A New Mixed Nonpolynomial Spline Method for the Numerical Solutions of Time Fractional Bioheat Equation

Ammar Muslim Abdullhussein*[1], Hameeda Oda Al-Humedi[2]

[1]Department Mathematics, Open Education College in Basrah, Basrah, Iraq

[2]Department Mathematics, Education College for Pure Science, Basrah University, Basrah, Iraq

Abstract

In this paper, a numerical approximation for a time fractional one-dimensional bioheat equation (transfer paradigm) of temperature distribution in tissues is introduced. It deals with the Caputo fractional derivative with α order for time fractional derivative and new mixed nonpolynomial spline for second order of space derivative. We also analyzed the convergence and stability by employing Von Neumann method for the present scheme.

Keywords: Fractional bioheat equations, Caputo fractional derivative, new mixed nonpolynomial spline, stability.

الطريقة الجديدة Mixed Nonpolynomial Spline لإيجاد الحلول العددية لمسألة الانتشار الحراري الحيوي الكسورية بالنسبة للزمن

عمار مسلم عبد الحسين[1]* ، حميدة عودة الحميدي[2]

[1]قسم الرياضيات ، الكلية التربوية المفتوحة في البصرة ، البصرة ، العراق

[2] قسم الرياضيات ، كلية التربية للعلوم الصرفة ، جامعة البصرة ، العراق

الخلاصه

في هذه المقالة، تم حل عدديا لمسألة الأنتشار الحراري الحيوي الكسورية الزمانية لتوزيع درجة الحرارة في الأنسجة بأستخدام الطريقة الجديدة Mixed Nonpolynomial Spline وتم أستخدام طريقة الأشتقاق الكسوري Caputo من الرتبة α للمشتقة الزمانية والطريقة الجديدة بالنسبة للمشتقة المكانية ثم تمت مقارنة بين الحلول الدقيقة والعددية. ثم ناقشنا تحليل الاستقرارية بطريقة فون نيومان.

1. Introduction

In the human body, the skin is considered as the largest organ. The study of skin and thermal behavior of living tissues is very fundamental, and it can be mathematically described by Pennes' bioheat transport equation:

$$\rho_t c_t \frac{\partial T(x,t)}{\partial t} = \mu \frac{\partial^2 T(x,t)}{\partial x^2} + W_b c_b (T_a - T) + Q + q_m \tag{1}$$

Mathematical resolve of the complex thermal interaction between the vasculature and tissues is a topic of interest for numerous physiologists, physicians, and engineers [1]. Temperature distribution in skin tissues is important for medical applications such as skin cancer, skin burns, etc. [2]. At most, the

---

*Email: ammar.muslim1@yahoo.com

accurate solution of Pennes' equation does not exist and, therefore, approximations and numerical techniques must be used to solve this equation.

In recent years, fractional calculus has been adopted by scientists and engineers and applied in many fields, namely in the fields of materials and mechanics, medical science, fluid mechanics, viscoelasticity, physics, signal processing, anomalous diffusion, biological systems, finance, hydrology and many others [3, 4] presented the solution of fractional bioheat equation by adopting the shifted Grünwald finite difference approximation for Riemann-Louville space fractional derivative method, the HPM of the fractional derivative of space, and the Caputo fractional for the fractional time. It has been spotted that the time possessed to achieve hyperthermia in a location is reduced as the order fractional derivative decreases. [5] discussed the two cases of 1D and 2D Pennes bioheat model for the implementation of triangular and quadrilateral elements method. In the 2D case, both quadrilateral and triangular elements were investigated. Through test problems, the discretization error generated from this method was reported[6] discussed the approximate solution of fractional Pennes bioheat equation with constant and sinusoidal heat flux conditions on skin, using the implicit finite difference method where the fractional time derivative is of the Caputo form.[7]. showed a numerical solution for the time-fractional Pennes bioheat transfer equation on skin tissues and solved it by Fourier Sine transform of second order derivative and the Caputo for the fractional time. [8], discussed the 2-D fractional bioheat equation by Laplace transforms of second order derivative and performed the numerical solutions to search the temperature transfer in skin exposed to immediate surface heating. Some differentiations were shown to estimate the impact of the fractional order parameter on the temperature wave. [9], presented a numerical solution of the fractional bioheat equation by finite difference of second order derivative and the fractional derivative by Grünwald Letnikov for the fractional time. They discussed and analyzed the stability and convergence. [10], studied the fractional bioheat transfer equation and solved it using an approximate solution (numerically) by finite difference of second order derivative and the fractional time derivative by Caputo derivative. They also discussed the stability and convergence by this scheme. [11], discussed the 2D fractional bioheat equation using Galerkin FEM. He found the solution method in the cylindrical living tissue and noted the effects of thermal conductivities that have significant and more remarkable effects on temperature variation in living tissue. [12], studied the fractional bioheat equation when the time-space fractional derivative in the form and solve it by Caputo fractional derivative of order $\alpha \in (0, 1]$ and Riesz–Feller fractional derivative of order $\beta \in (1, 2]$ respectively. They obtained the results in terms of Fox's H-function with some specific cases using Fourier–Laplace transforms. [13], studied the fractional bioheat equation and solved the space-time fractional bioheat equation using fractional order Legendre functions of fractional space order derivative and the fractional time derivative by Caputo derivative. They observed that the quantity of the temperature at the skin surface is a strong function of the space fractional order and, conversely, the impact of the time fractional order is almost negligible.

