط –J اذا كان كل مقاس جزئي منة يكون شبه نقياً من النمط –J. أعطينا العديد من النتائج حول المفـهوم هذا.

1. Introduction

M is viewed as a right module over an arbitrary ring with identity. A submodule N of an R-module M is called pure in M if IN $=$ N $\cap$ IM for every ideal I of R [1]. An R-module M is a regular module if every submodule of M is pure [2]. A submodule N of an R-module M is called a J-pure if N is pure in J (M), i.e. for each ideal I of R, I J (M) N $=$ IN, where J (M) is the Jacobson radical of M**.** An R-module M is said to be J-regular module if every submodule of M is J-pure submodule. Equivalently, an R-module M is said to be J-regular module if for each $m \in$ J(M), $r \in$ R, there exists $t \in$ Rsuch that $r\,m = r\,t\,r$ [3].

First, recall that a submodule N of an R-module M is called a quasi – pure if, for each $x \in$ M and $x \notin$ N, there exists a pure submodule L of M such that N $\subseteq$ L and $x \notin$ L, and an R-module M is called quasi – regular module if every submodule of M is quasi – pure [4]. This paper is structured in two sections. In section one we introduce a comprehensive study of J-pure submodules. Some results are analogous to the properties of pure submodules. In section two, we study the concept of quasi J-regular modules.

2. Quasi J-pure Submodules

In this section we introduce the concept of quasi J-pure submodule. We investigate the basic properties of these types of submodules which are analogous to the properties of J-pure submodules.

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Definition (2.1):

Let M be an R-module. A submodule N of M is called a quasi J-pure submodule of M if for each $x \in$ M and $x \notin$ N, there exists a J-pure submodule L of M such that N $\subseteq$ L and $x \notin$ L.

Remarks and Examples (2.2)

(1) It is clear that every J-pure submodule is quasi J-pure. But the converse is not true in general. For example, let M = $Z_8 \oplus Z_2$ be a Z-module, and N = $<(\bar{4},\bar{0})>$ = $\{(\bar{0},\bar{0}), (\bar{4},\bar{0})\}$. It is easily checked that N is quasi J-pure submodule of M, since for each $x \in$ M and $x \notin$ N, there exists a J-pure submodule L of M containing N and $x \notin$ L. But N is not J-pure submodule of M, since $(\bar{4},\bar{0})$ = $2(\bar{2},\bar{0}) \in 2\,J(Z_8 \oplus Z_2) \cap N$, but $(\bar{4},\bar{0}) \notin 2.N = \{(\bar{0},\bar{0})\}$.

(2) In any R-module M, the submodule <0> is always quasi J-pure.

(3) It is clear that every quasi –pure is quasi J-pure but the converse is not true. For example, the submodule $\{\bar{0},\bar{2}\}$ in the Z-module Z_4 is quasi J-pure. Since it is J-pure, but $\{\bar{0},\bar{2}\}$ is not quasi – pure. Since there exists no pure submodule that contains $\{\bar{0},\bar{2}\}$.

Recall that an R-module M is called J-pure simple if M and <0> are the only J-pure submodules of M.

(4) Every J-pure simple R-module M does not contain quasi J-pure submodule except $< 0 >$ and M. For example, the Z-modules Q, Z_{P^∞}. Q as Z-module is J-pure simple, hence Q does not have J-pure submodule except <0> and Q. Since $Q \cap IJ(Q) = Q \cap IQ = IQ$, then Q is quasi J-pure since it is J-pure.

(5) If N_1 and N_2 are quasi J-pure submodules of an R-module M, then $N_1 \cap N_2$ is quasi J-pure submodule of M. To show this, let $x \in$ M and $x \notin N_1 \cap N_2$, then either $x \notin N_1$ or $x \notin N_2$. If $x \notin N_1$, since N_1 is quasi J-pure in M, then there exists a J-pure submodule L of M such that $N_1 \subseteq$ L and $x \notin$ L. Hence $N_1 \cap N_2 \subseteq$ L. Similarly if $x \notin N_2$.

(6) The sum of two quasi J-pure submodules may not be quasi J-pure. For example, consider the Z-module M = $Z_8 \oplus Z_2$. Let N = $<(\bar{4},\bar{1})>$ and L = $<(\bar{2},\bar{0})>$. It is easy to see that N and L are quasi –pure submodules in M. But N + L = $\{(\bar{0},\bar{0}), (\bar{2},\bar{0}), (\bar{4},\bar{0}), (\bar{6},\bar{0}), (\bar{4},\bar{1})\}$ is not quasi J-pure submodule in M.

