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Homotopy Transforms Analysis Method for Solving Fractional Navier-Stokes Equations with Applications

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Abstract
The presented work includes the Homotopy Transforms of Analysis Method (HTAM). By this method, the approximate solution of nonlinear Navier- Stokes equations of fractional order derivative was obtained. The Caputo's derivative was used in the proposed method. The desired solution was calculated by using the convergent power series to the components. The obtained results are demonstrated by comparison with the results of Adomain decomposition method, Homotopy Analysis method and exact solution, as explained in examples (4.1) and (4.2). The comparison shows that the used method is powerful and efficient.

Keywords: Caputo's Derivative; Homotopy Analysis Method; Laplace Transform Method; Navier- Stokes equations of fractional order derivative; Numerical Solution.

التحويلات لطريقة هموتوبي التحليلية لحل معادلات نافير– ستوك الكسورية وتطبيقاتها

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الخلاصة
العمل المقدم يتضن تحويلات لطريقة هموتوبي التحليلية وبهذه الطريقة تم الحصول على الحل التقريبي لمعادلات نافير– ستوك ذات المشتقات الكسورية غير الخطية. تم استخدام مشتقة كابيتو بالطريقة المقترحة. تم احتساب الحل المرغوب باستخدام متسلسلة القوى المتقاربة. تم مقارنة نتائج الطريقة المستخدمة في هذا البحث مع نتائج طريقة ادومين التحليلية, طريقة هموتوبي التحليلية و الحل المضبوط كما في المثال الاول والثاني. وأن المقارنة أوضحت قوة وكفاءة الطريقة المستخدمة.

1 Introduction

The equations of fractional derivatives have received attention due to their widespread use in various science topics, including mathematical biology, electrochemistry, and others [1, 2]. The Navier-Stokes equations were submitted and solved numerically. The main importance of this kind of equations is the pure mathematics and successful applications, as in water flow in pipe, blood flow, the analysis of pollution and many other fields. In order to ensure the exact solutions of fractional order non-linear differential equations, it is an important consideration to fulfill need for new methods to obtain the desired solutions. The consideration of exact solutions of nonlinear fractional order differential equations is a very difficult task and, therefore, the approximate and numerical methods were used to solve this kind of equations. The proposed method was reliable to solve various kinds of non-linear problems.

In the last few years, many researchers pointed the numerical solution of the fractional order

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differential equations. Some of the numerical methods were improved, such as the semi-explicit multi-step method [3], transform methods [4], natural homotopy perturbation method [5], homotopy perturbation Elzaki transform [6], Laplace transform [7], Adomian decomposition method [8-10], and homotopy analysis method [11].

Navier-Stokes equations started in 1822, when Navier derived equations for homogenous incompressible fluids from a molecular point of view. The continuum derivation of the Navior –Stokes equations was presented by Saint- Venant and Stokes (1843). These equations are foundations of various fields of sciences, including geology, biology, medicine and physics [12].

The structure of this paper is described as follows: In section 2, we give the concept of fractional calculus. The HTAM is presented in section 3. In section 4 Navier- Stokes equations of fractional order derivative are solved to illustrate the competence of the considered method. Finally, in section 6, we present our conclusions.

The objective of this paper is to make a combination of the Homotopy Analysis and Laplace Transform methods, called The Homotopy Transforms of Analysis method (HTAM), to provide approximate solutions to the Navier- Stokes equations of fractional order derivative with variable coefficients of the form [13]:

$$D_t^{\alpha}\underline{u} + (\underline{u}\cdot\nabla)\underline{u} = -\frac{1}{\rho}\nabla P + \upsilon\nabla^2\underline{u}, \quad \text{... (1.1)}$$

Where $\underline{u}$ is the vector of the velocity, ρ is the density, υ refers to the kinematics viscosity, P is the pressure, and t is the time variable.

2. Preliminaries and notations

This section consists of some basic definitions and properties for the fractional calculus theory.

