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On Dense Subsemimodules and Prime Semimodules

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Abstract

In this paper, we study the class of prime semimodules and the related concepts, such as the class of π semimodules, the class of Dedekind semidomains, the class of prime semimodules which is invariant subsemimodules of its injective hull, and the compressible semimodules. In order to make the work as complete as possible, we stated, and sometimes proved, some known results related to the above concepts.

Keywords: Semimodule, Semiring, Dense subsemimodule, Invertible ideal, Prime semimodule, Dedekind semidomain.

حول شبه المقاسات الجزئية الكثيفة و شبه المقاسات الاولية

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الخلاصة

في هذا البحث, ندرس صنف شبه المقاسات الأولية والمفاهيم ذات الصلة, مثل صنف شبه المقاسات من النمط π, صنف شبه ساحات من النمط ديديكاند, صنف شبه المقاسات الأولية التي تمثل شبه مقاسات جزئية لا متغيرة في غلافها التبايني, وشبه مقاسات القابلة للضغط. من أجل جعل العمل كاملاً قدر الإمكان ، ذكرنا ، وفي بعض الوقت ، أثبتنا بعض النتائج المعروفة المتعلقة بالمفاهيم المذكورة أعلاه.

Introduction

Throughout this paper, R will denote a commutative semiring with identity, and M is an R-semimodule.

This paper consists of three sections. In Section one, we introduce some definitions and remarks which we will use in the paper. In Section two, we introduce the concept of density of semimodules. A non-zero R-subsemimodule of an R-semimodule is said to be dense in M, if $M = \sum_{\phi} \phi(N)$, where the sum is taken over all $\phi \in \mathrm{Hom}(N, M)$. We use the density concept to define the class of π semimodules, as M is said to be π semimodule if each non-zero subtractive subsemimodule of M is dense in M.

In Section three, we define the concept of prime semimodules, analogous to that in modules [4], where M is said to be prime if ann(N) =ann(M), for each non-zero subtractive subsemimodule N of M. Similar to that in modules [1], we will show that every π semimodule is a prime semimodule.

The aim of this paper is to discuss the converse of this statement in the case of semimodules having injective hull. Also we generalize some types of prime modules for semimodules, such as the compressible type.

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1. Preliminaries

In this section, we introduce some of definitions, remarks, and examples that might be needed in the main results.

Definition 1.1.[19] A nonempty set R with two operations of addition and multiplication (denoted by + and $\cdot$, respectively) is called a **semiring,** provided that:

1. $(R, +)$ is a commutative monoid (A monoid is a semigroup with identity)with identity element 0;
2. $(R,\cdot)$ is a monoid with identity element $1 \neq 0$;
3. Multiplication distributes over addition, i.e. $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$; for all $a, b, c \in R$.
4. The element 0 is the absorbing element of the multiplication, i.e. $r \cdot 0 = 0$ for all $r \in R$.

The semiring R is said to be commutative if its multiplication is commutative.

Definition 1.2.[18] A non-empty subset I of a semiring R will be called an **ideal** of R if $a, b \in I$ and $r \in R$ imply $a + b \in I$, ra , and $ar \in I$.

Definition 1.3.[6] A semiring R is said to be a **semidomain** if $ab = 0$, $(a, b \in R)$ then either $a = 0$ or $b = 0$.

Definition 1.4.[10] A semiring R is **semisubtractive** if, for all $x, y \in R$, then $x + h = y$ or $x = y + h$ for some $h \in R$.

Definition 1.5.[10] Let R be a semiring, a **left *R*-semimodule** is a commutative monoid $(M, +)$ with additive identity 0 for which we have a function $R \times M \longrightarrow M$ defined by $(r, x) \mapsto rx$ (scalar multipli-cation), which satisfies the following condition, for all $x, y \in M$ and for all $r, s \in R$:

1. $(rs)x = r(sx)$
2. $r(x + y) = rx + ry$
3. $(r + s)x = rx + sy$
4. $0_R\, m = 0 = 0x$

If the condition $1x = x$ for all $x \in M$ holds, then the semimodule M is said to be unitary.

