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Improved High order Euler Method for Numerical Solution of Initial value Time- Lag Differential Equations

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Abstract

The goal of this paper is to expose a new numerical method for solving initial value time-lag of delay differential equations by employing a high order improving formula of Euler method known as third order Euler method. Stability condition is discussed in detail for the proposed technique. Finally some examples are illustrated to verify the validity, efficiency and accuracy of the method.

Keywords: Time-Lag, Euler method, Differential Equation.

طريقة اويلر المطورة ذات الرتبة العليا للتعامل العددي لمسائل القيم الابتدائية للمعادلات التفاضلية التباطؤية

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الخلاصة

الهدف الاساسي من هذا البحث هو عرض طريقة عددية جديدة لحل مسائل القيم الابتدائية للمعادلات التفاضلية التباطؤية من خلال تسخير صيغة ذو رتبة عليا مطورة من طريقة اويلر العددية المسماة بطريقة اويلر ذو الرتبة الثالثة. وقد تم مناقشة شرط استقرارية التقنية المعروضة بالتفصيل. واخيرا تم توضيح بعض الامثلة للتحقق من صلاحية و كفاءة والدقة التي تصل الى الرتبة الثالثة للطريقة.

1. Introduction

Delay differential equations (DDEs) and ordinary differential equations are similar, but they differ in evaluation. The first equations determination involve past values of the independent variable [1]. The solution of delay differential equations requires existence of not only the current state, but also of the previous state at certain time. DDEs have numerous applications in (physiological, pharmaceutical kinetics, chemical kinetics, population dynamics, and the navigational control of ships and air craft) [2].

In this paper, we only concerned with retarded initial value delay differential problems (IVRDDE) of the following form

$$\left. \begin{aligned} y'(x) &= f(x, y(x), y(x - \tau)) + r(x) \\ y(x_0) &= y_0 \\ y(x) &= \phi(x) \quad x_0 - \tau \leq x \leq x_0 \end{aligned} \right\} x \geq x_0 \quad \dots\dots\dots(1)$$

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$y(x)$ is an n - vector valued function, $\tau > 0$ is a constant delay (retardation), and $\phi(x)$ is the initial function, which is supposed to be piecewise continuous for $x_0 - \tau \leq x \leq x_0$, and $r(x)$ is a source term function. Several numerical methods introduced for solving delay differential equations (DDEs) such as n's method of steps [1], modified Euler method and Runge- Kutta method [3], A domain decomposition method [4], Haar wavelets method [5], Zhou[6] present the Euler – Maruyama method as a numerical method for solving nonlinear stochastic delay differential equation, and Ekinci [7] discussed neutral initial value problem numerically .

Euler method is one of the most common methods that used to solve initial value problem (ordinary and delay differential equations) many authors and researchers have considered this method (see, e.g [3,8]), also many other authors have improved on the popular method of Euler developed between 1768 and 1770 [9] because of its ease of implementation. In this work, we will try to apply a new Euler improvement method that called three orders Euler method as an efficient modern numerical method to deal with our problem retarded delay differential equations (RDDEs).

2. Third Order Euler Method (TOEM)

Euler method and all single- step methods are depended on the principle of discretization. These methods have the common feature that no attempt is made to approximate the exact solution over continuous range of the independent variable. The general form of newly third order Euler method is:

$$y_{i+1}(x) = y_i(x) + hf(x_i + \frac{1}{2}h, y_i(x)) + \frac{1}{2}hf(x_i + \frac{1}{2}h, y_i(x) + \frac{1}{3}hf(x_i, y_i(x))) + o(h^4) ..(2)$$

The properties of the increment function φ_{TOEM} the right-hand side of TOEM is in general very crucial to its stability and convergence characteristics. These properties are also investigated to be able to ascertain how efficient is the new method being proposed [3]. For any initial value retarded delay differential equation given by eq.(1), we are interested in finding the numerical solution $y(x)$ for our problem (IVRDDEs) using TOEM.

