On The Diskcyclic Criterions

Rana Abdulkareem*[1], Zeana Zaki Jamil[2]

[1]Department of Mathematics, College of Education for Pure Science/ Ibn Al Haitham,, University of Baghdad, Baghdad, Iraq

[2]Department of Mathematics, College of Science, Baghdad University, Baghdad, Iraq

Abstract

The aim of this work is to give the new types for diskcyclic criterion. We also introduced the case if there is an equivalent relation between a diskcyclic operator $T \oplus T$ and T that satisfies the diskcyclic criterion. Moreover, we discussed the condition that makes T, which satisfies the diskcyclic criterion, a diskcyclic operator.

Keywords*:* Diskcyclic operator, Diskcyclic criterion

<div dir="rtl">

حول معايير الدوارية القرصية

رنا عبد الكريم*[1] ، زينة زكي جميل[2]

[1]قسم الرياضيات, كلية التربية للعلوم الصرفة/ابن الهيثم, جامعة بغداد, بغداد, العراق

[2]قسم الرياضيات, كلية العلوم, جامعة بغداد, بغداد, العراق

الخلاصه

الهدف من هذا العمل هو إعطاء انواع جديدة لمعايير الدوارية القرصية, وكذلك يبين اذ ماكانت هناك علاقة تكافؤية بين   بمعيار الدواري القرصي هو T الذي معيار الدواري القرصي. كما سنناقش الشرط الوافي الذي يجعل دواري قرصيT دواري قرصي و $T \oplus T$

</div>

Introduction

Let H be a separable infinite dimensional Hilbert space and let $T \in B(H)$ be a linear bounded operator. T is said to be a diskcyclic if the orbit of $(T, x) := \{\alpha T^n x : \alpha \in \mathbb{C} ; 0 < |\alpha| \le 1, n \ge 0 \}$ is dense in H [4]. A necessary condition for diskcyclic operators was found by Jamil in 2002 [4], and called the diskcyclic criterion. In 2016, Bamerni [1] provided another version of the diskcyclic criterion, which is simpler than the main diskcyclic criterion.

Jamil and Helal proved, in 2013, the equivalent between diskcyclic criterion and other theorems (Three open set conditions for disk cyclic) [3]. Also, Bamerni [2] proved another equivalent statement to diskcyclic criterion.

We will find and improve a new type of diskcyclic criterion and provide a characterization theorem for $T \oplus T$ to be diskcyclic operator.

This paper consists of two sections:

In section one we introduce theorems, some of which exist while the others will be achieved, that offer the necessary conditions for an operator to be diskcyclic operator.

In section two we present equivalent statements between theorems in section one, with $T \oplus T$ to be diskcyclic operator. Also, we discuss the case when a diskcyclic operator satisfies the diskcyclic criterion. We used the abbreviation B to express the unit ball.

_____

*Email: rorrrana_2006@yahoo.com

1- Diskcyclic Criterion

This section takes a new look at the diskcyclic criterion. But first we will refer to the first theorem which gives the conditions for the diskcyclic criterion.

Theorem 1.1: (Diskcyclic Criterion) [4]

Suppose that T $\in$ *B(H)*. If there exists a dense linear subspace *X,Y* in *H* and a right inverse operator S to T (not necessarily bounded), such that:

1) $S(Y) \subset Y$ and $TS = I_y$.

2) There exists an increasing sequence of positive integers $\{n_k\}$ in $\mathbb{N}$, as $k \to \infty$ such that

a) $\lim_{k\to\infty} \|S^{n_k} y\| = 0$ for all $y \in Y$

b) $\lim_{k\to\infty} \|T^{n_k}x\| \|S^{n_k} y\| = 0$ for all $x \in X, \ y \in Y$.

then T is a diskcyclic operator.

The following theorem gives a second version to diskcyclic criterion.

Theorem 1.2: (Second Diskcyclic Criterion) [1].