The time fractional for Pennes' bioheat transfer equation and all the constants within it are introduced in section 2. We present the mathematical background concerned with the fractional definitions in section 3. In section 4, a new mixed spline form for the second space derivative is derived. In sections 5 and 6, we derive and apply the time fractional derivative by Caputo fractional derivative and a new mixed nonpolynomial spline form for space derivative of Pennes' bioheat transfer equation. Section 7 contains a stability analysis for Pennes bioheat transfer equation. In section 8 we apply and find the numerical solutions for time fractional Pennes bioheat equation by a new mixed nonpolynomial spline method.

2. Pennes Bioheat Transfer Equation with Time Fractional Derivative

The problem of the time fractional Pennes bioheat transfer equation for the modeling of skin tissue heat transfer is expressed in previous works [6, 14, 15, 16], as follows:

$$\rho_t c_t \frac{\partial^\alpha T(x,t)}{\partial t^\alpha} = \mu \frac{\partial^2 T(x,t)}{\partial x^2} + W_b c_b (T_a - T) + Q + q_m \qquad (2)$$

$$T(x, 0) = T_0, \qquad x \in (0, L), \qquad (3)$$

$$\frac{\partial}{\partial x} T(x, 0) = g(x), \qquad x \in (0, L), \qquad (4)$$

$$T(0, t) = h(t), \qquad t > 0, \qquad (5)$$

2.1 Nomenclature

$\alpha \in (0,1)$ is the fractional order of time,

x is the distance from the skin surface,

$\rho_t = 1000$ is a constant representing the density tissue (kg/m^3) ,

$c_t = 4000$ is a constant representing the specific heat of tissue (Jk/kg),

$\mu = 0.5$ is the tissue thermal conductivity (J/sm),

$W_b = 0.0005$ is the mass flow rate of blood per unit volume of tissue (kg/sm^3),

$c_b = 4000$ is the specific heat of blood (J/kg),

$Q_m = 420$ is the metabolic heat generation per unit volume (J/m^3),

$T_a = 37$ represents the arterial blood temperature,

T is the temperature of tissue,

Q is the source of metabolic heat,

$W_b c_b (T_a - T)$ represents the blood perfusion. It is worth mentioning that the W_b constant was obtained experimentally by Pennes for a human forearm.

3. Definitions

Definition (1): The Riemann-Liouville fractional derivative of order $\alpha \in (n-1, n)$ is $n \in \mathbb{N}, t > a$, defined by [2-6], [8-14], [16], [17-21] as:

$^{RL}D_a^\alpha f(t) = \frac{1}{\Gamma(n-1)} \frac{d^n}{d\psi^n} \int_a^t (t-\psi)^{n-\alpha-1} f(\psi) d\psi$, $\alpha \in (n-1, n)$

Definition (2): The Caputo fractional derivative of order $\alpha \in (n-1, n), n \in \mathbb{N}, t > a$ is defined by [2-6], [8-14], [16], [19-21] as:

$^{C}D_a^\alpha f(t) = \frac{1}{\Gamma(n-1)} \int_a^t (t-\psi)^{n-\alpha-1} \frac{d^n}{d\psi^n} f(\psi) d\psi$, $\alpha \in (n-1, n)$

New Mixed Nonpolynomial Spline Form for the Second Derivative

Now we introduce the new nonpolynomial spline method which depends on a mixed spline $Q_i(x, t_k)$, which can be written in the form:

$$Q_i(x, t_k) = a_i(t_k) + b_i(t_k) e^{\cos \omega (x-x_i)} + c_i(t_k) e^{\sin \omega (x-x_i)}, \quad i = 0(1)(n-1), \tag{6}$$

where $a_i(t_i), b_i(t_i)$ and $c_i(t_i)$ are unknown coefficients with respect to time and ω is the frequency of the trigonometric part of the spline functions.

To find the coefficients of (6) in terms of S_i, S_{i+1}, F_i and F_{i+1}, at first we define

$$Q_i(x_i, t_k) = S_i \ , \quad Q_i(x_{i+1}, t_k) = S_{i+1} \quad , \quad Q_i^{(2)}(x_i, t_k) = F_i \ , \quad Q_i^{(2)}(x_{i+1}, t_k) = F_{i+1} \tag{7}$$

where $Q_i^{(2)}(x_{i+1}, t_k) = \frac{\partial^2 Q_i(x, t_k)}{\partial x^2}$

Then by using (6) and (7), we obtain

$a_i(t_k) + b_i(t_k) e + c_i(t_k) = S_i$

$a_i(t_k) + b_i(t_k) e^{\cos \theta} + c_i(t_k) e^{\sin \theta} = S_{i+1}$

$$-\omega^2 b_i(t_k) e + \omega^2 c_i(t_k) = F_i \tag{8}$$

where $\theta = h\omega$. From solving equation (8), we get the following expressions