The following propositions give some properties of quasi J-pure submodules.

Proposition (2.3)

Let M be an R-module and N be a J-pure submodule of M. If B is a quasi J-pure submodule of N, then B is a quasi J-pure submodule of M.

Proof

Let $x \in$ M with $x \notin$ B, then either $x \in$ N or $x \notin$ N. Assume that $x \in$ N, but B is a quasi J-pure submodule in N, so there exists a J-pure submodule L in N such that B $\subseteq$ L and $x \notin$ L. Thus we have L is J-pure in N and N is J-pure in M, so by 3, remark 2.3, L is J-pure in M. Therefore, B is quasi J-pure submodule of M. Now, if $x \notin$ N, then there is nothing to prove, since N is a J-pure submodule in M containing B and $x \notin$ N.

Proposition (2.4)

Let R be a good ring, M be an R-module and N be a J-pure submodule of M. If B is a submodule of M containing N, then N is a quasi J-pure submodule of B.

Proof

Since N is J-pure submodule of M and N $\subseteq$ B, since R is a good ring. So, as previously shown [3, proposition 2.4], N is a J-pure submodule of B, which implies that N is quasi J-pure submodule of B.

Recall that the A submodule N of an R- module M is called a small submodule of M (notation $N <<$ M), if for any submodule A of M such that M $= N + A$, then $A =$ M [5].

Proposition (2.5)

Let M be an R-module and N be a quasi J-pure submodule of M. If H is a small submodule of N, then $\frac{N}{H}$ is a quasi J-pure submodule of $\frac{M}{H}$.

Proof

Let $x + H \in \frac{M}{H}$ with $x + H \notin \frac{N}{H}$. Then $x \notin$ N and $x \in$ M. But N is quasi J-pure in M. So there exists a J-pure submodule L of M such that N $\subseteq$ L and $x \notin$ L. This implies that $\frac{N}{H} \subseteq \frac{L}{H}$ and $x + H \notin$

$\frac{L}{H}$. But L is J-pure submodule in M, hence by 3, proposition 2.5, $\frac{L}{H}$ is J-pure submodule in $\frac{M}{H}$. Therefore, $\frac{N}{H}$ is a quasi J-pure submodule in $\frac{M}{H}$.
The following proposition gives a characterization of quasi J-pure submodules.

Proposition (2.6)

Let M be an R- module and N be asubmodule of M. Then N is a quasi J-pure submodule of M if and only if there exists a collection of submodules $\{N_\alpha\}_{\alpha \in \Lambda}$, where Λ is an index set, such that for each $\alpha \in \Lambda$, N_α are J-pure submodules of M and $N = \cap_{\alpha \in \Lambda} N_\alpha$.

Proof

Assume that N is a quasi J-pure submodule of M. If N is a J-pure submodule of M then there is nothing to prove. If N is not J-pure submodule of M, Since N is quasi J-pure submodule of M, then there exists a collection of J-pure submodules $\{N_\alpha\}_{\alpha \in \Lambda}$ such that $N \subseteq \cap_{\alpha \in \Lambda} N_\alpha$,where Λ is an index set. To show that $\cap_{\alpha \in \Lambda} N_\alpha \subseteq N$, let $x \in \cap_{\alpha \in \Lambda} N_\alpha$, then $x \in N_\alpha$ for each $\alpha \in \Lambda$. Suppose that $x \notin N$. Since N is quasi J-pure submodule of M, then x is not contained in any J-pure submodule that contains N. So, $x \notin N_\alpha$, which is a contradiction. Therefore, $x \in N$ and hence $\cap_{\alpha \in \Lambda} N_\alpha \subseteq N$. That is, $\cap_{\alpha \in \Lambda} N_\alpha = N$.

Conversely, suppose that $N = \cap_{\alpha \in \Lambda} N_\alpha$, where N_α is a J-pure submodule of M for each $\alpha \in \Lambda$ and N_α containing N. Let $x \in M$ and $x \notin N$. Since $N = \cap_{\alpha \in \Lambda} N_\alpha$, so there exists $\beta \in \Lambda$ such that $x \notin N_\beta$. Thus $N \subseteq N_\beta$ and $x \notin N_\beta$. That is, N is quasi J-pure submodule in M.

Proposition (2.7)

Let M_1 and M_2 be two R-modules. If A is quasi J-pure submodule of M_1 and B is quasi J-pure submodule of M_2, then $A \oplus B$ is quasi J-pure submodule of $M = M_1 \oplus M_2$.