Definition 2.1 [5, 14]: Let $f(t)$ be a real valued function, $t > 0$, then $f(t)$ belongs to the space C_μ, $\mu \in R$ if there is a real number $p > \mu$, $\exists\, f(t) = t^p f_1(t)$, where $f_1(t) \in C[0,\infty]$ and it belongs to the space C_μ^r if $f^{(r)} \in C_\mu$, $r \in N \cup \{0\}$.

Definition 2.2 [5, 14]: The fractional integral of Riemann-Liouville of $\alpha > 0$ of $f(t) \in C_\mu$, $\mu > -1$ is:

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \quad t > 0 \quad \text{… (2.1)}$$

$$J^0 f(t) = f(t)$$

Definition 2.3 [14]: The fractional derivative of Caputo of $f(t) \in C_\mu$ is :

$$D_*^\alpha f(t) = J^{r-\alpha} D^r f(t) = \frac{1}{\Gamma(r-\alpha)} \int_0^t (t-\tau)^{r-\alpha-1} f^{(r)}(\tau)d\tau, \quad t > 0 \quad \text{… (2.2)}$$

For $r - 1 < \alpha \le r$, $r \in N$, $t > 0$, $f \in C_{-1}^r$.

Definition 2.4 [7]: The fractional Laplace transform of the Caputo derivative is:

$$\mathcal{L}[D_*^\alpha w(z,t)] = s^\alpha \mathcal{L}[w(z,t)] - \sum_{j=0}^{r-1} s^{(\alpha-j-1)} w^{(j)}(z,0), \quad r-1 < \alpha \le r. \quad \text{… (2.3)}$$

3. The Homotopy Transforms Analysis Method (HTAM) **[14]**

To explain the main idea, let us study the fractional partial differential equation:

$$D_t^\alpha w(z,t) = f(w, w_z, w_{zz}), \quad 0 < \alpha \le 2, \ t \ge 0 \quad \text{... (3.1)}$$

according to the initial conditions

$$w(z,0) = g_1(z), \ w_t(z,0) = g_2(z) \quad \text{... (3.2)}$$

The function f refers to the linear or nonlinear sense and D_t^α is a time fractional derivative operator.

We re-write eq.(3.1) as:

$$D_t^\alpha w(z,t) = L(w, w_z, w_{zz}) + N(w, w_z, w_{zz}) + C(z,t), \quad 0 < \alpha \le 2, \ t \ge 0 \quad \text{... (3.3)}$$

where L refers to the linear part and N is the nonlinear part. The function C is analytic.

Using the differentiation property of Laplace transform, we have:

$$\mathcal{L}[D_t^\alpha w(z,t)] = \mathcal{L}[L(w, w_z, w_{zz}) + N(w, w_z, w_{zz}) + C(z,t),]$$

$$s^\propto \mathcal{L}[w(z,t)] - \sum_{j=0}^{1} s^{\propto-j-1} w^{(j)}(z,0)$$

$$= \mathcal{L}[L(w, w_z, w_{zz})] + \mathcal{L}[N(w, w_z, w_{zz})] + \mathcal{L}[C(z,t)], \quad \text{... (3.4)}$$

By simplifying, we have

$$s^{\propto}\mathcal{L}[w(z,t)]-\sum_{j=0}^{1}s^{\propto-j-1}w^{(j)}(z,0)=\mathcal{L}[L(w,w_z,w_{zz})]+\mathcal{L}[N(w,w_z,w_{zz})]+\mathcal{L}[C(z,t)],$$

$$s^{\propto}\mathcal{L}[w(z,t)]-s^{\propto-1}g_1(z)-s^{\propto-2}g_2(z)=\mathcal{L}[L(w,w_z,w_{zz})]+\mathcal{L}[N(w,w_z,w_{zz})]+\mathcal{L}[C(z,t)],$$

$$\mathcal{L}[w(z,t)]-\frac{s^{\propto-1}}{s^{\propto}}g_1(z)-\frac{s^{\propto-2}}{s^{\propto}}g_2(z)=\frac{1}{s^{\propto}}\mathcal{L}[L(w,w_z,w_{zz})]+\frac{1}{s^{\propto}}\mathcal{L}[N(w,w_z,w_{zz})]+\frac{1}{s^{\propto}}\mathcal{L}[C(z,t)],$$

$$\mathcal{L}[w(z,t)]=\frac{s^{\propto-1}}{s^{\propto}}g_1(z)+\frac{s^{\propto-2}}{s^{\propto}}g_2(z)+\frac{1}{s^{\propto}}\mathcal{L}[L(w,w_z,w_{zz})]+\frac{1}{s^{\propto}}\mathcal{L}[N(w,w_z,w_{zz})]+\frac{1}{s^{\propto}}\mathcal{L}[C(z,t)], \quad \dots(3.5)$$