$$a_i(t_k) = S_i - \left(\left(\frac{-e + e^{\cos \theta}}{(e^{\cos \theta} - e)(e^{1+\sin \theta} - 2e + e^{\cos \theta})} \right)(S_{i+1} - S_i) + \frac{(1 - e^{\sin \theta})}{\omega^2 (e^{1+\sin \theta} - 2e + e^{\cos \theta})} F_i \right) e -$$

$$\frac{e}{(e^{1+\sin \theta} - 2e + e^{\cos \theta})}(S_{i+1} - S_i) + \frac{(e^{\cos \theta} - e)}{\omega^2 (e^{1+\sin \theta} - 2e + e^{\cos \theta})} F_i$$

$$b_i(t_k) = \left(\left(\frac{-e + e^{\cos \theta}}{(e^{\cos \theta} - e)(e^{1+\sin \theta} - 2e + e^{\cos \theta})} \right)(S_{i+1} - S_i) + \frac{(1 - e^{\sin \theta})}{\omega^2 (e^{1+\sin \theta} - 2e + e^{\cos \theta})} F_i \right)$$

$$c_i(t_k) = \frac{e}{(e^{1+\sin \theta} - 2e + e^{\cos \theta})}(S_{i+1} - S_i) + \frac{(e^{\cos \theta} - e)}{\omega^2 (e^{1+\sin \theta} - 2e + e^{\cos \theta})} F_i \tag{9}$$

Therefore, by (9) and the continuity condition at knots (x_i, t_k), such that

$$Q_{i-1}^{(1)}(x_i, t_k) = Q_i^{(1)}(x_i, t_k) \tag{10}$$

From equations (6), (9) and (10), we yield the following relation

$$\delta F_{i-1} + \gamma F_i = a S_i + b S_{i-1} + c\, S_{i+1} \tag{11}$$

where $\qquad \gamma = e - e^{\cos\theta}$, $\qquad \delta = -\gamma\cos\theta\, e^{\sin\theta} - \sin\theta\, e^{\cos\theta}(1 - e^{\sin\theta})$,
$b = \omega^2\cos\theta\, e^{\sin\theta} e - \omega^2\sin\theta\, e^{\cos\theta}$, $c = \omega^2 e$ and $a = -b - c$.

4. Caputo fractional derivative for the time fractional derivative

The discrete approximation of time-fractional derivative at time point $t = t_{k+1}$, can be achieved as follows [9, 10]

$$\frac{\partial^\alpha T(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^{t_{r+1}} \frac{\partial T(x,\tau)}{\partial \tau}(t_{r+1} - \tau)^{-\alpha} d\tau + O(\tau^{2-\alpha})$$

$$= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{r} \int_{t_k}^{t_{k+1}} \frac{\partial T(x,\tau)}{\partial \tau}(t_{r+1} - \tau)^{-\alpha} d\tau + O(\tau^{2-\alpha})$$

$$= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{r} \frac{T(x,t_{k+1}) - T(x,t_k)}{\tau} \int_{t_k}^{t_{k+1}}(t_{r+1} - \tau)^{-\alpha} d\tau + O(\tau^{2-\alpha})$$

By using the forward Euler scheme to discretize the Caputo time fractional derivative, let $t_r = r\Delta t = r\tau$, $r = 0(1)K$, in which $\tau = \frac{G}{K}$ is the time step size.

$$\frac{\partial^\alpha T(x,t)}{\partial t^\alpha} = \frac{1}{\tau^\alpha \Gamma(2-\alpha)} \sum_{k=0}^{r} u_j[T(x,t_{r-k+1}) - T(x,t_{r-k})] + O(\tau^{2-\alpha})$$

$$= \frac{1}{\tau^\alpha \Gamma(2-\alpha)} \left[T^{r+1} + T^r(u_1 - 1) + \sum_{k=1}^{r-1} T^{r-k}(u_{k+1} - u_k) - u_r T^0 \right] + O(\tau^{2-\alpha}) \qquad (12)$$

where $u_k = (k+1)^{1-\alpha} - (k)^{1-\alpha}$, for all $k = 0(1)r$ and $u_0 = 1$.

5. Derivation of a Caputo fractional and new mixed nonpolynomial spline forms for time fractional Pennes bioheat transfer equation

From (7) and (12) and by putting in (2), equation (2) can be rewritten as the following system of algebraic equation:

$$A T_i^{r+1} + B T_i^r + A \sum_{k=1}^{r-1} T_i^{r-k} w_k - A u_r T_i^0 = F_i^r + Q_i^r + C \qquad (13)$$

where $A = \frac{\rho_t c_t}{\mu\tau^\alpha \Gamma(2-\alpha)}$, $B = A(u_1 - 1) + W_b c_b$, $C = W_b c_b T_a + q_m$ and $w_k = (u_{k+1} - u_k)$

Eq. (13) can lead to

$$A T_{i-1}^{r+1} + B T_{i-1}^r + A \sum_{k=1}^{r-1} T_{i-1}^{r-k} w_k - A u_r T_{i-1}^0 = \mu F_{i-1}^r + Q_{i-1}^r + C \qquad (14)$$

