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Asymptotic Criteria of Neutral Differential Equations with Positive and Negative Coefficients and Impulsive Integral Term

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Abstract

In this paper, the asymptotic behavior of all solutions of impulsive neutral differential equations with positive and negative coefficients and with impulsive integral term was investigated. Some sufficient conditions were obtained to ensure that all nonoscillatory solutions converge to zero. Illustrative examples were given for the main results.

Keywords: Neutral Differential Equations, Asymptotic Behavior, impulses effect, Impulsive Integral Term, Forcing Term.

السلوك المحاذي للمعادلات التفاضلية المحايدة ذات المعاملات الموجبة والسالبة وبشرط تكاملي نابض

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الخلاصة

في هذا البحث، تم دراسة السلوك المحاذي لجميع الحلول للمعادلات التفاضلية المحايدة ذات المعاملات الموجبة والسالبة وحد التكامل النابض. تم الحصول على الشروط الكافية لتقارب جميع الحلول غير المتذبذبة إلى الصفر. تم إعطاء أمثلة توضيحية للنتائج.

Introduction

The differential equations with impulses effect describe the process of evolution that rapidly changes its state at certain moments. Therefore, this type of differential equations is suitable for the mathematical simulation of the evolutionary process in which the parameters are subject to relatively long periods of smooth variation followed by a rapid short-term change, and this is a jump in their values. The wide possibility of applications determines the increasing interest in impulsive differential equations. The importance of the need to study differential equations with impulsive effect is due to the fact that these equations are more comprehensive in their use of mathematical modeling, where gaps in the model can be addressed by limiting these gaps in specific points called the points of impulses effect. They were appeared in many real processes and phenomena, such as control theory, biology, mechanics, medicine, electronic, economic, etc.

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For instance, their applications include neural networks [1-3], control theory [4], economics [5, 6], communication security [7, 8], population dynamics [9] and medicine [10]. As a result, many papers were published for numerous impulsive neutral differential equations [11-13].

Guan and Shen [14] investigated the asymptotic criteria for the impulsive neutral differential equation of Euler form, as follows:

$$\left. \begin{aligned} [y(t) - P(t)y(\sigma t)]' + \frac{Q(t)}{t}y(\alpha t) &= 0, \quad t \neq t_k \\ y(t_k^+) &= c_k y(t_k) - (1 - c_k) \int_{\alpha t_k}^{t_k} \frac{Q(u)}{u} y(u) du, \quad k \in Z_+ \end{aligned} \right\}$$

Pandian and Balachandran [15] studied the following differential equation with impulsive condition:

$$\left. \begin{aligned} [y(t) + P(t)g(y(\tau(t)))]' + \frac{Q(t)}{t}f(y(\sigma(t))) - \frac{R(t)}{t}f(y(\alpha(t))) &=, t \neq t_k, \quad k = 1, 2, \dots \\ y(t_k^+) &= c_k y(t_k) + (1 - c_k) \left(\int_{\sigma(t_k)}^{t_k} \frac{Q(\sigma^{-1}(u))}{u} f(y(u)) du - \int_{\alpha(t_k)}^{t_k} \frac{Q(\alpha^{-1}(u))}{u} f(y(u)) du \right) \end{aligned} \right\}$$

Under some sufficient conditions, they showed that all solutions of this equation tend to constant or to zero.

The aim of this paper is to obtain some sufficient conditions to guarantee the convergence of all solutions of neutral differential equation with positive and negative coefficients and impulsive integral term of the form:

$$\left. \begin{aligned} [y(t) - P(t)y(\tau(t))] + Q(t)y(\sigma(t)) - R(t)y(\alpha(t)) &= 0, \quad t \neq t_k \quad k = 1, 2, \dots \\ y(t_k^+) &= a_k y(t_k) - (1 - a_k) \int_{\sigma^{-1}(\alpha(t_k))}^{t_k} Q(u)y(\sigma(u)) du, \quad t = t_k \quad k = 1, 2, \dots \end{aligned} \right\} \quad (1.1)$$

$$\left. \begin{aligned} [y(t) - P(t)y(\tau(t))] + Q(t)y(\sigma(t)) - R(t)y(\alpha(t)) &= f(t), t \neq t_k, \quad k = 1, 2, \dots \\ y(t_k^+) &= a_k y(t_k) + (1 - a_k) \left(\int_{t_k}^{\sigma^{-1}(\alpha(t_k))} Q(u)y(\sigma(u)) du \right), t = t_k, \quad k = 1, 2, \dots \end{aligned} \right\} \quad (1.2)$$