Suppose that T $\in$ *B(H)*. If there exists a dense linear subspace *X,Y* in *H* and a right inverse operator S to T (not necessarily bounded), and there exists an increasing sequence of positive integers $\{n_k\}$ in $\mathbb{N}$, as $k \to \infty$ such that :

1)　S such that : $S(Y) \subset Y$ and $T^{n_k}S^{n_k} \to I_y$.

2)　a) $\|S^{n_k} y\| \to 0$ for all $y \in Y$.

b) $\|T^{n_k}x\| \|S^{n_k} y\| \to 0$ for all $x \in X, y \in Y$.

then T is a diskcyclic operator.

In the next theorem, we show that an operator that satisfies $\{\alpha_{n_k} T^{n_k}x\}$ is dense in *H*, where T is an operator that satisfies certain conditions.

Theorem 1.3: (**Diskcyclic Criterion)(III)[1]**

Suppose that T $\in$ *B(H)*. If there exists a dense linear subspaces *X,Y* in *H* such that there exists an increasing sequence of positive integers $\{n_k\}$ in $\mathbb{N}$ and $\{\alpha_{n_k}\} \subset (0,1]$ for all $k \in \mathbb{N}$, a dense subset $X \subset H$ such that $\|\alpha_{n_k} T^{n_k}x\| \to 0$ for all $x \in X$, as $k \to \infty$

1) a dense subset $Y \subset H$ and a sequence of mappings $S_{n_k}: Y \to H$ such that

2) a) $\left\|\frac{1}{\alpha_{n_k}} S_{n_k}y\right\| \to 0$　for all $y \in Y$

　b) $T^{n_k}S_{n_k}y \to y$　for all $y \in Y$

then $\{\alpha_{n_k} T^{n_k}x\}$ is dense in *H* for some $x \in H$. In particular, x is a diskcyclic vector for T.

In order to give new a version for diskcyclic criterion, we need the following theorem which is a fundamental in what follow.

Theorem 1.4: [4, theorem (4.2.4)]

Let T $\in$ *B(H)*. The following statements are equivalent:

1)　*T* is a diskcyclic operator.

2)　For each non-empty open sets *U, V,* there are $\alpha \in \mathbb{C}$; $0 < |\alpha| \le 1$ and $n \in \mathbb{N}$ such that $T^n(\alpha U) \cap V \ne \emptyset$.

3)　For each $x,y \in H$, there are sequences $\{x_k\}$ in $H, \{n_k\}$ in $\mathbb{N}$, and $\{\alpha_k\}$ in $\mathbb{C}$; $0 < |\alpha_k| \le 1$ for all *k*, such that $x_k \to x$ and $T^{n_k}\alpha_k x_k \to y$.

4)　For each $x,y \in H$ and each neighborhood *W* for zero in *H*, there are $z \in H, n \in \mathbb{N}$, and $\alpha \in \mathbb{C}$; $0 < |\alpha| \le 1$, such that $x - z \in W$ and $T^n\alpha z - y \in W$.

Theorem 1.5: (Diskcyclic Criterion)(IV)

Suppose that T $\in$ *B(H)*. If there exists an increasing sequence of positive integers $\{n_k\}$ in $\mathbb{N}$ and $\{\alpha_{n_k}\} \subset (0,1]$, for which there are a dense subsets *Y, X* in *H* and a sequence of mappings, $S_{n_k}: Y \to H$, as $k \to \infty$ such that:

1)　　$\alpha_{n_k} T^{n_k}x \to 0$ for all $x \in X$

2)　　a) $\frac{1}{\alpha_{n_k}} S_{n_k}y \to 0$　for all $y \in Y$

　b) $T^{n_k}S_{n_k}y \to y$　for all $y \in Y$,

then T is diskcyclic operator.

Proof Let *U, V* be non empty open sets of *H*, and let $x \in X \cap U, \ y \in Y \cap V$.