By multiplying (13) and (14) by γ and δ, respectively, then adding these equations, we get

$$A(\delta T_{i-1}^{r+1} + \gamma T_i^{r+1}) + B(\delta T_{i-1}^r + \gamma T_i^r) + A \sum_{k=1}^{r-1}(\delta T_{i-1}^{r-k} + \gamma T_i^{r-k})w_k - A u_r(\delta T_{i-1}^0 + \gamma T_i^0) =$$
$$\mu(\delta F_{i-1}^r + \gamma F_i^r) + \delta Q_{i-1}^r + \gamma Q_i^r + C(\delta + \gamma) \qquad (15)$$

From (11) and by substituting the value $\delta F_{i-1}^r + \gamma F_i^r$ in (15), the last equation can be rewritten in the following form

$$A(\delta T_{i-1}^{r+1} + \gamma T_i^{r+1}) + (B\delta - \mu b)T_{i-1}^r + (B\gamma - \mu a)T_i^r + A \sum_{k=1}^{r-1}(\delta T_{i-1}^{r-k} + \gamma T_i^{r-k})w_k -$$
$$A u_r(\delta T_{i-1}^0 + \gamma T_i^0) = c\mu T_{i+1}^r + \delta Q_{i-1}^r + \gamma Q_i^r + C(\delta + \gamma) \qquad (16)$$

Equation (16) contains $(n-1)$ linear algebraic equations by $(n+1)$ unknowns T_i^r, $i = 0(1)n$. Thus, we need two boundary equations when $i = 0$ and $i = n$ and by using Taylor series for these nodes, we get the following two equations:

$$A\left((\delta + \gamma)T_0^{r+1} - h\delta T_{0(1)}^{r+1}\right) + (B\delta - \mu b)\left(T_0^r - hT_{0(1)}^r\right) + (B\gamma - \mu a)T_0^r + A \sum_{k=1}^{r-1}\left((\delta + \gamma)T_0^{r-k} - \delta h T_{0(1)}^{r-k}\right)w_k - A u_r\left((\delta + \gamma)T_0^0 - \delta h T_{0(1)}^0\right) = c\mu T_1^r + (\delta + \gamma)Q_0^r - hQ_{0(1)}^r + C(\delta + \gamma)$$
$$(17)$$

$$A(\delta T_{n-1}^{r+1} + \gamma T_n^{r+1}) + (B\delta - \mu b)T_{n-1}^r + (B\gamma - \mu a)T_n^r + A \sum_{k=1}^{r-1}(\delta T_{n-1}^{r-k} + \gamma T_n^{r-k})w_k - A u_r(\delta T_{n-1}^0 + \gamma T_n^0) = c\mu T_{n+1}^r + \delta Q_{n-1}^r + \gamma Q_n^r + C(\delta + \gamma) \qquad (18)$$

where $T_{0(1)} = \frac{\partial T}{\partial x}\Big|_{x=0}$, and $T_{n(1)} = \frac{\partial T}{\partial x}\Big|_{x=n}$

6. Stability analysis

The analysis of stability for the proposed scheme can be achieved by using Von Neumann method. Consider

$$T_i^r = \xi_r e^{mi\theta h} \qquad (19)$$

where $m = \sqrt{-1}$, θ is real and ξ is a complex number. We rewrite (16) as follows:
$$A\left(\delta T_{i-1}^{r+1} + \gamma T_i^{r+1}\right) + (B\delta - \mu b)T_{i-1}^r + (B\gamma - \mu a)T_i^r + A\sum_{k=1}^{r-1}\left(\delta T_{i-1}^{r-k} + \gamma T_i^{r-k}\right)w_k = c\mu T_{i+1}^r + R$$
(20)

where $R = Au_r\left(\delta T_{i-1}^0 + \gamma T_i^0\right) + \delta Q_{i-1}^r + \gamma Q_i^r + C(\delta + \gamma)$, which can be neglected [15],[22,23]. By substituting (19) into (20), we obtain

$$A\left(\delta \xi_{r+1}e^{m(i-1)\theta h} + \gamma \xi_{r+1}e^{mi\theta h}\right) + (B\delta - \mu b)\xi_r e^{m(i-1)\theta h} + (B\gamma - \mu a)\xi_r e^{mi\theta h} +$$
$$A\sum_{k=1}^{r-1}\left(\delta \xi_{r-k}e^{m(i-1)\theta h} + \gamma \xi_{r-k}e^{mi\theta h}\right)w_k = c\mu \xi_r e^{m(i+1)\theta h} \qquad (21)$$

By dividing (21) by $\xi_r e^{mi\theta h}$, we obtain

$$A\xi_{r+1}\left(\delta e^{-m\theta h} + \gamma\right) + A\xi_{r-k}\sum_{k=1}^{r-1}\left(\delta e^{-m\theta h} + \gamma\right)w_k$$
$$= \xi_r\left(c\mu e^{m\theta h} - B\left(\delta e^{-m\theta h} + \gamma\right) + \mu\left(be^{-m\theta h} + a\right)\right)$$

$$A\xi_{r+1}(\delta + \gamma) + A(\delta + \gamma)\xi_{r-k}\sum_{k=1}^{r-1}w_k < \xi_r\left(c\mu - B(\delta + \gamma) + \mu(b + a)\right)$$

$$|\xi_{r+1}| < \left|\frac{c\mu}{A(\delta + \gamma)} - \frac{B}{A} + \frac{\mu(b + a)}{A(\delta + \gamma)} - \sum_{k=1}^{r-1}\xi_{-k}w_k\right||\xi_r|$$