Definition 1.6.[12] A non-empty subset N of a left R-semimodule M is called **subsemimodule** of M if N is closed under addition and scalar multiplication, that is N is a semimodule itself (denoted by $N \hookrightarrow M$).

Definition 1.7.[6] Let M be an R-semimodule. A **subtractive** subsemimodule (or k-subsemimodule) N is a subsemimodule of M such that if $x, x + y \in N$, then $y \in N$. We define subtractive ideals (k-ideals) of a semiring R in an analogous manner.

Definition 1.8. Let S be a non-empty subset of an R-semimodule M. Then the intersection of all subsemimodules of M containing S is a subsemimodule of M, called a subsemimodule generated by S and denoted by RS. It is easy to verify that

$$RS = \{\sum_{i=1}^{k} r_i s_i \mid r_i \in R, s_i \in S, k \in \mathbb{N}\}.$$

The expression $\sum_{i=1}^{k} r_i s_i$ is called a linear combination of the elements $s_1, s_2, \cdots, s_k$. If $S = \{s_1, s_2, \cdots, s_m\}$, then

$$RS = \{\sum_{i=1}^{m} r_i s_i \mid r_i \in R, s_i \in S\}.$$

Especially, if $S = \{s\}$, then we denote RS by Rs, i.e., $Rs = \{rs \mid r \in R\}$.

If $RS = M$, then S is called a **generating set** for M. An R-semimodule having a finite generating set is called finitely generated, if $Rs = M$ then M is called cyclic. A non-empty subset S of M is called a **free** set if for each $\{s_1, s_2, \cdots, s_m\} \subseteq S$, the linear combination $\sum_{i=1}^{m} r_i s_i = 0$ implies $r_i = 0$, $\forall i$, where $r_i \in R$. An R-semimodule M is called **free** semimodule if M has a free generating subset S. In this case, S is said to be a **basis** for M.

Remark 1.9. If a semiring R is a ring then any R-semimodule is an R-module.

Proof: Let M be a semimodule over a ring R. Then M is a commutative monoid (commutative semigroup with identity) which satisfies all the conditions in Definition 1.5. To show that M is an R-module, we need only to prove that for all $m \in M$ there exists $-m \in M$ such that $m + (-m) = -m + m = 0$. Now let $m \in M$, since R is a ring, i.e. R is a ring with identity 1. Hence $-1 \in R$, and so $-1(m) \in M$. Thus $-m \in M$, $\forall m \in M$. Therefore M is a group, and hence M is an R-module.

Remark 1.10. The only subtractive ideals of the semiring $(\mathbb{N}, +,\cdot)$ are the cyclic ideals.

Proof: Let I be a non-cyclic ideal of $\mathbb{N}$, and let a be the smallest non-zero element of I and b is the first element of I which is greater than a and not multiple of a. Then $b = a + k$ for some $k \in \mathbb{N}$ and $k \notin$

I(by the choice of a and b), hence I is not subtractive. On the other hand, it is clear that any cyclic ideal of $\mathbb{N}$ is subtractive.

Remark 1.11. Let A be a subsemimodule of the $\mathbb{N}$-semimodule $\mathbb{N}$, and let a_0 be the smallest non-zero element of A, then either $A = \mathbb{N}a_0 = \{ na_0 | n \in \mathbb{N}\}$ or $A = \{0, a_0, a_0 + 1, a_0 + 2, \cdots\}$.

Proof: Assume that $A \neq \mathbb{N}a_0$, then $A \subset \mathbb{N}a_0$, if b_0 is the smallest element greater than a_0 such that $b_0 \in A$, $b_0 \notin \mathbb{N}a_0$, then $\mathbb{N}a_0 \cup \mathbb{N}b_0 \cup \mathbb{N}(a_0 + b_0) \subset A$. Similarly proceeding, we have $A = \{0, a_0, a_0 + 1, a_0 + 2, \cdots\}$.