The nonlinear operator is:

$$N[\emptyset(z,t;q)]=L[\emptyset(z,t;q)]-\frac{s^{\propto-1}}{s^{\propto}}g_1(z)-\frac{s^{\propto-2}}{s^{\propto}}g_2(z)+\frac{1}{s^{\propto}}\mathcal{L}\big[L\big(\emptyset(z,t;q),\emptyset_z(x,t;q),\emptyset_{zz}(z,t;q)\big)\big]+\frac{1}{s^{\propto}}\mathcal{L}\big[N\big(\emptyset(z,t;q),\emptyset_z(x,t;q),\emptyset_{zz}(z,t;q)\big)\big]+\frac{1}{s^{\propto}}\mathcal{L}[C(z,t)], \quad \dots(3.6)$$

where $\emptyset(z,t;q)$ is a real function of z, t and q. We combine a homotopy:

$$(1-q)\mathcal{L}[\emptyset(z,t;q)-w_0(z,t)]=qh\Bigg[L[\emptyset(z,t;q)]-\frac{s^{\propto-1}}{s^{\propto}}g_1(z)-\frac{s^{\propto-2}}{s^{\propto}}g_2(z)+\frac{1}{s^{\propto}}\mathcal{L}\big[L\big(\emptyset(z,t;q),\emptyset_z(z,t;q),\emptyset_{zz}(z,t;q)\big)\big]+\frac{1}{s^{\propto}}\mathcal{L}\big[N\big(\emptyset(z,t;q),\emptyset_z(z,t;q),\emptyset_{zz}(z,t;q)\big)\big]+\frac{1}{s^{\propto}}\mathcal{L}[C(z,t)]\Bigg], \quad \dots(3.7)$$

The variable q is a parameter of embedding $q\in[0,1]$. If $q=0$ and $q=1$, then $\mathcal{L}[\emptyset(z,t;0)]=\mathcal{L}[w_0(z,t)]$ and $\mathcal{L}[\emptyset(z,t;1)]=\mathcal{L}[w(z,t)]$. Thus, when q increases from 0 to 1, $\emptyset(z,t;q)$ varies from $w_0(z,t)$ to $w(z,t)$. By extending $\emptyset(z,t;q)$ to Taylor series with respect to q, we obtain:

$$\mathcal{L}[\emptyset(z,t;q)]=\mathcal{L}[w_0(z,t)]+\sum_{r=1}^{\infty}\mathcal{L}[w_r(z,t)]\,q^r \quad \dots(3.8)$$

and

$$\mathcal{L}[w_r(z,t)]=\frac{1}{r!}\frac{\partial^r}{\partial x^r}\,\mathcal{L}[\emptyset(z,t;q)]\;\Big|_{q=0} \quad \dots(3.9)$$

We define the following vectors:

$$\overline{w}_r(z,t)=\{\mathcal{L}[w_0(z,t)],\mathcal{L}[w_1(z,t)],\mathcal{L}[w_2(z,t)],\quad \dots,\quad \mathcal{L}[w_r(z,t)]\} \quad \dots(3.10)$$