Proof

Let $(x_1, x_2) \in M = M_1 \oplus M_2$ with $(x_1, x_2) \notin A \oplus B$, then either $x_1 \notin A$ or $x_2 \notin B$. Assume that $x_1 \notin A$, since A is quasi J-pure in M, so there exists a J-pure submodule L_1 in M_1 such that L_1 containing A and $x_1 \notin L_1$. But L_1 is J-pure in M_1, so by 3, proposition 2.6, $L_1 \oplus M_2$ is J-pure in M. Also $L_1 \oplus M_2$ containing $A \oplus B$ and $(x_1, x_2) \notin L_1 \oplus M_2$.
Similarly, if $x_2 \notin B$, then there exists a J-pure submodule in M containing $A \oplus B$ and does not contain (x_1, x_2). Therefore, $A \oplus B$ is quasi J-pure submodule in M.

The converse of proposition (2.7) is true under certain conditions, as in the following:

Proposition (2.8)

Let M_1 and M_2 be R-modules, N be a submodule in M_1, and K be a submodule in M_2 such that $ann_R(M_1) + ann_R(M_2) = R$. If $N \oplus K$ is quasi J-pure submodule in $M = M_1 \oplus M_2$, then N is quasi J-pure in M_1 and K is quasi J-pure submodule in M_2.

Proof

To show that N is quasi J-pure in M_1, let $x \in M_1$ and $x \notin N$. Then $(x, 0) \notin N \oplus K$. Since $N \oplus K$ is quasi J-pure submodule in M, so there exists a J-pure submodule H in M such that $N \oplus K \subseteq H$ and $(x, 0) \notin H$. Since $ann_R(M_1) + ann_R(M_2) = R$, then by a part of the proof of a previous work [6, Proposition (4.2), CH.1], any submodule of $M = M_1 \oplus M_2$ can be written as a direct sum of two submodule of M_1 and M_2. Thus $H = A \oplus B$ for some submodules A and B of M_1 and M_2, respectively. It follows, by remark and example (2.2) in an earlier study [3], that A is J-pure submodule in M_1 and B is J-pure submodule in M_2. Since $N \oplus K \subseteq A \oplus B$, so $N \subseteq A$ and $K \subseteq B$. But $(x, 0) \notin H = A \oplus B$, then $x \in A$. Therefore, N is J-pure submodule in M_1.
Similarly, K is quasi J-pure submodule in M_1.

Remark (2.9)

The condition $ann_R(M_1) + ann_R(M_2) = R$ is necessary in proposition (2.8). For example, the module $Z_8 \oplus Z_2$ is a Z-module. Clearly, $ann_Z(Z_8) + ann_Z(Z_2) = 2Z \neq Z$. As we have seen in remark and example (2.2), the submodule $< (\bar{4}, \bar{0}) > = < \bar{4} > \oplus < \bar{0} >$ is quasi J-pure submodule in $Z_8 \oplus Z_2$. But $< \bar{4} >$ is not quasi J-pure submodule in Z_8. Because $2 \in Z_8$, $2 \notin < \bar{4} >$, there exists no J-pure submodule L of Z_8 containing $N = < \bar{4} >$ and $2 \notin L$.

Recall that an R-module M is called a multiplication module if for each submodule N of M there exists an ideal I of R such that $N = IM$ [7].

Proposition (2.10)

Let M be a faithful finitely J-generated multiplication R-module and let N be a submodule of M. The following statements are equivalent:

(1) N is a quasi J-pure submodule of M.

(2) $[N:_R M]$ is a quasi J-pure ideal of R.

Proof

(1) ⇒ (2) Let $r \in R$ and $r \notin [N:_R M]$. Then $rM \nsubseteq N$, so there exists $x \in M$, such that $rx \notin N$. But N is a quasi J-pure submodule in M, then there exists a J-pure submodule K of M such that $N \subseteq K$ and $rx \notin K$. Since M is faithful finitely generated J-multiplication, so it is clear that if K is J-pure submodule in M, then $[K:_R M]$ is J-pure ideal of R. Also $rx \notin K$ for each $x \in M$, then $r \notin [K:_R M]$. Hence $[K:_R M]$ is J-pure ideal of R such that $[N:_R M] \subseteq [K:_R M]$ and $r \notin [K:_R M]$. That is, $[N:_R M]$ is quasi J-pure ideal of R.

3. Basic Results for Quasi J-regular modules

In this section, we introduce and study the class of quasi J-regular modules.

Definition (3.1)

An R-module M is called quasi J-regular module if every submodule of M is quasi J-pure.