Remark 1.12. Let R be a commutative semiring with identity. A set $S \subseteq R$ is said to be a multiplicatively closed set of R provided that "if $a, b \in S$, then $ab \in S$". The localization of R at S (R_S) is defined in the following way:

First define the equivalent relation $\sim$ on $R \times S$ by $(a, b) \sim (c, d)$, if $sad = sbc$ for some $s \in S$. Then put R_S as the set of all equivalence classes of $R \times S$ and define the addition and multiplication on R_S, respectively, by $[a, b] + [c, d] = [ad + bc, bd]$ and $[a, b] \cdot [c, d] = [ac, bd]$, where $[a, b]$ is also denoted by a/b, by which we mean the equivalence class of (a, b). It is, then, easy to see that R_S with the above mentioned operations of addition and multiplication on R_S is a semiring [15].

Definition 1.13. In Remark 1.12, if S is the set of all not zero divisors of R, then the **total quotient** semiring Q(R) of the semiring R is defined as the localization of R at S. Note that Q(R) is also an R-semimodule. For more details, see previous articles [11, 13].

Definition 1.14. A subset I of the total quotient semiring Q(R) of R is called **fractional** ideal of a semiring R, if the following hold:

1. I is an R-subsemimodule ofQ(R), that is, if $a, b \in I$ andr $\in R$, then $a + b \in I$ andra $\in I$.
2. There exists a non-zero divisor element $d \in R$ such thatdI $\subseteq R$.

Let I, J be two fractional ideals of a semiring R. Then

$$JI = \{b_1a_1, b_2a_2, \dots, b_na_n : b_i \in J, a_i \in I\}.$$

It is clear that any ideal I of R is a fractional ideal of a semiring R.

Definition 1.15. Let I be a fractional ideal of a semiring R, then I is called **invertible** if there exists a fractional ideal J of R such that $JI = R$. Note that J is unique and we denote that by I^{-1}. For more details, see for example earlier works [10, 11].

2. π Semimodules

Let Ω be a family of R-semimodules. The R-semimodule M, as an R-module [14, Ex.17(b), page 241]) is said to be **generator** for the family Ω if for each $N \in \Omega$,

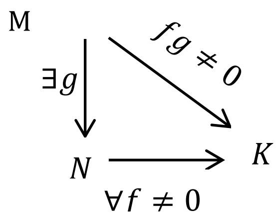
$$N = \sum_{\phi \in Hom(M,N)} \phi(M)$$

In some cases, for simplicity, we put $H = Hom(M, N)$.

The following theorem gives a different form for generators.

Theorem 2.1. Let M be an R-semimodule and Ω be a family of R-semimodules. Then the following statements are equivalent:

1. M is a generator for Ω.
2. For all R-semimodules N and K in Ω, and $f \in Hom(N, K)$ with $f \neq 0$, $\exists g \in Hom(M, N)$ such that $fg \neq 0$, see the diagram below.



Proof: (1) $\Rightarrow$ (2). Since $f \neq 0$, there is $a \in N$ with $f(a) \neq 0$. As M is a generator, there is a representation $a = \sum_{i=1}^{n} g_i(m_i)$, $g_i \in Hom(M, N)$, $m_i \in M$. Hence we have

$$0 \neq f(a) = \sum_{i=1}^{n} fg_i(m_i),$$

and consequently there is a g_i with $fg_i \neq 0$.

(2) $\Rightarrow$ (1). Suppose that $\sum_{\phi\in H}\phi(M)\neq N$, ($H=Hom(M,N)$, $N\in\Omega$), then let $\nu: N\longrightarrow N/\sum_{\phi\in H}\phi(M)$ be the natural epimorphism. Since $\nu\neq 0$, there is a $g\in H$ with $\nu g\neq 0$. Consequently, we have $g(M)\nsubseteq\sum_{\phi\in H}\phi(M)$, *in contradiction to the definition of* $\sum_{\phi\in H}\phi(M)$. *This completes the proof.*