We differentiate eq. (3.7) r-times with respect to q, then by setting $q=0$, $h=-1$ and dividing by r!, then:

$$\mathcal{L}[w_r(z,t)]=x_r\mathcal{L}[w_r(z,t)]-R_r\big(\overline{w}_{r-1}(z,t)\big) \quad \dots(3.11)$$

and

$$R_r\big(\overline{w}_{r-1}(z,t)\big)=\mathcal{L}[w_r(z,t)]-\frac{1}{s^{\propto}}\left(\frac{1}{(r-1)!}\frac{\partial^{r-1}}{\partial q^{r-1}}\Big[\mathcal{L}\big[L\big(\emptyset(z,t;q),\emptyset_z(z,t;q),\emptyset_{zz}(z,t;q)\big)\big]+\mathcal{L}\big[N\big(\emptyset(z,t;q),\emptyset_z(z,t;q),\emptyset_{zz}(z,t;q)\big)\big]\Big]\right)_{q=0}-\left(\frac{g_1(z)}{s^{\propto}}+\frac{g_2(z)}{s^2}+\frac{1}{s^{\propto}}\mathcal{L}[C(z,t)]\right)(1-x_r) \quad \dots(3.12)$$

and

$$x_r = \begin{cases} 0, & r \le 1 \\ 1, & r > 1 \end{cases}$$

We take the inverse of Laplace transform of eq. (3.11), then the solution of (3.1) is:

$$w(z,t) = \sum_{r=0}^{\infty} w_r(z,t)$$

4. Applications and Results

The Navier- Stokes equation (1.1) in cylindrical coordinates for unsteady one dimensional motion of a viscous fluid is given by

$$D_*^{\alpha} w = p + \upsilon\left(\frac{\partial^2 w}{\partial z^2} + \frac{1}{z}\frac{\partial w}{\partial z}\right)$$

In this part, some applications to HTAM are introduced for solving Navier- Stokes equations of fractional order derivative.

Example 4.1: Firsty, we study the differential equation of fractional order:

$$D_*^{\alpha} w = p + \frac{\partial^2 w}{\partial z^2} + \frac{1}{z}\frac{\partial w}{\partial z} \quad \text{... (4.1)}$$

According to the initial conditions

$$w(z,0) = 1 - z^2 \quad \text{... (4.2)}$$

the Laplace transform of eq.(4.1) is:

$$\mathcal{L}[w(z,t)] - \frac{1}{s}(1-z^2) - \frac{1}{s^{\propto}}\mathcal{L}[P] - \frac{1}{s^{\propto}}\ \mathcal{L}\left[\frac{\partial^2 w}{\partial z^2} + \frac{1}{z}\frac{\partial w}{\partial z}\right] = 0 \quad \text{... (4.3)}$$

The nonlinear term is:

$$N[\emptyset(z,t;q)] = \mathcal{L}[\emptyset(z,t;q)] - \frac{1}{s}(1-z^2) - \frac{1}{s^{\propto}}\mathcal{L}[P] - \frac{1}{s^{\propto}}\ \mathcal{L}\left[\frac{\partial^2}{\partial z^2}\emptyset(z,t;q) + \frac{1}{z}\frac{\partial}{\partial z}\emptyset(z,t;q)\right], \quad \text{... (4.4)}$$

thus

$$R_r\left(\vec{w}_{r-1}(z,t)\right) = \mathcal{L}[w_{r-1}(z,t)] - \frac{1}{s^{\propto}}\ \mathcal{L}\left[\frac{\partial^2}{\partial z^2}w_{r-1} + \frac{1}{z}\frac{\partial}{\partial z}w_{r-1}\right] - (1-x_r)\ \frac{1}{s}(1-z^2) - \frac{1}{s^{\propto+1}}\mathcal{L}[P](1-x_r) \quad \text{... (4.5)}$$

The r^{th} - order deformation equation is:

$$\mathcal{L}[w_r(z,t) - x_r w_{r-1}(z,t)] = hR_r\left(\vec{w}_{r-1}(z,t)\right) \quad \text{... (4.6)}$$

Next step is to apply the inverse Laplace transform:

$$w_m(z,t) - x_m w_{m-1}(z,t) = h\,\mathcal{L}^{-1}\left[R_m\left(\vec{w}_{m-1}(z,t)\right)\right] \quad \text{... (4.7)}$$

By solving the above eq. (4.7), for r=1, 2, 3, …, then

$$w_0(z,t) = 1 - z^2$$

$$w_1(z,t) = -\frac{h(-4+P)}{\Gamma(\alpha+1)}t^{\propto}$$

$$w_2(z,t) = -\frac{h(-4+P)(1+h)}{\Gamma(\alpha+1)}t^{\propto}$$

$$w_3(z,t) = -\frac{h(-4+P)(1+h)^2}{\Gamma(\alpha+1)}t^{\propto}$$

$$w_4(z,t) = -\frac{h(-4+P)(1+h)^3}{\Gamma(\alpha+1)}t^{\propto}.$$

And so on.