Now, $r = 0$
$$|\xi_1| \implies \left|\frac{c\mu}{A(\delta+\gamma)} - \frac{B}{A} + \frac{\mu(b+a)}{A(\delta+\gamma)}\right||\xi_0| \implies |\xi_1| < |\xi_0| \qquad (22)$$

Now, $r = i$
$$|\xi_{i+1}| < \left|\frac{c\mu}{A(\delta+\gamma)} - \frac{B}{A} + \frac{\mu(b+a)}{A(\delta+\gamma)}\sum_{k=1}^{i-1}\xi_{-i}w_i\right||\xi_i| \implies |\xi_{i+1}| < |\xi_i| \qquad (23)$$

where $\left|\frac{c\mu}{A(\delta+\gamma)} - \frac{B}{A} + \frac{\mu(b+a)}{A(\delta+\gamma)}\right| < 1$, and $\left|\frac{c\mu}{A(\delta+\gamma)} - \frac{B}{A} + \frac{\mu(b+a)}{A(\delta+\gamma)} - \sum_{k=1}^{r-1}w_k\right| < 1$ for all values of r.

Therefore, (22) and (23) lead to $|\xi_{r+1}| < |\xi_0|$ for all values of r. Hence, the new mixed nonpolynomial spline form is stable.

7. Numerical experiment

Now, in this section, we apply the proposed method for solving two cases of problems (2)-(5). We show the reliability and applicability of this method by contrasting the numerical results of it with exact solutions for each case. In all the following examples, the new mixed nonpolynomial spline method is used.

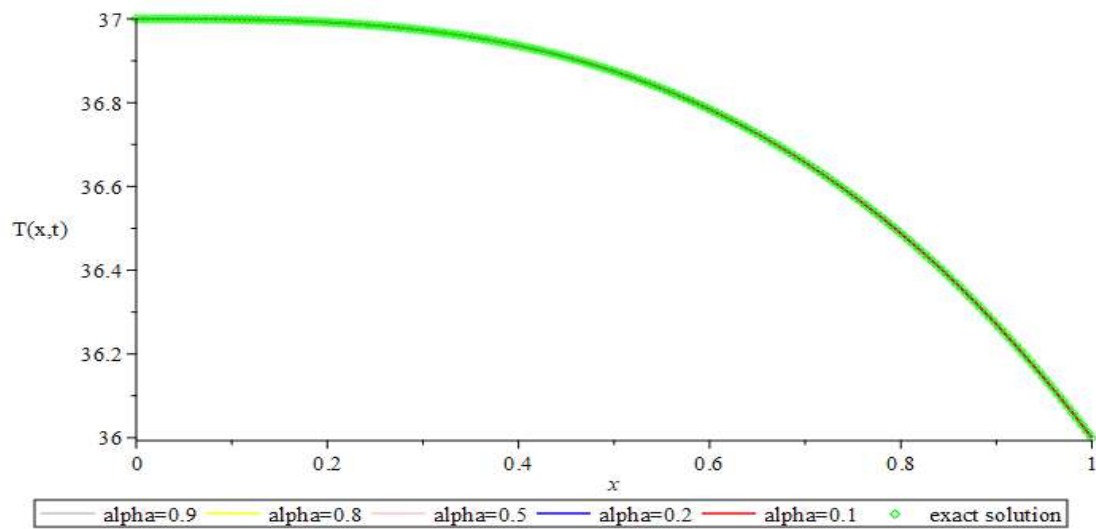**Example 1:** Consider Pennes' bioheat equation (2) with the conditions:
$$T(x,0) = 37 - x^3, \qquad \frac{\partial}{\partial x}T(x,0) = -3x^2, \quad T(0,t) = 37,$$

where, by choosing the source function Q, the exact solution is given as follows:
$$T(x,t) = xt^2 - x^3 + 37.$$

Table 1-The Errors of Numerical approximations for distinct values of $\alpha = 0.1, 0.2\ 0.5, 0.8, 0.9, 1$, and $t = 1, \tau = 0.001$ for Example 1.