Birge Zimmermann-Huisgen [3] introduced the definition of **self-generator** for R-modules. In this paper, we recall this definition for R-semimodules. M is called a self-generator if M generates each of its subsemimodules. In other words, an R-semimodule M is called self-generator, if for any subsemimodule N of M,

$$N=\sum_{\phi\in Hom(M,N)}\phi(M).$$

In this section, we study the semimodules which can be generated by each of their non-zero subsemimodules. This is a "dual proplem" of self-generator concept. Now, for any two R-semimodules M_1, M_2, let $\pi(M_1,M_2)=\sum_\phi\phi(M_1)$ where the sum is taken over all $\phi\in Hom(M_1,M_2)$. If N is a sub-semimodule of M, then we may put $\pi(N)$ instead of $\pi(N,M)$. Note that if $M_2=R$ then $\pi(M_1,R)$ is just **the trace** of M_1. For more details see a previously published study [1, page 7].

Now we introduce the following definition.

Definition 2.2. A non-zero subsemimodule N of an R-semimodule M is said to be **dense** in M, if N generates M. This means that $M=\sum_\phi\phi(N)$, where the sum is taken over all $\phi\in Hom(N,M)$. In other words, N is dense in M if $\pi(N)=M$.

Geometrically, N is dense in M if M can be covered by images of homomorphisms from N into M. Note that N is dense in M, iff $\forall m\in M$, $\exists\phi_1,\phi_2,\cdots,\phi_n\in Hom(N,M)$, and $\exists x_1,x_2,\cdots,x_n\in N$ such that $m=\sum_{i=1}^{n}\phi_i(x_i)$. A subsemimodule N of M is said to be dense in M, if N generates M, i.e

$$M=\sum_{\phi\in Hom(N,M)}\phi(N)$$

In the following lemma, we give other forms of dense subsemimodules, with the proof as in Theorem 2.1.

Lemma 2.3. Let N be a non-zero subsemimodule of an R-semimodule M. Then the following statements are equivalent:

1. N is dense in M.
2. For any R-semimodule K, and $\forall f\in Hom(M,K)$ with $f\neq 0$, $\exists g\in Hom(N,M)$, such that $fg\neq 0$.

Proposition 2.4. Let N be a non-zero subsemimodule of an R-semimodule M. If N is dense in M, then ann(N) =ann(M).

Proof: We have $ann(M)\subseteq ann(N)$, thus it is enough to show that $ann(N)\subseteq ann(M)$. Let $r\in ann(N)$. Sine N is dense in M, then by definition 2.2, $\forall m\in M$, $\exists\phi_1,\phi_2,\cdots,\phi_n\in Hom(N,M)$, and $\exists x_1, x_2,\dots,x_n\in N$ such that $m=\sum_{i=1}^{n}\phi_i(x_i)$. Then $rm=\sum_{i=1}^{n}\phi_i(rx_i)$, but $r\in ann(N)$, hence $rx_i=0$, and $rm=0$. Therefore, $r\in ann(M)$ and $ann(N)=ann(M)$.

Remark 2.5.

1. $Hom_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z})=0$.
2. $Hom_{\mathbb{N}}(\mathbb{Q},\mathbb{N})=0$.

Proof: For(1), assume that $0\neq f(1)=n$ and m is any integer with $g.c.d(m,n)=1$, then $n=f(1)=f(m/m)=mf(1/m)$, $\rightarrow f(1/m)=n/m\notin\mathbb{Z}$ (which is not possible). Hence f(1) must equal zero. Therefore $Hom_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z})=0$. Using the same way we prove(2).

The following example shows that the condition in Proposition 2.4 is not sufficient.

Example 2.6. Let $M=\mathbb{Z}\oplus\mathbb{Q}$ be considered as a $\mathbb{Z}$-semimodule, where $\mathbb{Z}$ and $\mathbb{Q}$ are the groups of integers and rationals, respectively. Let N=0+ $\mathbb{Q}$ be a non-zero subsemimodule of M. It is clear that, $ann(N)=ann(M)=(0)$. If $(n,0)\in M$, $n\neq 0$. From Remark 2.5, we have $Hom(\mathbb{Q},\mathbb{Z})=0$, then $(n,0)\notin\pi(N)$. Thus N is not dense in M.