Concluding that

$$w(z,t) = w_0(z,t) + w_1(z,t) + w_2(x,t) + w_3(z,t) + w_4(z,t) + \cdots$$

Then

$$w(z,t)=1-z^2-\frac{h(-4+P)}{\Gamma(\alpha+1)}t^{\propto}-\frac{h(-4+P)(1+h)}{\Gamma(\alpha+1)}t^{\propto}-\frac{h(-4+P)(1+h)^2}{\Gamma(\alpha+1)}t^{\propto}-\frac{h(-4+P)(1+h)^3}{\Gamma(\alpha+1)}t^{\propto}+\cdots$$

$$w(z,t)=1-z^2-\frac{h(-4+P)}{\Gamma(\alpha+1)}t^{\propto}[1+(1+h)+(1+h)^2+(1+h)^3+\cdots] \quad \text{... (4.8)}$$

If h less than zero then the geometric series [11] will be convergent at infinity, the solution takes the form

$$w(z,t)=1-z^2+\frac{(-4+P)}{\Gamma(\alpha+1)}t^{\propto} \quad \text{... (4.9)}$$

If $\propto=1$, then the exact solution of eq.(4.1) $w(z,t)=1-z^2+(-4+P)$.

This solution is exactly the same solution obtained by Ragab *et al.* [13] and by Momani and Odibat [9].

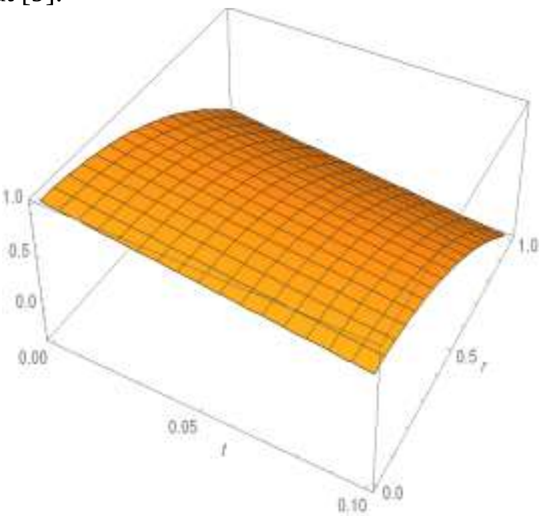

Figure 1-Solution plots of Eq. (4.9) for $(\alpha=1)$. Example 4.1

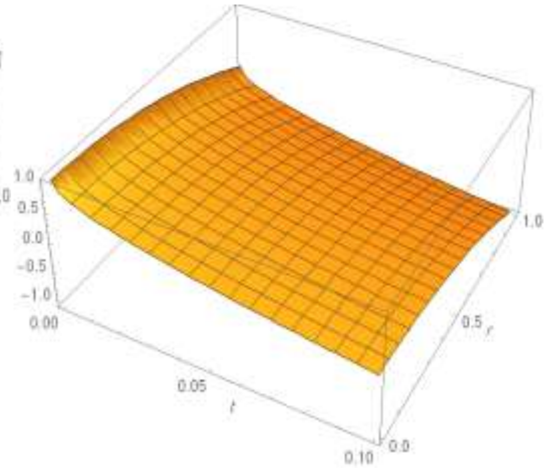

Figure 2-Solution plots of Eq. (4.9) for $(\alpha=0.5)$. Example 4.1

Example 4. 2: studying the fractional derivatives differential equation:

$$D_*^{\alpha}w=\frac{\partial^2 w}{\partial z^2}+\frac{1}{z}\frac{\partial w}{\partial z} \quad \text{... (4.10)}$$