By (2(a)) we get $x + \frac{1}{\alpha_{n_k}} S_{n_k} y \to x \in U$, where, $\alpha_{n_k} T^{n_k} \left(x + \frac{1}{\alpha_{n_k}} S_{n_k} y \right) = \alpha_{n_k} T^{n_k} x + T^{n_k} S_{n_k} y \to$ $y \in V$. Thus for k is large enough $\alpha_{n_k} T^{n_k}(U) \cap V \neq \emptyset$. Then by theorem (1.4), T is a diskcyclic operator.

Theorem 1.6: (Outer Diskcyclic Criterion)

Suppose that $T \in B(H)$. There exists an increasing sequence of positive integers $\{n_k\}$ in $\mathbb{N}$. If there exists a dense linear subspace Y, while for every $y \in Y$ there is a dense linear subspace X_y in H and there exist mappings $S_{n_k} : Y \to H$, as $k \to \infty$ such that:

1)　　　$T^{n_k} S_{n_k} y \to y$ for all $y \in Y$.

2)　a) $\| T^{n_k} x \| \| S_{n_k} y \| \to 0,$　　　for all $y \in Y$ and $x \in X_y$

　　b) $\| S_{n_k} x \| \to 0$ for all $x \in X_y$. then T is a diskcyclic operator.

Proof

Let g, h $\in H$ and W be a 0-neighborhood, and let $\epsilon > 0$ such that $\epsilon B \subset W$.

Since Y is a dense set in H, we take $y \in Y$ such that $\|h - y\| \leq \frac{\epsilon}{4}$ and $\|y - T^{n_k} S_{n_k} y\| \leq \frac{\epsilon}{4}$.

And since X_y is a dense set for every $y \in Y$, let $x \in X_y$ such that $\|g - x\| \leq \frac{\epsilon}{2}$.

By (2) there exists $n_k > 0$ such that $\|S_{n_k} y\| \leq \frac{\epsilon}{2}$ and $\| T^{n_k} x \| \| S_{n_k} y \| \leq \frac{\epsilon^2}{4}$.

We put $\alpha = \frac{2}{\epsilon} \| S_n y \|$ thus $0 < \alpha \leq 1$. Hence $(1 \backslash \alpha) \| S_{n_k} y \| = \frac{\epsilon}{2}$.

Let $z = x + (1 \backslash \alpha) S_{n_k} y$.

Then, $\|g - z\| = \|g - x - (1 \backslash \alpha) S_{n_k} y \| \leq \|g - x\| + (1 \backslash \alpha) \| S_{n_k} y \| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

By using the fact that $T^{n_k} S_{n_k} y \to y$, we have

$\|h - \alpha T^{n_k} z\| = \|h - \alpha T^{n_k} x - T^{n_k} S_{n_k} y \| \leq \|h - T^{n_k} S_{n_k} y\| + \alpha \| T^{n_k} x \|$

　　　　$= \|h - y\| + \|y - T^{n_k} S_{n_k} y\| (2 \backslash \epsilon) \| S_n y \| \| T^n x \| \leq \epsilon.$

Then $g - z \in W$ and $h - \alpha T^{n_k} z \in W$, so T is a diskcyclic operator by theorem (1.4).

Since the proof of the next result is similar to the proof of (1.6), hence it is omitted.

Theorem 1.7: (An Inner Diskcyclic Criterion)

Suppose that $T \in B(H)$. There exists an increasing sequence of positive integers $\{n_k\}$ in $\mathbb{N}$. If there exists a dense linear subspace Y in H and for every $y \in Y$ there is a dense linear subspace X_y in H such that:

1)　　　there exists function $S_{y,n_k} : X_y \to H$, as $k \to \infty$ such that $T^{n_k} S_{y,n_k} x \to x$ for all $x \in X_y$

2)　a) if $y \in Y$ and $x \in X_y$, then $\|T^{n_k} y\| \|T^{n_k} S_{y,n_k} x\| \to 0$

　　b) $\| S_{n_k} x \| \to 0$ for all $x \in X$.

then T is a diskcyclic operator.