Recall that an R-module M is called F-regular if each submodule of M is pure. Equivalently, an R-module M is said to be F-regular R- module if for each $m \in M, r \in R$, there exists $t \in R$ such that $r\,m = r\,t\,r\,m$.[8,9]

Remarks and Examples (3.2):

(1) It is clear that every J-regular R-module is quasi J-regular module. But the converse is not true in general.

(2) If M is J-pure simple R-module, then M is not quasi J-regular. For example, the Z-modules Q, Z_{p^∞}.

(3) It is clear that every F-regular R-module is quasi J-regular, but the converse is not true in general. For example, if M = Z_4 as Z-module. M is quasi J-regular since it is J-regular, but it is not F −regular, by remarks and examples (3.3) in an earlier work [3].

Recall that an R-epimorphism $\varphi : M \to M'$ is called small epimorphism if $\mathrm{Ker}\varphi \ll M$ [5].

Proposition (3.3)

Let M be an R-module. Then M is quasi J-regular if and only if $\frac{M}{N}$ is quasi J-regular for every small submodule N of M.

Proof

Let N be a small submodule of M and K be any submodule of M containing N. Since M is quasi J-regular then K is quasi J-pure in M. So, by proposition (2.5), $\frac{K}{N}$ is quasi J-pure in $\frac{M}{N}$. Therefore, $\frac{M}{N}$ is quasi J-regular.

The converse is clear by taking N = $< 0 >$.

Corollary (3.4)

Let M and M′ be two R-modules and $f: M \longrightarrow M'$ be a small epimorphism. If M is quasi J-regular, then $\frac{M}{Kerf}$ is quasi J-regular.

Proof

Since $f: M \longrightarrow M'$ is an epimorphism and $Kerf \ll M$ and M are quasi J-regular, then $\frac{M}{Kerf}$ is quasi J-regular by proposition (3.3).

Recall that a non-zero R-module M is called a hollow if every proper submodule of M is a small [10].

Corollary (3.5)

Let M be a hollow R-module and M′ be any R-module. If $f: M \longrightarrow M'$ is an epimorphism, then $\frac{M}{Kerf}$ is quasi J-regular.

Proof: It is clear.

Corollary (3.6)

Let M_1 and M_2 be R-modules such that $\mathrm{ann}_R(M_1) + \mathrm{ann}_R(M_2) = R$. Then $M = M_1 \oplus M_2$ is quasi J-regular if and only if M_1 and M_2 are quasi J-regular.

Proof

Assume that $M = M_1 \oplus M_2$ is quasi J-regular and let N_1 be a submodule of M_1, then we have to show that N_1 is quasi J-pure in M_1. Let $x \in M_1$ and $x \notin N_1$, then there exists $N_1 \subseteq L_1$ such that $x \notin L_1$. $(x, 0) \in M = M_1 \oplus M_2$, $(x, 0) \notin N = N_1 \oplus N_2$. Since M is quasi J-regular, then $N = N_1 \oplus N_2$ is quasi J-pure. Then there exists a J-pure submodule L of M such that $N \subseteq L$ and $(x, 0) \notin L$. Since $ann_R(M_1) + ann_R(M_2) = R$, then $L = L_1 \oplus L_2$ where L_1 and L_2 are submodules in M_1 and M_2, respectively, by remarks and examples (2.2) in the above mentioned study [3]. $N_1 \subseteq L_1$ and $x \notin L_1$. Thus M_1 is quasi J-regular.
Similarly M_2 is quasi J-regular.

Conversely, assume that M_1 and M_2 are quasi J-regular and $M = M_1 \oplus M_2$. Let N be a submodule of $M = M_1 \oplus M_2$. Since $ann_R(M_1) + ann_R(M_2) = R$, then $N = N_1 \oplus N_2$ where N_1 is a submodule in M_1 and N_2 is a submodule of M_2. Since M_1 and M_2 are quasi J-regular, then N_1 and N_2 are quasi J-pure, hence by proposition (2.8), $N_1 \oplus N_2$ is quasi J-pure. Therefore, $N = N_1 \oplus N_2$ is quasi J-pure. Thus M is quasi J-regular.

Recall that a ring R is called a quasi –regular ring if every ideal in R is quasi –pure [4].

Definition (3.7)

Let R be a ring, then R is called a quasi J-regular ring if every ideal in R is quasi J-pure.

Remarks and Examples (3.8)

(1) It is clear that every J-regular ring is quasi J-regular ring. But the converse is not true in general. We have no example.

(2) Every regular ring is quasi J-regular, but the converse is not true in general. For example, Z_9 is quasi J-regular ring but not regular.

(3) Let R be an integral domain. If R is quasi J-regular, then R is a field.