where a_k are positive real numbers and t_k are the moments of impulses effect. Let $P \in PC([t_0, \infty); R^+)$, where $PC(X; Y) = \{h: X \rightarrow Y: h(t) \text{ is continuous for } t \in X \text{ and } t \neq t_k\}$. We assume that $h(t_k^+) \lim_{t \rightarrow t_k^+} h(t_k)$ and $h(t_k^-) = \lim_{t \rightarrow t_k^-} h(t_k)$ exist with $h(t_k^-) = h(t_k)$. Let $Q, R \in C([t_0, \infty); R^+)$, $k = 1, 2, \dots$, and $\tau, \sigma, \alpha < t$ are continuous strictly increasing functions with $\lim_{t \rightarrow \infty} \tau(t) = \infty, \lim_{t \rightarrow \infty} \alpha(t) = \infty, \lim_{t \rightarrow \infty} \sigma(t) = \infty$. The functions $\tau^{-1}(t), \sigma^{-1}(t), \alpha^{-1}(t)$ are the inverse of the functions $\tau(t), \alpha(t), \sigma(t)$, respectively,

We define the initial function $y(t) = \omega(t), t \in [\rho(t_0), t_0], \rho(t) = \min\{\tau(t), \alpha(t), \sigma(t)\}, t \geq t_0$, where $\omega(t) \in PC(\rho(t_0), R)$.

A function $y(t)$ is said to be a solution of (1.1) which satisfies the initial condition if

H1: $y(t) = \omega(t)$ for $\rho(t_0) \leq t \leq t_0$ and $y(t)$ is continuous for $t \geq t_0$ and $t \neq t_k, k = 1, 2, \dots$.

H2: $y(t) - P(t)y(\tau(t))$ is continuously differentiable $t \geq t_0$ and $t \neq t_k, t \neq \tau^{-1}(t_k), t \neq \sigma^{-1}(t_k), t \neq \alpha^{-1}(t_k) k = 1, 2, \dots$, which satisfies the first equation.

H3: $y(t_k^+)$ and $y(t_k^-)$ exist with $y(t_k^-) = y(t_k)$ and satisfy the impulsive differential equation in (1.1) or (1.2), where

$$y(t_k^+) = \lim_{t \rightarrow t_k^+} y(t_k) \text{ and } y(t_k^-) = \lim_{t \rightarrow t_k^-} y(t_k).$$

2. Asymptotic behavior of INDEPNC (1.1)

In this section, some sufficient conditions are obtained to ensure that all the solutions of impulsive neutral differential equations with positive and negative coefficients (INDEPNC) in the form of Eq.(1.1) converge to zero.

Theorem 2.1

Assume that $0 \leq P(t) \leq p_1 < 1, P(t_k^+) \geq P(t_k), 0 < a_k \leq 1$, and $\tau(t_k)$ is not an impulsive point, let

$$W(t) = y(t) - P(t)y(\tau(t)) + \int_{\sigma^{-1}(\alpha(t))}^t Q(u)y(\sigma(u))du, \quad t \in (t_k, t_{k+1}]. \quad (2.1)$$

where $t_k < \sigma^{-1}(\alpha(t)) \leq t \leq t_{k+1}$, in addition to:

$$Q(\sigma^{-1}(\alpha(t))) (\sigma^{-1}(\alpha(t)))' - R(t) \geq 0, \quad t \in (t_k, t_{k+1}]. \quad (2.2)$$

$$\lim_{t \rightarrow \infty} \int_{\sigma^{-1}(\alpha(t))}^t Q(u)du = 0. \quad (2.3)$$

Then every nonoscillatory solution of Eq.(1.1) converges to zero as $t \rightarrow \infty$.

Proof

Suppose that Eq.(1.1) has a nonoscillatory impulsive solution $y(t) > 0, y(\tau(t)) > 0, y(\sigma(t)) > 0, y(\alpha(t)) > 0, t \in (t_k, t_{k+1}], k = 1, 2, \dots$. Differentiate (2.1) for every interval $(t_k, t_{k+1}]$, $k = 1, 2, \dots$ and using Eq.(1.1) we get

$$\begin{aligned} W'(t) &= [y(t) - P(t)y(\tau(t))] + Q(t)y(\sigma(t)) - Q(\sigma^{-1}(\alpha(t)))y(\alpha(t))(\sigma^{-1}(\alpha(t)))' \\ &= -Q(t)y(\sigma(t)) + R(t)y(\alpha(t)) + Q(t)y(\sigma(t)) - Q(\sigma^{-1}(\alpha(t)))y(\alpha(t))(\sigma^{-1}(\alpha(t)))' \\ &= -\left(Q(\sigma^{-1}(\alpha(t))) (\sigma^{-1}(\alpha(t)))' - R(t)\right) y(\alpha(t)) \leq 0. \end{aligned} \quad (2.4)$$

Hence $W(t)$ is nonincreasing for $t_k < t \leq t_{k+1}$, $k = 1, 2, \dots$.