2-　　　Some Equivalent Relations on Diskcyclic Operator.

Since every operator that satisfies diskcyclic criterion is diskcyclic operator but the reverse is incorrect, as shown in the proposition (3.2.8) in [1], and if $T \oplus T$ is diskcyclic operator (then T is diskcyclic) but the reverse is incorrect as evident in proposition (4.3.17) in [1], then the aim of this section is to find the equivalent between the statement that $T \oplus T$ is diskcyclic operator and the theorems in section one, as we showed in theorem (2.1). Since (4.3.15) in [1] proved the parts $(a \Rightarrow b)$ and $(b \Rightarrow c)$ are trivials, then we omit its proof. Also, we discuss the case when a diskcyclic operator satisfies the diskcyclic criterion.

Theorem 2.1

Let H be a separable, infinite, dimensional Hilbert space and let $T \in B(H)$.Then the followings　　are equivalent :

a) $T \oplus T$ is a diskcyclic operator.

b) T satisfies the Diskcyclic Criterion (IV).

c) T satisfies the Diskcyclic Criterion (III).

d) T satisfies the Second Diskcyclic Criterion.

e) T satisfies the Outer Diskcyclic Criterion.

f) T satisfies the Inner Diskcyclic Criterion

Proof

c) $\Rightarrow$ d): Let T satisfies the diskcyclic criterion (III) with respect to the sequence $\{n_k\}$ in $\mathbb{N}$ and $\{\alpha_{n_k}\} \subset (0,1]$ for all $k \in \mathbb{N}$, a sequence maps $S_{n_k}: Y \to H$, as $k \to \infty$ and dense subsets X, $Y \subset H$ such that $\|\alpha_{n_k} T^{n_k} x\| \to 0$ for all $x \in X$. If we assume $S^{n_k} = S_{n_k}$.

So since $\{\alpha_{n_k}\}$ is a bounded sequence and $\left\|\frac{1}{\alpha_{n_k}} S_{n_k} y\right\| \to 0$,

then $\alpha_{n_k} \left\|\frac{1}{\alpha_{n_k}} S_{n_k} y\right\| \to 0$, hence $\|S^{n_k} y\| \to 0$ for all $x \in X$. Also

$\|\alpha_{n_k} T^{n_k} x\| \left\|\frac{1}{\alpha_{n_k}} S_{n_k} y\right\| \to 0$, then $\|T^{n_k} x\| \|S^{n_k} y\| \to 0$ for all $x \in X$, $y \in Y$

d) $\Rightarrow$ e): Let T satisfies the second diskcyclic criterion. If we assume that $X = X_y$ for all $y \in Y$, where Y is a subspace is dense in H and $S^{n_k} = S_{n_k}$, then it immediately follows that T satisfies the Outer diskcyclic criterion

e) $\Rightarrow$ f): Let T satisfies the Outer diskcyclic criterion. If we assume that $Y = X_y$ for all $y \in Y$, where Y is adense subspace in H and $S_{n_k} = S_{y,n_k}$