α	$L_2 - error$	$L_\infty - error$
0.1	0.444927073364448205733992e-3	0.38854140888807493627e-3
0.2	0.216773185314852148820715e-3	0.18910537292943955978e-3
0.5	0.273991536829268677845054 6e-4	0.23550524225658979990e-4
0.8	0.509413429305641001567308 1e-5	0.39586567273706585300e-5
0.9	0.353910935231989960657543 5e-5	0.25460438955123084000e-5
1	0.277130791454935580771907 2e-5	0.18239450176970080000e-5

Figure 1-Comparison between exact solution and numerical approximations for distinct values of α and $t = 1, \tau = 0.001$.
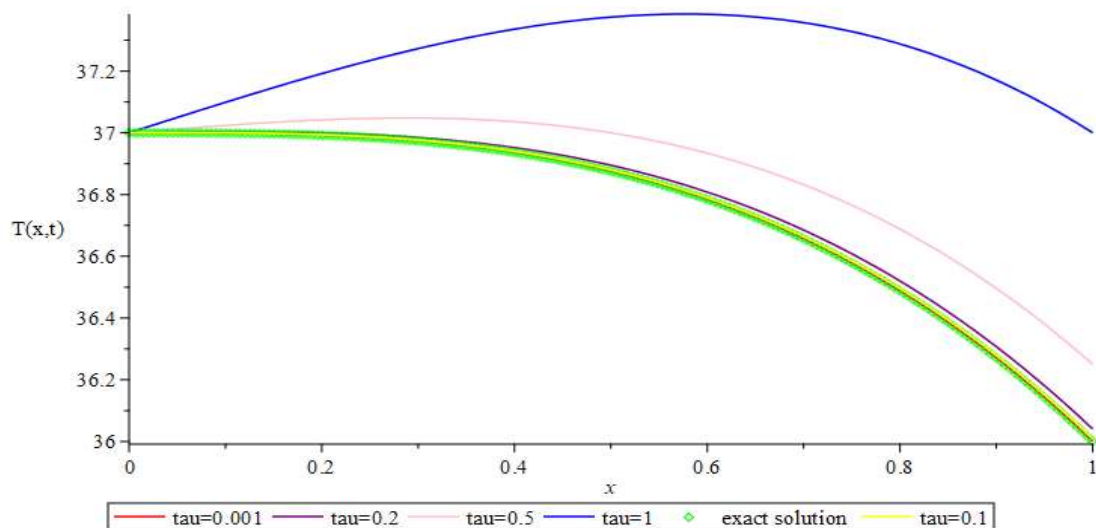
Figure 2-Comparison between exact solution and numerical approximations for Example1 for different τ, and $\alpha = 0.8$.
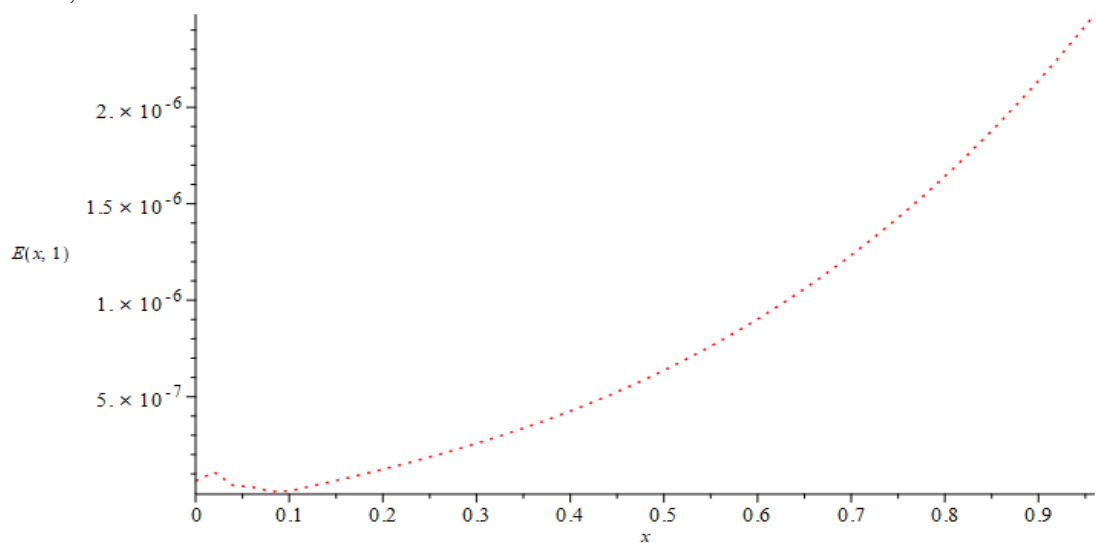
Figure 3-The errors of numerical approximations for Example1 at $\alpha = 0.8$ and $\tau = 0.001$.

Example 2: Consider Pennes bioheat equation (2) with the following conditions:
$$T(0,t) = t^3 + 37, \quad T(x,0) = sin(x^2) - x^{3+\alpha} + 37,$$
$$\frac{\partial}{\partial x}T(x,0) = 2x\cos(x^2) - (3+\alpha)x^{2+\alpha} ,$$
where, by choosing the source function Q, the exact solution is given as follows:
$$T(x,t) = sin(x^2) - (x^{3+\alpha} + t^3) + 37,$$

Table 2-The Errors of Numerical approximations for distinct values of Example 2 at $\alpha = 0.1, 0.3, 0.5, 0.7, 0.9, 1$ and $t = 1, \tau = 0.001.$