Next, we aim at $W(t_k^+) \leq W(t_k)$ for $k = 1, 2, \dots$. We have $0 < a_k \leq 1$ and $\tau(t_k) \neq t_i, i < k = 1, 2, \dots$ then:

$$\begin{aligned} W(t_k^+) &= y(t_k^+) - P(t_k^+)y(\tau(t_k)) + \int_{\sigma^{-1}(\alpha(t_k))}^{t_k} Q(u)y(\sigma(u))du \\ &\leq a_k y(t_k) - (1 - a_k) \int_{\sigma^{-1}(\alpha(t_k))}^{t_k} Q(u)y(\sigma(u))du - P(t_k)y(\tau(t_k)) + \int_{\sigma^{-1}(\alpha(t_k))}^{t_k} Q(u)y(\sigma(u))du, \\ &= a_k y(t_k) + a_k \int_{\sigma^{-1}(\alpha(t_k))}^{t_k} Q(u)y(\sigma(u))du - P(t_k)y(\tau(t_k)) \leq W(t_k). \end{aligned}$$

Hence $W(t)$ is nonincreasing on $[t_0, \infty)$. Hence $-\infty \leq \lim_{t \rightarrow \infty} W(t) < \infty$. We claim that $y(t)$ is bounded, and $-\infty < \lim_{t \rightarrow \infty} W(t) < \infty$.

If $y(t)$ is unbounded then there exists a sequence $\{t_n\}$ such that $\lim_{n \rightarrow \infty} t_n = \infty, \lim_{n \rightarrow \infty} y(t_n) = \infty, y(t_n) = \max\{y(s), \tau(t_0) \leq s \leq t_n\}$. Then by (2.1) it follows that:

$$W(t_n) \geq y(t_n) - p_1 y(t_n) = (1 - p_1)y(t_n) \quad (2.5)$$

As $n \rightarrow \infty$, we get $\lim_{n \rightarrow \infty} W(t_n) = \infty$, which is a contradiction. Hence $y(t)$ is bounded. It follows that $W(t)$ is bounded.

If $W(t) \leq 0$,

$$\begin{aligned} 0 \geq W(t) &\geq y(t) - P(t)y(\tau(t)) > y(t) - y(\tau(t)), \quad t \in (t_k, t_{k+1}] \\ y(t) &< y(\tau(t)), \quad t \in (t_k, t_{k+1}] \end{aligned}$$

Hence $y(t)$ is bounded and decreasing on every interval $(t_k, t_{k+1}]$. Let $\lim_{t \rightarrow \infty} y(t) = l \geq 0$. We claim that $l = 0$, otherwise $l > 0$, hence

$$\begin{aligned} W(t) &\geq y(t) - P(t)y(\tau(t))W(t) \geq y(t) - p_1 y(\tau(t)) \\ \lim_{t \rightarrow \infty} W(t) &\geq l - p_1 l = (1 - p_1)l > 0, \text{ which is a contradiction. Thus } \lim_{t \rightarrow \infty} W(t) = \lim_{t \rightarrow \infty} y(t) = 0. \end{aligned}$$

However, in this case $W(t) \rightarrow 0$ is impossible since $W(t)$ is negative and nonincreasing.

If $W(t) \geq 0$, let $y(t) \leq M, \lim_{t \rightarrow \infty} W(t) = L \geq 0, W(t) \geq L$. We claim that $L = 0$, otherwise $L > 0$.

Using (2.3) in (2.1) we get

$$\begin{aligned} W(t) &= y(t) - P(t)y(\tau(t)) + \int_{\sigma^{-1}(\alpha(t))}^t Q(u)y(\sigma(u))du, \quad t \in (t_k, t_{k+1}] \\ &\leq y(t) + M \int_{\sigma^{-1}(\alpha(t))}^t Q(u)du < y(t) + \varepsilon, \quad \varepsilon > 0. \end{aligned}$$

Then for large enough t we get

$$W(t) \leq y(t), \quad t \in (t_l, t_{l+1}], \quad t_l \geq t_k. \tag{2.6}$$

Choose $\gamma(t) > t$, such that for large enough $t_j \geq t_l$, we obtain $W(\gamma(t)) - W(t) = 0$, for $t \in (t_j, t_{j+1}]$

By integrating (2.4) from t to $\gamma(t)$ we get

$$\begin{aligned} W(\gamma(t)) - W(t) &= - \int_t^{\gamma(t)} \left(Q(\sigma^{-1}(\alpha(s))) (\sigma^{-1}(\alpha(s)))' - R(s) \right) y(\alpha(s)) ds \\ &\leq - \int_t^{\gamma(t)} \left(Q(\sigma^{-1}(\alpha(s))) (\sigma^{-1}(\alpha(s)))' - R(s) \right) W(\alpha(s)) ds \\ &\leq -W(\alpha(\gamma(t))) \int_t^{\gamma(t)} \left(Q(\sigma^{-1}(\alpha(s))) (\sigma^{-1}(\alpha(s)))' - R(s) \right) ds \end{aligned}$$