According to the initial conditions

$$w(z,0)=z \quad \text{... (4.11)}$$

the Laplace transform of eq. (4.10) is:

$$\mathcal{L}[w(z,t)]-\frac{1}{s}z-\frac{1}{s^{\propto}}\ \mathcal{L}\left[\frac{\partial^2 w}{\partial z^2}+\frac{1}{z}\frac{\partial w}{\partial z}\right]=0 \quad \text{... (4.12)}$$

The nonlinear term is

$$N[\emptyset(z,t;q)]=\mathcal{L}[\emptyset(z,t;q)]-\frac{1}{s}z-\frac{1}{s^{\propto}}\ \mathcal{L}\left[\frac{\partial^2}{\partial z^2}\emptyset(z,t;q)+\frac{1}{z}\frac{\partial}{\partial z}\emptyset(z,t;q)\right], \quad \text{... (4.13)}$$

thus

$$R_r(\vec{w}_{r-1}(z,t))=\mathcal{L}[w_{r-1}(z,t)]-\frac{1}{s^{\propto}}\ L\left[\frac{\partial^2}{\partial z^2}w_{r-1}(z,t)+\frac{1}{z}\frac{\partial}{\partial z}w_{r-1}(z,t)\right]-\ \frac{1}{s}z\,(1-x_r) \quad (4.14)$$

The r^{th} - order deformation equation is :

$$\mathcal{L}[w_r(z,t)-x_r w_{r-1}(z,t)]=hR_r(\vec{w}_{r-1}(z,t)) \quad \text{... (4.15)}$$

By taking the inverse Laplace transform of eq. (4.15), we have:

$$w_r(z,t)-x_r w_{r-1}(z,t)=h\,\mathcal{L}^{-1}[R_r(\vec{w}_{r-1}(z,t))] \quad \text{... (4.16)}$$

Solving the above eq. (4.16), for r=1, 2, 3, …, then:

$$w_0(z,t)=z$$

$$w_1(z,t)=-\frac{h\,t^{\propto}}{z\,\Gamma(1+\alpha)}$$

$$w_2(z,t) = -\frac{h(1+h)\,t^{\propto}}{z\,\Gamma(1+\alpha)} + \frac{h^2\,t^{2\propto}}{z^3\,\Gamma(1+2\alpha)}$$

$$w_3(z,t) = -\frac{h(1+h)^2\,t^{\propto}}{z\,\Gamma(1+\alpha)} + \frac{h^2(1+h)\,t^{2\propto}}{z^3\,\Gamma(1+2\alpha)} - \frac{9h^3\,t^{3\propto}}{z^5\,\Gamma(1+3\alpha)}$$

$$w_4(x,t) = -\frac{h\,(1+h)^3 t^{\propto}}{z\,\Gamma(1+\alpha)} + \frac{h^2\,(1+h)^2 t^{2\propto}}{z^3\,\Gamma(1+2\alpha)} - \frac{3h^3(1+h)9\,t^{3\propto}}{z^5\,\Gamma(1+3\alpha)} + \frac{9h^4\,25t^{3\propto}}{z^7\,\Gamma(1+4\alpha)}$$

Concluding that

$$w(z,t) = w_0(z,t) + w_1(z,t) + w_2(z,t) + w_3(z,t) + w_4(z,t) + \cdots$$

$$= z - \frac{h\,t^{\propto}}{z\,\Gamma(1+\alpha)} - \frac{h(1+h)\,t^{\propto}}{z\,\Gamma(1+\alpha)} + \frac{h^2\,t^{2\propto}}{z^3\,\Gamma(1+2\alpha)} - \frac{h(1+h)^2\,t^{\propto}}{z\,\Gamma(1+\alpha)} + \frac{h^2(1+h)\,t^{2\propto}}{z^3\,\Gamma(1+2\alpha)} - \frac{9h^3\,t^{3\propto}}{z^5\,\Gamma(1+3\alpha)} - \frac{h\,(1+h)^3 t^{\propto}}{z\,\Gamma(1+\alpha)} + \frac{h^2\,(1+h)^2 t^{2\propto}}{z^3\,\Gamma(1+2\alpha)} - \frac{3h^3(1+h)9\,t^{3\propto}}{z^5\,\Gamma(1+3\alpha)} + \frac{9h^4\,25t^{3\propto}}{z^7\,\Gamma(1+4\alpha)} + \cdots$$