Theorem (1.1) [3]:

The existence of such numerical solution $y(x)$ is guaranteed and unique provided that $f(x, y(x), y(x - \tau))$

- is continuous in the infinite strip $I = \{x_0 \leq x \leq T, |y| < \infty\}$
- and is, more specifically, Lipschitz continuous in the depended variables $y(x)$ and $y(x - \tau)$ over the same region I, i.e \exists a positive constant L such that

$$\forall(x, y(x), y(x - \tau)), (x, \hat{y}(x), \hat{y}(x - \tau)) \in I \dots\dots\dots(3)$$

$$|f(x, y(x), y(x - \tau)) - f(x, \hat{y}(x), \hat{y}(x - \tau))| \leq L(|y(x) - \hat{y}(x)| + |y(x - \tau) - \hat{y}(x - \tau)|) \dots\dots(4)$$

Lemma (1.1):-

Let $\{\delta_i \ i = 0(1)n\}$ be a set of real numbers. If there exist finite constants Γ and Π such that

$$|\delta_{i+1}| \leq \Gamma|e_i| + \Pi \ i = 0(1)n - 1, \text{ then } |\delta_i| \leq \frac{r^i - 1}{r - 1} \Pi + r^i |e_0|, r \neq 1.$$

Proof: (see [8])

2.1. Stability of Third Order Euler Method for Solving (IVRDDE)

In this section we state the theorem that needed to prove the stability of the proposed TOEM in solving our problem.

Theorem (2.1.1):

Suppose the retarded delay differential satisfies hypotheses of theorem (1.1), then the TOEM is stable.

proof: Let Y_n and Z_n be two sets, of solutions generated recursively by the

TOEM with the initial conditions $y(x_0) = y_0$, and $z(x_0) = z_0$

$$|y_0 - z_0| = \delta_0$$

let $\delta_n = y_n - z_n, n \geq 0$ (5)

and

$$y_{n+1} = y_n + h\varphi_{TOEM}(x_n, y_n(x), y_n(x - \tau); h) \dots\dots\dots(6)$$

$$z_{n+1} = z_n + h\varphi_{TOEM}(x_n, z_n(x), z_n(x - \tau); h) \dots\dots\dots(7)$$

This implies that

$$y_{n+1} - z_{n+1} = y_n - z_n + h(\varphi_{TOEM}(x_n, y_n(x), y_n(x - \tau); h) - \varphi_{TOEM}(x_n, z_n(x), z_n(x - \tau); h)) \dots(8)$$

using eq(5) and triangle inequality, we have $|\delta_{n+1}| = (1 + hL)|\delta_n|, n \geq 0$

If we assume $(\Gamma = 1 + hL)$ and $\Pi = 0$ using lemma (1.1) implies that $|\delta_n| \leq k |\delta_0|$

where $k = e^{L\tau} < \alpha$

which implies the stability of the proposed method.

3. Third Order Euler Method for solving Initial Value Retarded Delay Differential Equations

In this section, we present third order Euler method to numerically solve the following initial value delay differential equations

$$\left. \begin{aligned} y'(x) &= f(x, y(x), y(x - \tau)) + r(x) \\ y(x_0) &= y_0 \\ y(x) &= \phi(x) \quad x_0 - \tau \leq x \leq x_0 \end{aligned} \right\} x \geq x_0 \dots\dots\dots(9)$$

Eq.(9) can be solved if we use the initial function as follows

$$y' = f(x, y(x), \phi(x - \tau)) + r(x) \dots\dots\dots(10)$$

with initial condition $y(x_0) = y_0$

For solving the above IVRDDEs in (9), and (10) using TOEM are written as

$$y_{i+1} = y_i + hf(x_i + \frac{1}{2}h, y_i(x_i) + \frac{1}{2}hf(x_i + \frac{1}{2}h, y_i(x_i) + \frac{1}{3}hf(x_i, y_i(x_i), y_i(x_i - \tau)), y_i(x_i - \tau) + \frac{1}{2}hf(x_i + \frac{1}{2}h, y_i(x_i) + \frac{1}{3}hf(x_i, y_i(x_i), y_i(x_i - \tau)), y_i(x_i - \tau) + \frac{1}{3}hf(x_i, y_i(x_i), y_i(x_i - \tau))) \dots\dots\dots(11)$$

For each (i=0, 1, ..., n)

4. Algorithm TOEM

Step 1 Set $h = \frac{\tau}{n}$

Step 2 Input the initial values or initial function $\phi(x)$ or y_0 .

Step 3 Choose $x_{i+1} = x_i + h$ for, $i = 0, \dots, n$.

Step 4 compute eq.(11) for, $i = 0, \dots, n$.

Step 5 Compute the exact solution for $y_i, i = 1, \dots, n$.

Step 6 Set the output numerical and exact y_i in a table

5. Numerical Examples

In order to illustrate the advantages and the accuracy of the proposed method for solving this kind of problems we have applied the method to different examples.