For large enough t , it follows that:

$$0 \leq -L \int_t^{\gamma(t)} \left(Q(\sigma^{-1}(\alpha(s))) (\sigma^{-1}(\alpha(s)))' - R(s) \right) ds$$

Which is a contradiction. Thus, $L = 0$. Assume that $\limsup_{t \rightarrow \infty} y(t) = M_1 \geq 0$. So there exists a sequence $\{t_m\}$ such that $\lim_{m \rightarrow \infty} t_m = \infty$, $\lim_{m \rightarrow \infty} y(t_m) = M_1$. We claim that $M_1 = 0$, otherwise $M_1 > 0$. Then it follows that

$$W(t_m) \geq y(t_m) - p_1 y(\tau(t_m)).$$

As $m \rightarrow \infty$, $0 \geq M_1 - p_1 M_1 = (1 - p_1)M_1$, it yields that $M_1 \leq 0$, which is a contradiction.

Hence $\limsup_{t \rightarrow \infty} y(t) = 0$, which implies that $\lim_{t \rightarrow \infty} y(t) = 0$. \square

Theorem 2.2

Let $W(t)$ be defined as in (2.1) and let $a_k \geq 1$, $1 < p_2 \leq P(t) \leq p_3$, $P(t_k^+) \leq P(t_k)$ and $\tau(t_k)$ is not an impulsive points, in addition to:

$$\int_T^\infty Q(t)dt < \infty, \tag{2.7}$$

$$R(u) - Q(\sigma^{-1}(\alpha(u))) (\sigma^{-1}(\alpha(u)))' \geq 0, \tag{2.8}$$

$$\int_T^\infty \left(R(u) - Q(\sigma^{-1}(\alpha(u))) (\sigma^{-1}(\alpha(u)))' \right) du = \infty. \tag{2.9}$$

Then every nonoscillatory solution of Eq.(1.1) converges to zero as $t \rightarrow \infty$.

Proof

Suppose that $y(t) > 0, y(\tau(t)) > 0, y(\sigma(t)) > 0, y(\alpha(t)) > 0, t \in (t_k, t_{k+1}]$. Differentiate (2.1) and use (1.1) for every interval $(t_k, t_{k+1}]$, $k = 1, 2, \dots$

$$\begin{aligned} W'(t) &= [y(t) - P(t)y(\tau(t))] + Q(t)y(\sigma(t)) - Q(\sigma^{-1}(\alpha(t)))y(\alpha(t))(\sigma^{-1}(\alpha(t)))' \\ &= -Q(t)y(\sigma(t)) + R(t)y(\alpha(t)) + Q(t)y(\sigma(t)) \\ &\quad - Q(\sigma^{-1}(\alpha(t)))y(\alpha(t))(\sigma^{-1}(\alpha(t)))' \\ &= \left(R(t) - Q(\sigma^{-1}(\alpha(t))) (\sigma^{-1}(\alpha(t)))' \right) y(\alpha(t)) \geq 0. \end{aligned} \tag{2.10}$$

Hence $W(t)$ is nondecreasing for $t_k < t \leq t_{k+1}$ for $k = 1, 2, \dots$

Next, we need to prove that $W(t_k^+) \geq W(t_k)$ for $k = 1, 2, \dots$. Since $a_k \geq 1$ and $\tau(t_k) \neq t_i, i < k, k = 1, 2, \dots$, it follows that:

$$\begin{aligned} W(t_k^+) &= y(t_k^+) - P(t_k^+)y(\tau(t_k)) + \int_{\sigma^{-1}(\alpha(t_k))}^{t_k} Q(u)y(\sigma(u))du \\ &\geq a_k y(t_k) - (1 - a_k) \int_{\sigma^{-1}(\alpha(t_k))}^{t_k} Q(u)y(\sigma(u))du - P(t_k)y(\tau(t_k)) + \int_{\sigma^{-1}(\alpha(t_k))}^{t_k} Q(u)y(\sigma(u))du \\ &= a_k y(t_k) + a_k \int_{t_k}^{\sigma^{-1}(\alpha(t_k))} Q(u)y(\sigma(u))du - P(t_k)y(\tau(t_k)) \\ &\geq W(t_k). \end{aligned}$$

So, $W(t)$ is nondecreasing on $[t_0, \infty)$.