Note that, in Example 2.6, if we put $M=\mathbb{N}\oplus\mathbb{Q}$ considered as a $\mathbb{N}$-semimodule, where $\mathbb{N}$ is a semi-group of natural numbers, we will get $N=0+\mathbb{Q}$ is not dense in $M=\mathbb{N}\oplus\mathbb{Q}$.

The following lemma shows that the condition of Proposition 2.4 is sufficient to make a subsemi-module dense if a subsemimodule is cyclic.

Lemma 2.7. Let Ra be a non-zero cyclic subsemimodule of an R-semimodule M, then the following statements are equivalent:

1. $M=\pi(Ra)$

2. ann(M) =ann(Ra)
3. $\forall m \in M$, $\exists$ is a homomorphism $\phi_m: Ra \longrightarrow M$ such that $\phi_m(a) = m$.

Proof: From Proposition 2.4, (1) gives (2). Suppose that (2) holds and $m \in M$.We define $\phi_m: Ra \longrightarrow M$ as follows: $\phi_m(ra)= rm$, in particular $\phi_m(a) = m$. The assumption implies that ϕ_m is well-defined. Finally, suppose that (3) holds , then it is clear that $\pi(Ra) \subseteq M$, let $m \in M$ by (3), then for all $\phi \in Hom(Ra, M)$, we have $m \in \sum_{\phi} \phi(Ra)$, thus $M \subseteq \pi(Ra)$. Thus $M = \pi(Ra)$.

After defining the concept of a dense subsemimodule, as previously described in the modules [1, page 11], we are ready now to give the concept of a π semimodule, which is a dual, in some sense, to the concept of self-generator semimodule, given in modules.

Definition 2.8. An R-semimodule M is said to be a $\boldsymbol{\pi}$ **semimodule** if for each non-zero subtractive subsemimodule N of M, $\pi(N) = M$, i.e. each non-zero subtractive subsemimodule of M is dense in M.

Note that M is a π semimodule if it is generated by each of its nonzero subtractive subsemimodule, while M is a self-generator if it generates each of its subtractive subsemimodules.

Example 2.9. Here we introduce some examples to explain π semimodules:
1. Any simple semimodule is a π semimodule.
2. Let $\mathbb{N}$ be the semiring of natural numbers, and let $a\mathbb{N}$ be a any non-zero ideal in $\mathbb{N}$. Define a $\mathbb{N}$-homomorphism $f: a\mathbb{N} \longrightarrow \mathbb{N}$ by putting $f(an) = n, \forall an \in a\mathbb{N}$. In particular,$f(a) = 1$. Hence $a\mathbb{N}$ is dense in $\mathbb{N}$. Thus $\mathbb{N}$ is a π semimodule.
3. Let $\mathbb{Q}^+$ be the $\mathbb{N}$-semimodule of non-negative rational numbers, and let K be any non-zero subsemimodule of $\mathbb{Q}^+$. Then $\exists a/b \in K$ with $a, b \neq 0$. Let $m/n \in \mathbb{Q}^+$. Define a map $f: K \longrightarrow \mathbb{Q}^+$ by putting $f(x) = (bm/an)x, \forall x \in K$. It is clear that f is an $\mathbb{N}$ -homomorphism and $f(a/b) = m/n$. Thus K is dense in $\mathbb{Q}^+$, and $\mathbb{Q}^+$ is a π $\mathbb{N}$-semimodule.
4. Let p be a prime number, and let $\mathbb{N}_{(p)}$ be the set of rationals of the form m/n, with m and n are in $\mathbb{N}$ and n is not divisible by p. Then $\mathbb{N}_{(p)}$ is a subsemigroup of $\mathbb{Q}^+$. As a $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ [17]. We put

$$\mathbb{N}_{p^\infty} = \mathbb{Q}^+/\mathbb{N}_{(p)}.$$

Then $\mathbb{N}_{p^\infty}$ is a $\mathbb{N}$-semimodule. It is known that each proper non-zero subsemigroup of $\mathbb{N}_{p^\infty}$ is cyclic of the form $\mathbb{N}_{p^n}$ [17]. Note that, since each element of $f(\mathbb{N}_{p^n})$ where $f \in Hom(\mathbb{N}_{p^n}, \mathbb{N}_{p^\infty})$ is of order less than or equal to p^n, then $\mathbb{N}_{p^n}$ is not dense in $\mathbb{N}_{p^\infty}$. Thus $\mathbb{N}_{p^\infty}$ is not a π semimodule.