Example 1: Consider the following first order initial value retarded delay

differential equation
$$y'(x) = \frac{1}{2}e^{\frac{x}{2}}y\left(\frac{x}{2}\right) + \frac{1}{2}y(x)$$

with initial function
$$y(x) = e^x \quad -\frac{1}{2} \leq x \leq 0$$

and initial condition $y(0) = 1$

Table-1 shows the absolute error comparison between the exact solution and approximate solution using algorithm TOEM using MATLAB program.

Table 1- (Numerical results using third order Euler method TOEM).

X	Third-Order Euler(TOEM)	Exact Solution	$ Exact - TOEM $
0	1	1	0.0000
0.1000	1.1066	1.1052	0.0014
0.2000	1.2230	1.2214	0.0016
0.3000	1.3519	1.3499	0.0020
0.4000	1.4943	1.4918	0.0025
0.5000	1.6517	1.6487	0.0030
0.6000	1.8257	1.8221	0.0036
0.7000	2.0181	2.0138	0.0043
0.8000	2.2307	2.2255	0.0052
0.9000	2.4658	2.4596	0.0062
1.0000	2.7256	2.7183	0.0073

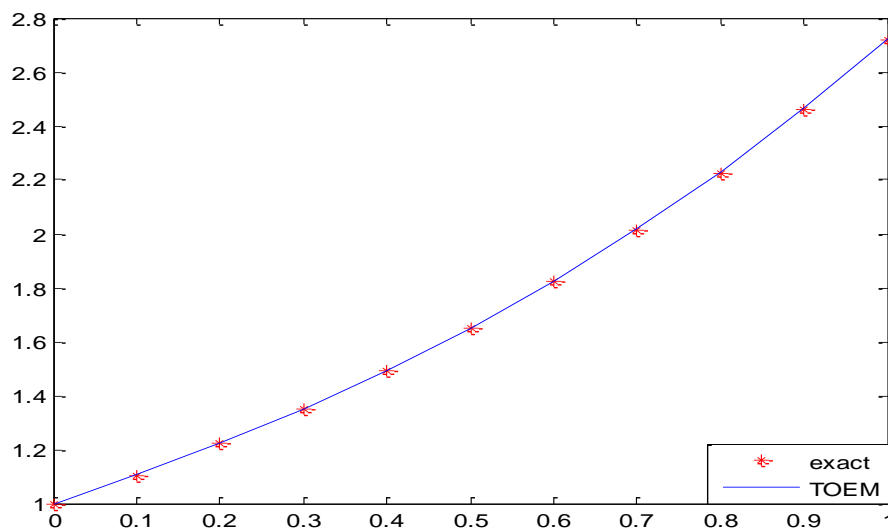


Figure 1-The Exact and Numerical results by (TOEM) for Example 1.

Example 2: Consider the following first order initial value retarded delay differential equation

$$y'(x) - \sqrt{y(x-1)} = \frac{1}{2}$$
 with initial function
$$y(x) = \frac{1}{4}(x+2)^2 \quad -1 \leq x \leq 0$$

and initial condition $y(0) = 1$

Table-2 shows the absolute error comparison between the exact solution and the numerical solution using algorithm TOEM in Matlab program.

Table 2-(Numerical results using Third order Euler Method for DDEs).

X	numerical results by TOEM	Exact solution	$ Exact - TOEM $
0	1	1	0.0000
0.1000	1.1049	1.1025	0.0024
0.2000	1.2122	1.2100	0.0022
0.3000	1.3245	1.3225	0.0020
0.4000	1.4419	1.4400	0.0019
0.5000	1.5643	1.5625	0.0018
0.6000	1.6916	1.6900	0.0016
0.7000	1.8240	1.8225	0.0015
0.8000	1.9614	1.9600	0.0014
0.9000	2.1039	2.1025	0.0014
1.0000	2.2513	2.2500	0.0013