Hence $-\infty < \lim_{t \rightarrow \infty} W(t) \leq \infty$. We claim that $y(t)$ is bounded, otherwise there exists a sequence $\{t_n\}$ such that $\lim_{n \rightarrow \infty} t_n = \infty$, $\lim_{n \rightarrow \infty} y(t_n) = \infty$, $y(t_n) = \max\{y(s), t_{-1} \leq s \leq \tau^{-1}(t_n)\}$, $t_{-1} = \min\{\tau(t_0), \sigma^{-1}(\alpha(t_0))\}$. Condition (2.7) implies that $\lim_{t \rightarrow \infty} \int_{\sigma^{-1}(\alpha(\tau^{-1}(t)))}^{\tau^{-1}(t)} Q(u) du = 0$, and $\lim_{t \rightarrow \infty} \int_{\sigma^{-1}(\alpha(t))}^t Q(u) du = 0$. From (2.1) it follows that:

$$\begin{aligned} W(\tau^{-1}(t_n)) &= y(\tau^{-1}(t_n)) - P(\tau^{-1}(t_n))y(t_n) + \int_{\sigma^{-1}(\alpha(\tau^{-1}(t_n)))}^{\tau^{-1}(t_n)} Q(u)y(\sigma(u))du, \\ &\leq y(t_n) - P(\tau^{-1}(t_n))y(t_n) + y(t_n) \int_{\sigma^{-1}(\alpha(\tau^{-1}(t_n)))}^{\tau^{-1}(t_n)} Q(u)du, \\ &< (1 - p_2 + \varepsilon)y(t_n). \end{aligned}$$

As $n \rightarrow \infty$ and for $\varepsilon \in (0, p_2 - 1)$, the last inequality implies that $\lim_{t \rightarrow \infty} W(t) = -\infty$, which is a contradiction. So, $y(t)$ is bounded, thus $W(t)$ is bounded. Let $\liminf_{t \rightarrow \infty} y(t) = l \geq 0$, if $l > 0$, then there exists $\mu > 0$ such that $y(t) \geq \mu > 0$ for $t \in (t_l, t_{l+1}]$, $l \geq k$. Integrating (2.4) from T to t , where $t_l < t \leq t_{l+1}$, $T \geq t_1$, yields

$$\begin{aligned} W(t) - W(T) &= \int_T^t \left(R(u) - Q(\sigma^{-1}(\alpha(u))) (\sigma^{-1}(\alpha(u))) \right) y(\alpha(u)) du \\ &\geq \mu \int_T^t \left(R(u) - Q(\sigma^{-1}(\alpha(u))) (\sigma^{-1}(\alpha(u))) \right) du \end{aligned}$$

Regardless of the sign of $W(t)$, as $t \rightarrow \infty$, by using (2.9), the last inequality leads to a contradiction.

We conclude that $\liminf_{t \rightarrow \infty} y(t) = 0$. It remains to show that $\limsup_{t \rightarrow \infty} y(t) = 0$

Let $\limsup_{t \rightarrow \infty} y(t) = M_1 > 0$, hence for large enough t , we get $y(t) \leq M_1$. Let $\lim_{t \rightarrow \infty} W(t) = l$.

If $W(t) > 0$

$$\begin{aligned} W(t_n) &= y(t_n) - P(t_n)y(\tau(t_n)) + \int_{\sigma^{-1}(\alpha(t_n))}^{t_n} Q(u)y(\sigma(u))du, \\ &\leq y(t_n) - p_2y(\tau(t_n)) + M_1 \int_{\sigma^{-1}(\alpha(t_n))}^{t_n} Q(u)du, \\ \lim_{n \rightarrow \infty} W(t_n) &= l \leq -p_2M_1 + \varepsilon M_1 = (\varepsilon - p_2)M_1 \end{aligned}$$

For $\varepsilon \in (0, p_2)$, the last inequality leads to a contradiction, unless $M_1 = 0, l = 0$. However, this is impossible since $W(t)$ is positive and nondecreasing.

If $W(t) < 0$, we claim that $l = 0$, otherwise $l < 0$, hence

$$\begin{aligned} 0 > W(\tau^{-1}(t_n)) &= y(\tau^{-1}(t_n)) - P(\tau^{-1}(t_n))y(t_n) + \int_{\sigma^{-1}(\alpha(\tau^{-1}(t_n)))}^{\tau^{-1}(t_n)} Q(u)y(\sigma(u))du, \\ &\geq y(\tau^{-1}(t_n)) - p_3y(t_n) \end{aligned}$$

As $n \rightarrow \infty$, it follows that $l \geq \limsup_{n \rightarrow \infty} y(\tau^{-1}(t_n))$, which is a contradiction. Hence $l = 0$. On the other side, if $M_1 > 0$, then from the following equation:

$$W(t) = y(t) - P(t)y(\tau(t)) + \int_{\sigma^{-1}(\alpha(t))}^t Q(u)y(\sigma(u))du$$

$$\text{we have: } W(\tau^{-1}(t)) \leq y(\tau^{-1}(t)) - p_2y(t) + \int_{\sigma^{-1}(\alpha(\tau^{-1}(t)))}^{\tau^{-1}(t)} Q(u)y(\sigma(u))du$$

Taking limit superior for both sides to the last inequality yields

$0 \leq M_1 - p_2M_1 + \varepsilon M_1 = (1 - p_2 + \varepsilon)M_1$ Then for every $\varepsilon \in (0, p_2 - 1)$ the last inequality leads to a contradiction. Hence $M_1 = 0$. This means that $\lim_{t \rightarrow \infty} y(t) = 0$.