Proof

Since R is quasi J-regular, then every ideal I of R is quasi J-pure. So by proposition (2.6), there exists a collection of J-pure ideals $\{I_\alpha\}_{\alpha \in \Lambda}$ where Λ is some index set, such that $I = \cap_{\alpha \in \Lambda} I_\alpha$. Since R is integral domain, so by a previous study [11, proposition 2.5], R is J-pure simple. Thus R has no J-pure ideals, except $< 0 >$ and R. That is, $\cap_{\alpha \in \Lambda} I_\alpha = < 0 >$ or R, and hence $I = < 0 >$ or I = R. Therefore, R is a field.

Recall that a submodule N of an R- module M is called a maximal submodule of M, if whenever K is a submodule of M with $K \supsetneq N$, then $K = M$ [5].

(4) If R is quasi J -regular ring, then every prime ideal of R that contain in JI where JI intersection of all maximal ideals.

Proof

Let P be a prime ideal in the ring R. Since R is quasi J-regular and $P \subseteq J(R)$, then by a previous study [11, proposition 2.5], $\frac{R}{P}$ is quasi J-regular. But $\frac{R}{P}$ is an integral domain, thus $\frac{R}{P}$ is a field by the above remark (3). Therefore, P is maximal.

(5) Let R be a quasi J-regular ring. If $JI = < 0 >$, then R is regular.

Proof

Since R is quasi J-regular, then by remark (4), every prime ideal of R is maximal. But $J I = < 0 >$, so by an earlier work [12], R is regular.

Recall that a proper ideal I of a ring R is said to be a prime ideal if for each $a, b \in R$ such that $a.b \in I$, then either $a \in I$ or $b \in I$ [5].

(6) Every quasi J-regular ring is nearly regular, where a ring R is called nearly ring if R/J I is regular ring [13].

Proof

Let R be a quasi J-regular ring. Then R/JI is quasi J-regular by the above mentioned study [11, proposition 2.5]. So by the above remark (4), every prime ideal of R/JI is maximal ideal, and since $J(R/J(R)) = 0$, therefore by an earlier work [13], R/JI is regular. Thus R is nearly regular.

(7) The converse of Remark (6) is not true in general. For example, the Z-module Z_8 is nearly regular but not quasi J-regular ring.

Theorem (3.9)

Let M be a faithful finitely generated multiplication R-module. Then M is quasi J-regular module if and only if R is quasi J-regular ring.

Proof

Let N be a submodule of M. Since M is a multiplication R-module, then $N = IM$ for some ideal in R. Since R is quasi J-regular, then $I = \cap_{\alpha \in \Lambda} I_\alpha$ by Proposition (2.6), where I_α is a J-pure ideal of R containing I. Thus $N = (\cap_{\alpha \in \Lambda} I_\alpha)M$. Since M is faithful multiplication, then $(\cap_{\alpha \in \Lambda} I_\alpha)M = \cap_{\alpha \in \Lambda} (I_\alpha M)_\alpha$ [14].

Claim: I M is J-pure in M. Since I is J-pure in R, then

$IJ(M) \cap N = IJ(M) \cap KM$ for some ideal K of R

$= (I \cap K)M$ since M is faithful multiplication.

$= (IK)M$ since R is quasi J-regular ring.

$= I(KM)$

$IJ(M) \cap N = IN$

Thus $N = KM$ is J-pure in M and M is quasi J-regular.

Conversely, let I be an ideal of the ring R. We have to show that I is quasi J-pure. I M is a submodule of M. Since M is quasi J-regular, then by proposition (2.6) $IM = \cap_{\alpha \in \Lambda} L_\alpha$ where L_α is J-pure submodules of M containing I M for each $\alpha \in \Lambda$. Put $L_\alpha = I_\alpha M$. Thus $IM = \cap_{\alpha \in \Lambda} L_\alpha = \cap_{\alpha \in \Lambda} I_\alpha M$, since M is faithful finitely generated, by the $\frac{1}{2}$ cancellation property [12]. Then, $I = \cap_{\alpha \in \Lambda} I_\alpha$.

Claim: I_α is J-pure in R and $I \subseteq I_\alpha$. Let $K \subsetneq J(R)$ be an ideal of R.

$(I_\alpha \cap K)M = I_\alpha M \cap KM$ Since M is faithful multiplication.

$= L_\alpha \cap KM$

$(I_\alpha \cap K)M = I_\alpha KM$

Thus $I_\alpha \cap K = I_\alpha K$ [8], which implies that I_α is J-pure in R. Also, since $I_\alpha M = IM \subseteq L_\alpha$, thus $I \subseteq I_\alpha$, so R is quasi J-regular.

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