A subsemimodule N of an R-semimodule M is called **invariant** subsemimodule if $f(N) \subseteq N, \forall f \in Hom(M, M)$, and N is called a **stable** subsemimodule if $f(N) \subseteq N$, $f \in Hom(N, M)$[2].

Remark 2.10. Let N be a non-zero subsemimodule of an R-semimodule M, then
1. $N \subseteq \pi(N) \subseteq M$.
2. N is a stable subsemimodule of M iff $\pi(N) = N$.
3. $\pi(N)$ is a stable subsemimodule of M.

Proof: (1) and (2) are clear. (3) Let $f: \pi(N) \longrightarrow M$. We want to show that $f(m) \in \pi(N), \forall m \in \pi(N)$. Since $m \in \pi(N)$, then $m = \sum_{i=1}^{n} \phi_i(x_i)$, where $\phi_i \in Hom(N, M)$, and $x_i \in N, \forall i \ 1 \leq i \leq n$. Thus

$$f(m) = \sum_{i=1}^{n} f\phi_i(x_i)$$

Since $f\phi_i \in Hom(N, M)$, then $f(m) \in \pi(N)$, so $f(\pi(N)) \subseteq \pi(N)$, $\forall f \in Hom(\pi(N), M)$. Then $\pi(N)$ is a stable subsemimodule of M.

The following proposition relates the concept of a π semimodule and the concept of stability.

Proposition 2.11. Let M be an R-semimodule, then M is a π semimodule iff M has no non-trivial stable subsemimodules.

Proof: Assume that M is a π semimodule, and N is a proper non-zero stable subsemimodule of M. By Remark 2.10, $\pi(N) = N$. Since M is π semimodule, hence $M = \pi(N) = N$, which is a contradiction.

Conversely, since $\pi(N)$ is a stable non-zero subsemimodule of M, see Remark 2.10, thus by assumption, $M = \pi(N)$. Therefore M is π semimodule.

Now, we study when an ideal is dense in semiring.

Remark 2.12. A non-zero ideal I of a semiring R is dense in R iff $trace(I) = R$.

Golan [9, page 39] proved thatan ideal I of a ring R is a direct summand iff $I = Re$ for some idempotent element e of R. Here, we use another proof for a semirings.

Lemma 2.13. An ideal I of R is a direct summand iff $I = Re$ for some idempotent element e of R.

Proof: (⇒) Assume that I is a direct summand of R, that is $R = I \oplus J$, then $1 = e + \acute{e}$ for some $e \in I$

and $\acute{e} \in J$. For each $x \in I$, $x = xe + x\acute{e}$. Since I is subtractive $x \in I \wedge xe + x\acute{e} \in I$, imply $x\acute{e} \in I$, hence $x\acute{e} \in I \cap J = \{0\}$. Then, $x = xe$ $(\forall x \in I)$, that is $I = Re$. Now if we put $x = e$ in the expression $x = xe$, we get $e = ee = e^2$, and e is idempotent.

$(\Leftarrow)$ Assume that e is an idempotent element of R and $I = Re$. If e is a non-zero divisor, then $\alpha: R \longrightarrow Re$, defined by $\mapsto re$, is an isomorphism, so Re is a direct summand of R. If e is a zero divisor, and $e\acute{e} = 0$ (for some $\acute{e} \in R$). Claim that $R = Re + R\acute{e}$ for some $\acute{e}$ such that $e\acute{e} = 0$. We need to consider that R is semisubtractive. In this case, either $e + \acute{e} = 1$ or $e = 1 + \acute{e}$ for some $\acute{e} \in R$. If $e + \acute{e} = 1$, then $R = Re + R\acute{e}$, and since $(e + \acute{e})e = e \rightarrow e^2 + \acute{e}e = e \rightarrow e + \acute{e}e = e \rightarrow \acute{e}e = 0$, then $R = Re \oplus R\acute{e}$. In the case that $e = 1 + \acute{e}$, we also get $\acute{e}e = 0$ and $Re \cap R\acute{e} = 0$. On the other hand, $re = r + r\acute{e}, \forall r \in R$. Now $re \in Re + R\acute{e} \wedge r\acute{e} \in Re + R\acute{e}$, by subtractivity, $r \in Re + R\acute{e} \rightarrow R = Re + R\acute{e} \rightarrow R = Re \oplus R\acute{e}$. Therefore. $I = Re$ is a direct summand of R.