$$= z - \frac{h\,t^{\propto}}{z\,\Gamma(\alpha+1)}[1 + (1+h) + (1+h)^2 + (1+h)^3 + \cdots] + \frac{h^2\,t^{2\propto}}{z^3\,\Gamma(2\alpha+1)}[1 + 2\,(1+h) + 3(1+h)^2 + \cdots] - \frac{9h^3\,t^{3\propto}}{z^5\,\Gamma(3\alpha+1)}[1 + 3(1+h) + 6(1+h)^2 + \cdots] + \cdots$$

If we take h as less than zero then the geometric series [11] will be convergent at infinity, the solution takes the form

$$w(z,t) = z + \sum_{j=1}^{\infty} \frac{1^2 \times 3^2 \times \ldots \times (2j-3)^2}{z^{2j-1}}\ \frac{t^{j\propto}}{\Gamma(j\alpha+1)} \qquad \ldots (4.17)$$

This solution is exactly the same solution obtained by Momani and Odibat [9].
If we put $\propto = 1$, then solution of eq. (4.10) is:

$$w(z,t) = z + \sum_{j=1}^{\infty} \frac{1^2 \times 3^2 \times \ldots \times (2j-3)^2}{z^{2j-1}}\ \frac{t^j}{j!}$$

This solution is exactly the same solution obtained by Biazar *et al.* [8] and Ragab *et al.* [13].

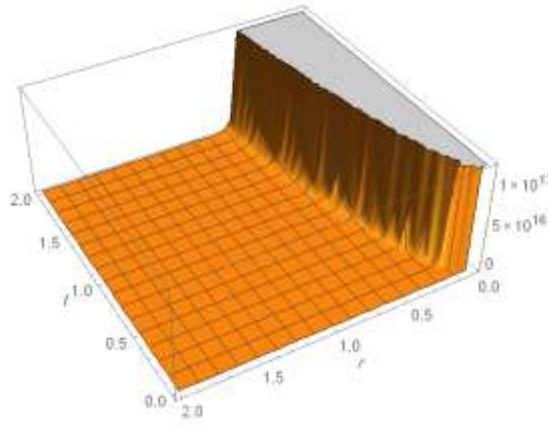

Figure 3-Solution plots of Eq. (4.17) for $(\alpha = 1)$. Example 4.2

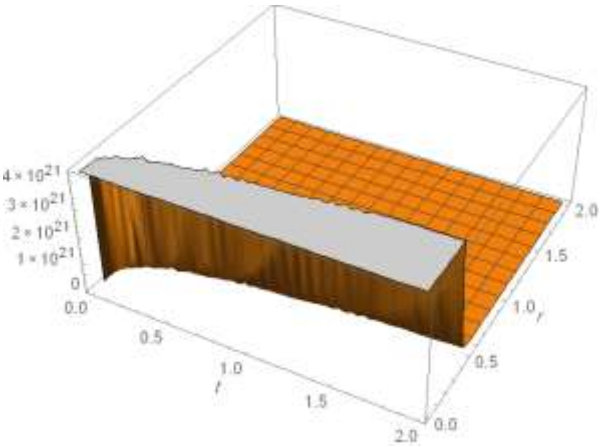

Figure 4-Solution plots of Eq. (4.17) for $(\alpha = 0.5)$. Example 4.2

Conclusions

In this work, the introduced method desegregated the Laplace Transforms with Homotopy Analysis method for fractional derivative equations. This combination produced a very powerful method called

The Homotopy Transforms Analysis Method (HTAM). The proposed method was used to solve the Navier- Stokes equations of fractional order derivative. HTAM showed accurate effective and simple results for solving various types of problems. We got that the approximation solution of proposed problem tends to exact solution rapidly. Finally, we conclude that HTAM may be a good tool to numerically solve the fractional order nonlinear partial differential equations.

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