α	$L_2 - error$	$L_\infty - error$
0.1	0.1715340679449874796505876e-3	0.13738288950461518936e-3
0.3	0.4393141587685349876491385e-4	0.34874785528640870720e-4
0.5	0.1152700471776546187571774e-4	0.90996711140071062000e-5
0.7	0.3234063693751174266862903e-5	0.25967473864369616500e-5
0.9	0.8801986532719872660697001e-6	0.69064900642968784000e-6
1	0.4766435393819385355082149e-6	0.37326780928752769000e-6
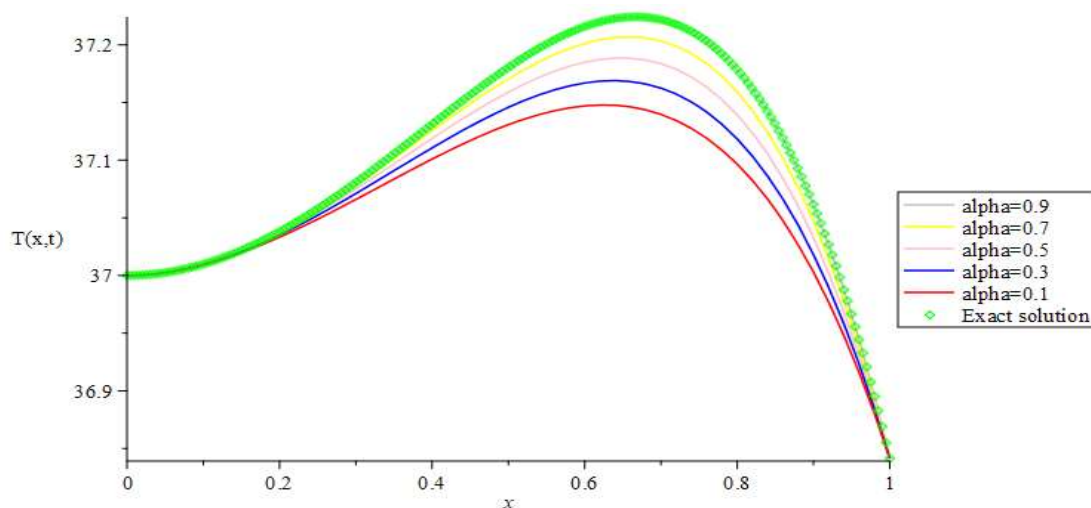
Figure 4-Comparison between exact solution and numerical approximations for Example 2 for at $\alpha = 0.1, 0.2, 0.5, 0.8, 0.9$ and $t = 1, \tau = 0.001.$
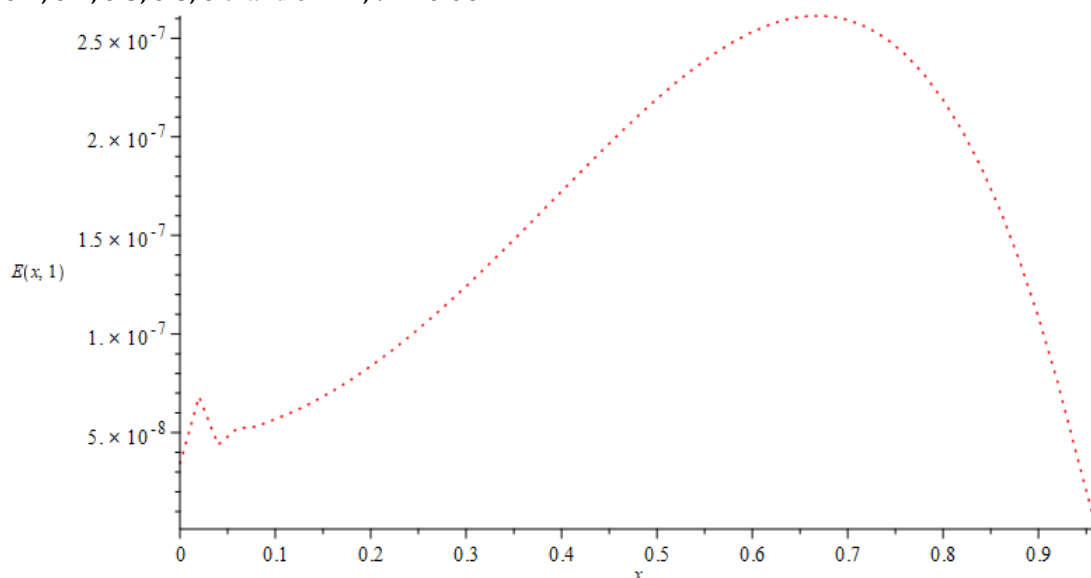
Figure 5-The errors of numerical approximations for Example2 at $\alpha = 0.9$ and $t = 1, \tau = 0.001.$

Conclusions

The objective of this article is to compare the achievement of the model approach based on our new mixed nonpolynomial spline method, which have been considered for finding the numerical solutions of time fractional Pennes' bioheat equation by using Caputo fractional derivative for the time fractional derivative and a new scheme for the derivative of second order in this equation. In general, it can be concluded from L_2 and L_∞ errors of the numerical approximations that the proposed method is powerful, effective, highly accurate and needed a small recurrence, as compared to the accurate solution. Furthermore, the present algorithm is simply applicable and the results clarified the activity of the suggested method. We discussed the stability of the fractional bioheat equation by the new mixed nonpolynomial spline method to clarify that the scheme is stable.