3. Asymptotic behavior of INDEPNCFT (1.2)

In this section, some sufficient conditions are obtained to ensure that all the solutions of first order impulsive neutral differential equations, with positive and negative coefficients and with forcing term (INDEPNCFT) of the form (1.2), converge to zero.

Let $W(t)$ be defined as

$$W(t) = y(t) - P(t)y(\tau(t)) + \int_{\sigma^{-1}(\alpha(t))}^t Q(u)y(\sigma(u))du - F(t) \tag{3.1}$$

For $t_k < t < \sigma^{-1}(\alpha(t)) \leq t_{k+1}$, $k = 1, 2, \dots$, such that

$$F(t) = \begin{cases} \int_{\tau(t)}^t f(s)ds, & t \in (t_k, t_{k+1}]; \\ a_k \int_{\tau(t_k)}^{t_k} f(s)ds, & t = t_k, \quad k = 1, 2, \dots \end{cases} \tag{3.2}$$

where $\int_{\tau(t)}^t f(s)ds$ is convergent for $t \geq T \geq t_0$ and $\lim_{t \rightarrow \infty} F(t) = 0$.

Theorem 3.1

Assume that $0 \leq P(t) \leq p_1 < 1$, $P(t_k^+) \geq P(t_k)$, $F(t_k^+) \geq F(t_k)$ and $\tau(t_k)$ is not an impulsive points. Therefore, the conditions (2.2) and (2.3) hold. Then every nonoscillatory solution of Eq.(1.2) converges to zero as $t \rightarrow \infty$.

Proof

Suppose that $y(t) > 0, y(\tau(t)) > 0, y(\sigma(t)) > 0, y(\alpha(t)) > 0, t \in (t_k, t_{k+1}], k = 1, 2, \dots$.

We differentiate (3.1) for each interval $(t_k, t_{k+1}]$, $k = 1, 2, \dots$ and use Eq.(1.2) to get Eq.(2.4).

Hence $W(t)$ is nonincreasing for $t_k < t \leq t_{k+1}$ for $k = 1, 2, \dots$

First, we need to show that $W(t_k^+) \leq W(t_k)$ for $k = 1, 2, \dots$. Since $0 < a_k \leq 1$ and $\tau(t_k) \neq t_i, i < k = 1, 2, \dots$ it follows that:

$$\begin{aligned} W(t_k^+) &= y(t_k^+) - P(t_k^+)y(\tau(t_k)) + \int_{\sigma^{-1}(\alpha(t_k))}^{t_k} Q(u)y(\sigma(u))du - F(t_k^+) \\ &\leq a_k y(t_k) - (1 - a_k) \int_{\sigma^{-1}(\alpha(t_k))}^{t_k} Q(u)y(\sigma(u))du - P(t_k)y(\tau(t_k)) + \int_{\sigma^{-1}(\alpha(t_k))}^{t_k} Q(u)y(\sigma(u))du \\ &\quad - F(t_k), \\ &= a_k y(t_k) + a_k \int_{\sigma^{-1}(\alpha(t_k))}^{t_k} Q(u)y(\sigma(u))du - P(t_k)y(\tau(t_k)) - a_k \int_{\tau(t_k)}^{t_k} f(s)ds \leq W(t_k). \end{aligned}$$

Hence $W(t)$ is nonincreasing on $[t_0, \infty)$. Hence $-\infty \leq \lim_{t \rightarrow \infty} W(t) < \infty$. We claim that $y(t)$ is bounded, and $-\infty < \lim_{t \rightarrow \infty} W(t) < \infty$.

If $y(t)$ is unbounded, then there exists a sequence $\{t_n\}$ such that $\lim_{n \rightarrow \infty} t_n = \infty$, $\lim_{n \rightarrow \infty} y(t_n) = \infty, y(t_n) = \max\{y(s), \tau(t_0) \leq s \leq t_n\}$. Then by (2.1) it follows that:

$$W(t_n) \geq y(t_n) - p_1 y(t_n) - F(t_n) = (1 - p_1)y(t_n) - F(t_n)$$

As $n \rightarrow \infty$ we get $\lim_{n \rightarrow \infty} W(t_n) = \infty$, but this is a contradiction. Therefore $y(t)$ is bounded, and this leads to $W(t)$ is bounded.