As in the modules, we give the following lemma without proof, since it is already included in the modules [9, page 61].

Lemma 2.14. A left R-semimodule is isomorphic to a direct summand of a free left R-semimodule iff it is projective.

Theorem 2.15. Let I be a non-zero subtractive ideal of R, then I is dense in R iff is a faithful finitely generated projective ideal.

Proof: $(\Rightarrow)$ Suppose that I is dense in R, by Remark 2.12, $1 = \sum_i \varphi_i(x_i)$, where $\varphi_i \in I^* = \mathrm{Hom}(I, R)$, $x_i \in I$, for finite i. Thus, $\forall x \in I, x = \sum_i x\varphi_i(x_i) = \sum_i x_i\varphi_i(x)$.

Hence I is finitely generated, and by the dual basis lemma, I is projective, [5]. Since $\mathrm{ann}(I) = \mathrm{ann}(R) = (0)$, thus I is faithful.

$(\Leftarrow)$ Suppose that I is a faithful finitely generated projective ideal. Since I is faithful, then $\mathrm{ann}(I) = 0$. Since I is projective, then by Lemma 2.14, we have I is a direct summand of R. Then, by Lemma 2.13, we have $I = Re$ for some idempotent element $e \in R$. Now, let $\alpha: R \longrightarrow Re$, defined by $r \mapsto re$, then α is an epimorphism and $R/\mathrm{Ker}\alpha \cong Re$, where $\mathrm{Ker}\alpha = \{r \in R | re = 0\} = \mathrm{ann}(e) = \mathrm{ann}(I)$. But I is faithful, then $\mathrm{er}\alpha = \mathrm{ann}(I) = 0$. Hence $R \cong I$, and $\mathrm{trace}(I) = R$. By Remark 2.12, I is dense in R.

Corollary 2.16. If I is a subtractive dense ideal of a semiring R, then I is an invertible in R.

Proof: Since I is a subtractive dense ideal of R, then by Theorem 2.15, we have I is a finitely generated projective ideal. As in the rings theory [8], we have I is invertible.

Proposition 2.17. If I is an invertible ideal of a semiring R, then I is dense in R.

Proof: Since I is an invertible, then we have $JI = R$, for some fractional ideal J of R. $J = \{x \in Q(R) | xI \subseteq R\}$, where $Q(R)$ is a total quotient semiring of R. Hence, each element of J can be thought of as an R-homomorphism in $\mathrm{Hom}(I, R)$. In fact, for each $r \in R$, $r = \sum_{i=1}^{n} x_i a_i$, $x_i \in I, a_i \in J$. i.e. $r = \sum_{i=1}^{n} \phi_{x_i}(a_i)$. Where if $x \in J$, then $\phi_x(a) = xa\ \forall a \in I$. Hence by Remark 2.12, we have I is dense in R.

An integral domain R is called a Dedekind domain if every non-zero ideal of R is invertible [16]. Similar to this, we construct concept of **Dedekind semidomain** as follows: A semidomain R (R is a semiring) is said to be a Dedekind semidomain if every non-zero subtractive ideal of R is invertible
in R

The following theorem is immediate from Corollary 2.16 and Proposition 2.17.

Theorem 2.18. Let R be a semiring, then R is a π R-semimodule iff R is a Dedekind semidomain.

Proof: $(\Rightarrow)$ Assume that R is π semimodule, then Ra is dense in R, $\forall a \in R$, and by Theorem 2.15, Ra is faithful and $\mathrm{ann}(Ra) = 0$. Hence, R is a semidomain. Moreover, every non-zero subtractive ideal I of R is dense, thus by Corollary 2.16, I is invertible. Then R is a Dedekind semidomain.