f) $\Rightarrow$ a): Let U_i, $V_i \subset H$ be non empty open sets with ($i = 1, 2$). We have to show that there exist $\alpha \in \mathbb{C}$; $0 < |\alpha| \leq 1$ and $m \in \mathbb{N}$ such that
$$\alpha T^m(U_i) \cap V_i \neq \emptyset, \alpha \in \mathbb{C}; 0 < |\alpha| \leq 1 \quad \text{for} \quad i = 1, 2.$$
If we fix $y_i \in Y \cap V_i$ then there exists $X_{y_i} \subset H$ dense such that
$$\lim_{k \to \infty} \|T^{n_k} y_i\| \|S_{y,n_k} x\| = 0, \text{for all } y \in Y \text{ for } i = 1, 2.$$
By passing to the subsequence, if necessary, we suppose, without loss of generality, that $\|T^{n_k} y_1\| \leq \|T^{n_k} y_2\|$ for all $k \in \mathbb{N}$. Thus
$$\lim_{k \to \infty} \|T^{n_k} y_1\| \|S_{y,n_k} x\| \leq \lim_{k \to \infty} \|T^{n_k} y_2\| \|S_{y,n_k} x\| = 0 \quad \text{for all } x \in X_{y_2}$$
Given that $x_i \in X_{y_2} \cap U_i$ we select $m \in \mathbb{N}$ (large enough) and $\epsilon > 0$ such that
$y_i + \epsilon B \subset V_i$, $T^m S_{y,m} x_i + \epsilon B_H \subset U_i$, and $\|T^m y_i\| \|S_{y,m} x_j\| < \epsilon^2$ for $i, j = 1, 2$.
Since $\|S_{y,n} x\| \to 0$, then we define $\alpha = \|S_{y,m} x_j\|/\epsilon$. This implies that $\|\alpha T^m y_i\| < \epsilon$; $i = 1, 2$.
And, therefore, $\alpha T^m \left(y_i + \frac{1}{\alpha} S_{y,m} x_i\right) = \alpha T^m y_i + T^m S_{y,m} x_i \in U_i \cap \alpha T^m(V_i)$, for $i = 1, 2$.

The following proposition characterizes that if T is a diskcyclic operator then T satisfies the diskcyclic criterion under dense generalized kernel, i.e. $\bigcup_{n=1}^{\infty} \ker T^n$ is a dense set in H.

Proposition 2.2

Let H be a separable, infinite dimensional, Hilbert space and let $T \in B(H)$ with a dense generalized kernel. Then T is a diskcyclic operator if and only if T satisfies the diskcyclic criterion.

Proof

By (1.5), the necessary condition was achieved.

Conversely, let T be a diskcyclic operator and $y \in H$. Since $0 \in H$ then by (1.4) there are $\{x_k\}$ in H, $\{n_k\}$ in $\mathbb{N}$, and $\{\alpha_{n_k}\}$ in $\mathbb{C}$; $0 < |\alpha_{n_k}| \leq 1$ for all $k \in \mathbb{N}$, such that $x_k \to 0$ and $T^{n_k} \alpha_{n_k} x_k \to y$.

Let $X := \bigcup_{n=1}^{\infty} \ker T^n$ and $Y := \text{Dorb}(T, x)$, then X and Y are dense sets in H. We define $S_{n_k}: Y \to H$, by $S_{n_k}(\lambda T^r y) = \alpha_{n_k} T^r(x_k)$; $r \in \mathbb{N}$

Then: $\alpha_{n_k} T^{n_k} x \to 0$ for all $x \in X$.

$\frac{1}{\alpha_{n_k}} S_{n_k}(\lambda T^r y) = \frac{1}{\alpha_{n_k}} \alpha_{n_k} T^r(x_k) \to 0$.

$T^{n_r} S_{n_r}(\lambda T^r y) = \lambda T^{n_r} S_{n_r}(T^r y) = \lambda T^{n_r}(\alpha_{n_k} T^r(x_k)) = \lambda T^r(\alpha_{n_k} T^{n_r}(x_k)) \to \lambda T^r y$.

So, T satisfies the diskcyclic criterion.

References
1. Bamerni N. **2016**. Cyclicity Of Bunded linear Operators on Separable Banach Spaces and Closed Subspaces; Ph.D. Thesis, University of Putra Malaysia.
2. Bamerni N. and Kilicman A. **2016**. K-diskclicity operators on Banach Space, *AIP Conference Proceedings*, **1739**(1): 020056.
3. Jamil Z. and Helal M. **2013**. Equivalent between the Criterion and the Three Open Set`s Condition`s in Disk Cyclicity, *Int.J.Contemp.Math.Scineces*, **8**(6): 257-261.
4. Jamil Z. **2002**. Cyclic Phenomena of Operators on Hilbert Space; Ph.D. Thesis, University of Baghdad