If $W(t) < 0$, then $0 > W(\tau^{-1}(t_n)) \geq y(\tau^{-1}(t_n)) - p_1 y(t_n) - F(\tau^{-1}(t_n))$ as $n \rightarrow \infty$, we get $\lim_{t \rightarrow \infty} W(t) = 0$. However, this is impossible since $W(t)$ is negative and nonincreasing.

If $W(t) > 0$, let $y(t) \leq M, \lim_{t \rightarrow \infty} W(t) = l \geq 0, W(t) \geq l$. We claim that $l = 0$, otherwise $l > 0$.

Then by

$$\begin{aligned} W(t) &= y(t) - P(t)y(\tau(t)) + \int_{\sigma^{-1}(\alpha(t))}^t Q(u)y(\sigma(u))du - F(t), \quad t \in (t_k, t_{k+1}] \\ &\leq y(t) + M \int_{\sigma^{-1}(\alpha(t))}^t Q(u)du - F(t) < y(t) - F(t) + \varepsilon, \quad \varepsilon > 0. \end{aligned}$$

Then for large enough t , we get

$$W(t) \leq y(t) - F(t), \quad t \in (t_j, t_{j+1}], \quad j \geq k.$$

Which implies that

$$W(t) \leq y(t), \quad t \in (t_j, t_{j+1}], \quad j \geq k.$$

Let $\gamma(t) > t$ such that $W(\gamma(t)) - W(t) = 0$, for large enough t .

By integrating (2.4) from t to $\gamma(t)$ we get

$$\begin{aligned} W(\gamma(t)) - W(t) &= - \int_t^{\gamma(t)} \left(Q(\sigma^{-1}(\alpha(s))) (\sigma^{-1}(\alpha(s)))' - R(s) \right) y(\alpha(s)) ds \\ &\leq - \int_t^{\gamma(t)} \left(Q(\sigma^{-1}(\alpha(s))) (\sigma^{-1}(\alpha(s)))' - R(s) \right) W(\alpha(s)) ds \\ &\leq -W(\alpha(\gamma(t))) \int_t^{\gamma(t)} \left(Q(\sigma^{-1}(\alpha(s))) (\sigma^{-1}(\alpha(s)))' - R(s) \right) ds \end{aligned}$$

For large enough t , we conclude that

$$0 \leq -L \int_t^{\gamma(t)} \left(Q(\sigma^{-1}(\alpha(s))) (\sigma^{-1}(\alpha(s)))' - R(s) \right) ds$$

Which is a contradiction. Thus $L = 0$. Assume that $\limsup_{t \rightarrow \infty} y(t) = M_1 \geq 0$. So there exists a sequence $\{t_m\}$ such that $\lim_{m \rightarrow \infty} t_m = \infty$, $\lim_{m \rightarrow \infty} y(t_m) = M_1$. We claim that $M_1 = 0$, otherwise $M_1 > 0$. Then it follows that

$$W(t_m) \geq y(t_m) - p_1 y(\tau(t_m)) - F(t_m).$$

As $m \rightarrow \infty$, $0 \geq M_1 - p_1 M_1 = (1 - p_1)M_1$, this yields $M_1 \leq 0$, which is a contradiction. Hence $\limsup_{t \rightarrow \infty} y(t) = 0$, which implies that $\lim_{t \rightarrow \infty} y(t) = 0$.

4. Examples

Example 4.1

Consider the impulsive neutral differential equation:

$$\left[y(t) - \frac{1}{4} e^{-\frac{t}{2}} y\left(\frac{t}{2} - \frac{1}{6}\right) \right]' + e^1 e^{-\frac{t}{2}} y\left(\frac{t}{2} + 1\right) - \frac{1}{4} e^{\frac{1}{6}} e^{-\frac{t}{2}} y\left(\frac{t}{2}\right) = 0,$$

$$t \neq k, k = 1, 2, \dots$$

$$y(t_k^+) = a_k y(t_k) - (1 - a_k) \left(\int_{\sigma^{-1}(\alpha(t))}^t Q(u) y(\sigma(u)) du \right), \quad t = k$$

where

$$a_k = \frac{k}{k+1} < 1, \quad k = 1, 2, \dots$$

$$\text{Let } P(t) = \begin{cases} \frac{1}{4} e^{-\frac{t}{2}}, & t \in (k, k + 1] \\ 0, & t = k \end{cases}$$

$$\tau(t_k) = \tau(k) = \frac{k}{2} - \frac{1}{6}$$

$$P(t_k^+) = P(k^+) = \lim_{t \rightarrow k^+} P(t) = \lim_{t \rightarrow k^+} \frac{1}{4} e^{-\frac{t}{2}} = \frac{1}{4} e^{-\frac{k}{2}} \geq 0$$

$$P(t_k) = 0, \text{ then } P(t_k^+) \geq P(t_k)$$

$$\text{Let } \sigma(t) = \frac{t}{2} + 1, \alpha(t) = \frac{t}{2} \text{ and } \sigma^{-1}(\alpha(t)) = t - 2$$