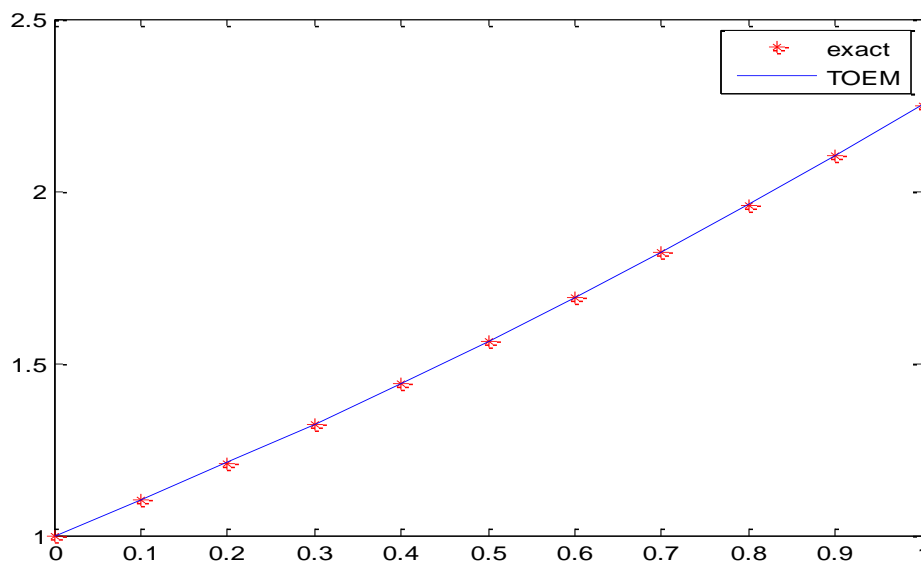


Figure 2-Comparison between the Exact and Numerical results by TOEM for Example 2.

Example 3: Consider the following first order initial value retarded delay differential equation

$$y'(x) - y(x) + y(x - \frac{\pi}{2}) = -\sin(x) \quad \text{with initial function } y(x) = \sin(x) \quad -\frac{\pi}{2} \leq x \leq 0$$

and initial condition $y(0) = 0$

Table-3 listed the exact and numerical results determined using our new algorithm TOEM by running MATLAB program. The absolute errors of the two results are listed too.

Table 3-(Numerical results using Third order Euler Method for Example 3).

X	Numerical results by TOEM	Exact solution	$ Exact - TOEM $
0	0	0	0.0000
0.1000	0.095	0.0998	0.0048
0.2000	0.1944	0.1987	0.0043
0.3000	0.2918	0.2955	0.0037
0.4000	0.3863	0.3894	0.0031
0.5000	0.4770	0.4794	0.0025

0.6000	0.5629	0.5646	0.0018
0.7000	0.6431	0.6442	0.0011
0.8000	0.7170	0.7174	3.9629e-004
0.9000	0.7836	0.7833	3.0742e-004
1.0000	0.8425	0.8415	0.0010

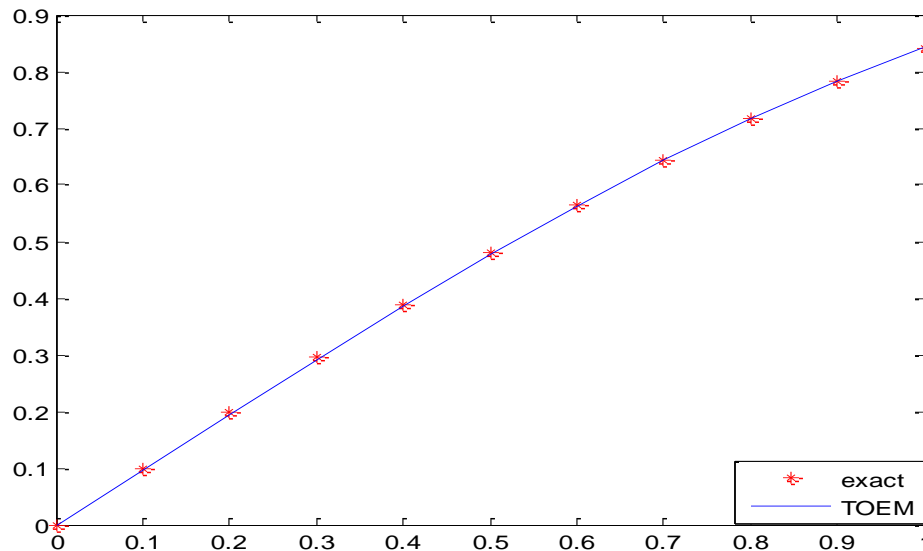


Figure 3-The numerical results by TOEM and the exact results of Example 3.

6. Conclusion

The third order Euler method proposed as a numerical method to solve our problem initial value retarded differential equation. From the analysis and the computational results listed in the tables and the figures, it is evident that the method is efficient. The test of stability function also reveals that the method is of order three. Therefore we conclude that the third order Euler method proposed is efficient and third order accurate.

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