Acknowledgements

References

1. Chato J.C. **1981**. Reflection on the history of heat and mass transfer in bioengineering, *Journal of Biomechanical Engineering*, **103**(2): 97-101.
2. Cui, Z., Chen, G. and Zhang R. **2014**. Analytical solution for the time-fractional Pennes bioheat transfer equation on skin tissue, *Advanced Materials Research*, 1049-1050 , 1471- 1474.
3. Damor, R. S., Kumar, S.and Shukla, A. K. **2013**. Numerical Solution of Fractional Bioheat Equation with Constant and Sinusoidal Heat Flux Condition on Skine Tissue, *American Journal of Mathematical Analysis,* **1**(2): 20-24.
4. Damor, R.S, Kumar, S. and Shukla,A. K. **2015**. Parametric Study of Fractional Bioheat Equation in Skin Tissue With Sinusoidalheat Flux, *Fractional Differential Calculus*, **5**(1): 43-53.
5. Damor, R. S., Kumar, S. and Shukla, A. K. **2016**. *Solution of fractional bioheat equation in terms of Fox's H function* , Springer Plus,Vol. 5.
6. Danaf, T. S. **2013**. Numerical solution for the linear time and space fractional diffusion equation, *Journal of Vibration and Control*, **21**: 1-9.
7. Dehghan, M. and Sabouri, M. **2012**. A spectral element method for solving the Pennes bioheat transfer equation by using trianglar and quadrilateral elements, *Applied Mathematical modelling*, **36**: 6031–6049.
8. Ferras, L. L. **2009**. Fractional Penne´s Bioheat Equation: Thertical and Numerical Studies, *Fractional Calculus and Applied Analysis an International Journal for Theory and Applications*, **18**(4): 1080-1106.
9. Hire, T. **2016**. Quintic Spline Technique for Time Fractional Fourth - Order Partial Differential Equation, *Numerical Methods for Partial Differential Equations*, **33**(2): 445-466.
10. Hosseini, S. M. and Ghaffari, R. **2014**. Polynomial and nonpolynomial spline methods for fractional sub-diffusion equations, *Applied Mathematical Modelling*, **38**(14): 3554-3566.
11. Khalil H. and Khan R. **2015**. Extended Spectral Method for Fractional order Three-dimensional Heat Condition Problem, *Progress in Fractional Differentiation and Applications an International Journal*, **1**(3): 165-185.
12. Magdy A. Ezzat, Noorah S. AlSowayan, Zeid I. A. Al-Muhiameed and Shereen M. Ezzat, **2014**. Fractional modelling of Pennes bioheat transfer equation, Springer, *Heat Mass transfer*, **50**(7): 907-914.
13. Meerschaert, M. M. and Tadjeran, C. **2006**. Finite difference approximations for two-sided space-fractional partial differential equations, *Applied Numerical Mathematics*, **56**(1): 80-90.
14. Mishra, T.N. and Rai, K.N. **2015**. Implicit finite difference approximation for time Fractional heat conduction under boundary Condition of second kind, *International Journal of Applied Mathematical Research*, **4**(1): 135-149.
15. Otieno, O. R., Manyonge, A., Maurice, O. and Daniel, O. **2015**. Finite Difference Analysis of 2-Dimensional Acoustic Wave with a Signal Function, *International Journal of Multidisciplinary Sciences and Engineering*, **6**(10).
16. Qin, Y. and Wu, K. **2016**. Numerical solution of fractional equation by quadratic spline collocation method, *Journal of nonlinear Sciences and Applications*, **9**(7): 5061-5072.

17. andey, H. R. 2015. A One- Dimensional Bio-Heat Transfer Equation with Galerkin FEM in Cylindrical Living Tissue, *Journal of Advanced College of Engineering and Management*, **1**: 45-50.
18. Ramadan, M., Lashien, I. and Zahra, W. K. **2007**. Polynomial and Nonpolynomial Spline Approaches to the Numerical Solution of Second Order Boundary Value Problem, *Applied Mathematics and Computation,* **184**(2): 476-484.
19. Roohi, R, Heydari, M. , Aslami,M. and Mohmoudi, M. **2018**. A Comprehensive Numerical Study of Space-Time Fractional Bioheat Equation Using Fractional- Order Legendre Functions, *The European Physical Journal*, **133**(10): 1-15.
20. Sarkar, N. **2015**. On A New Time-Fractional Order Bioheat Transfer Model, *International Research Journal of Engineering and Technology*, **2**(4): 1140-1142.
21. Singh, J., Gupta, P. K. and Rai, K.N. **2011**. Solutionof fractional bioheat equations by finite difference method and HPM, *Mathematical and Computer Modelling, Elsevier*, **54**(9): 2316 - 2325.
22. Tuzikiewicz, W. and Duda, M. **2015**. Bioheat transfer equation. The problem of FDM explicit scheme stability, *Journal of Applied Mathematics and Computational Mechanics*, **14**(4): 139-144.
23. Yang, Q. and Turner, I. **2010**. Numerical methods for fractioal derivative. *Applied Mathematical Modelling*, **34**(4): 200-218.