To show condition (2.2): $Q(\sigma^{-1}(\alpha(t))) (\sigma^{-1}(\alpha(t)))' - R(t)$

$$= e^1 e^{-\frac{(t-2)}{2}} - \frac{1}{4} e^{\frac{1}{6}} e^{-\frac{t}{2}} = (7.389056 - 1.042546) e^{-\frac{t}{2}} = 6.34651 \geq 0.$$

And condition (2.3):

$$\begin{aligned} \lim_{t \rightarrow \infty} \int_{\sigma^{-1}(\alpha(u))}^t Q(u) du &= \lim_{t \rightarrow \infty} \int_{t-2}^t e^1 e^{-\frac{u}{2}} du \\ &= \lim_{t \rightarrow \infty} \left[2e^1 e^{-\frac{t}{2}} (e^1 - 1) \right] = \lim_{t \rightarrow \infty} 9.341545 e^{-\frac{t}{2}} = 0 \end{aligned}$$

Hence all the conditions of theorem 2.1 hold. So, according to theorem 2.1, every solution of equation

$$(1.1) \text{ converges to zero as } t \rightarrow \infty. \text{ Thus } y(t) = \begin{cases} e^{-t}, & t \neq k \\ \frac{1}{k+1}, & t = k \end{cases} \text{ is the desired solution.}$$

Example 4.2

Consider the impulsive neutral differential equation

$$\left[y(t) - 6 \left(1 - \frac{1}{54} e^{-2t} \right) y(t-1) \right]' + \frac{1}{3} e^{\frac{10}{9}} e^{-2t} y \left(t + \frac{1}{9} \right) - \left(6 e^{\frac{8}{9}} - e^{-\frac{1}{9}} \right) y \left(t - \frac{1}{9} \right) = 0,$$

$t \neq k, k = 1, 2, \dots$

$$y(t_k^+) = a_k y(t_k) - (1 - a_k) \left(\int_{\sigma^{-1}(\alpha(t))}^t Q(u) y(\sigma(u)) du \right), \quad t_k = k$$

where $a_k = \frac{k+2}{k} > 1, k = 1, 2, \dots$

$$\tau(t_k) = \tau(k) = k - 1$$

$$p(t_k^+) = P(k^+) = \lim_{t \rightarrow k^+} 6 \left(1 - \frac{1}{54} e^{-2t} \right) = 6 \left(1 - \frac{1}{54} e^{-2k} \right)$$

$$P(t_k) = 10k, \text{ then } P(t_k^+) \leq P(t_k)$$

$$\text{Let } P(t) = \begin{cases} 6 \left(1 - \frac{1}{54} e^{-2t} \right), & t \in (t_k, t_{k+1}] \\ 10t, & t = t_k \end{cases}$$

$$\text{Let } \sigma(t) = t + \frac{1}{9}, \alpha(t) = t - \frac{1}{9} \text{ and } \sigma^{-1}(\alpha(t)) = t - \frac{2}{9}$$

To verify condition (2.7):

$$\int_T^\infty Q(t) dt = \int_T^\infty \frac{1}{3} e^{\frac{10}{9}} e^{-2t} dt = 0 < \infty$$

To show condition (2.8): $R(t) - Q(\sigma^{-1}(\alpha(t))) (\sigma^{-1}(\delta(t)))'$

$$= \left(6 e^{\frac{8}{9}} - e^{-\frac{1}{9}} \right) - \left(\frac{1}{3} e^{\frac{10}{9}} e^{-2(t-\frac{2}{9})} \right) = 13.699711 - 1.579239 e^{-2t} \geq 12.120472 > 0.$$

To verify condition (2.9):

$$\lim_{t \rightarrow \infty} \int_T^t \left(R(u) - Q(\sigma^{-1}(\alpha(u))) (\sigma^{-1}(\delta(u)))' \right) du = \lim_{t \rightarrow \infty} \int_T^t (13.699711 - 1.579239 e^{-2u}) du = \infty$$

Hence all the conditions of theorem 2.2 hold. So all solutions of equation (1.1) converge to zero as $t \rightarrow \infty$.

For instance the solution $y(t) = \begin{cases} e^{-t}, & t \neq k \\ \frac{2}{k+1}, & t = k \end{cases}$ does so.

Conclusions

In this paper, the impulsive neutral differential equations were considered. The impulses characteristics of the first order neutral differential equations with positive and negative coefficients were clarified. Some necessary and sufficient conditions that determine the asymptotic behavior of all solutions of equations (1.1) and (1.2) were obtained. Illustrative examples of the obtained results were explained.

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