$(\Leftarrow)$ The converse follows immediately from Proposition 2.17. Thus R is a π R-semimodule.

Remark 2.19. Let R be a semiring and $a \in R$. Then the principal ideal (a) is invertible iff a is not zero divisor.

Proof: $(\Rightarrow)$Assume that $0 \neq a \in R$, and $ab = 0$, for some $b \in R$. Since the principal ideal (a) is invertible, then $(a)J = R$, fore some fractional ideal J of R. Hence,

$$(ar_1)j_1 + (ar_2)j_2 + \cdots + (ar_n)j_n = 1$$
$$a(r_1j_1 + r_2j_2 + \cdots + r_nj_n) = 1$$

Then, $\exists x \in J$ such that $ax = 1 = xa$. But $x(ab) = x0 = 0$, $\rightarrow 0 = (xa)b = 1b = b$ and hence a is not a zero divisor.

($\Leftarrow$) Assume that a is not zero divisor element of R, and let (a) be a principal ideal of R. Since a is not zero divisor, then $J = (s/a)$ is a fractional ideal of R. Now, $R \subseteq IJ = (a)(s/a)$. Let $y \in IJ$, then

$$y = (r_1a)\left(\frac{s_1}{a}\right) + (r_2a)\left(\frac{s_2}{a}\right) + \cdots + (r_na)\left(\frac{s_n}{a}\right).$$

$$y = r_1s_1 + r_2s_2 + \cdots + r_ns_n \in R.$$

Hence, IJ is an ideal of R, and $IJ = R$. Therefore, I is invertible ideal of R.

The following two corollaries are immediate from Remark 2.19 and Proposition 2.17.

Corollary 2.20. Every principal ideal in a semiring R generated by a non-zero divisor is dense in R.

Corollary 2.21. Let R be a semiring, then the following statements are equivalent:

1. R is a semidomain.
2. Each non-zero principal ideal of R is an invertible ideal of R.
3. Each non-zero principal ideal of R is dense in R.

3. Prime Semimodules Having Injective Hull

In Proposition 2.4, we saw that for every dense subsemimodule N of M, ann(N) =ann(M), thus in a π R-semimodule M, for every non-zero subtractive subsemimodule N of M, ann(N) =ann(M). And in Lemma 2.7, we observed that a cyclic subsemimodule Ra is dense in M iff ann(Ra) =ann(M).

These observations lead us to study prime semimodules. Analogous to the concept of prime modules [4], we define a prime semimodules as follows:

Definition 3.1. An R-semimodule M is said to be **prime** semimodule if ann(N) =ann(M), for every non-zero subtractive subsemimodule N of M.

We observed that the class of prime semimodules contains the class of π semimodules. But the converse is false. Note that the $\mathbb{Z}$-semimodule $M = \mathbb{Z}\oplus\mathbb{Q}$ is easily seen to be a prime semimodule. Anyway, any direct summand of semimodule is subtractive, [11, page 184], hence $\mathbb{Q}$ is a subtractive subsemimodule of M which is not dense in M (see Example 2.6). Thus, M is not a π semimodule. One can ask when a prime semimodule can possibly be a π semimodule. We will show later that, in the class of quasi-injective semimodule, the two concepts of π semimodule and prime semimodule are equivalent.

It is well known that, for every R-module M, M can be embedded in an injective R-module. $\widehat{M}$ is called an **injective hull** of M, if $\widehat{M}$ is an essential extension of M, i.e $M \cap N \neq 0$ for every non-zero submodule N of $\widehat{M}$ [17].

It is well known, however, that injective hulls always exist if R is a ring. But, Golan[10] proved that injective hulls of non-zero R-semimodules need not exist for every semiring R[10, prop.17.21, page 198]. If R is a semiring then any cancellative R-semimodule can be embedded in an injective R-module $\widehat{M}$, [10, Ex.17.35, page 202]. Wang [19] proved that every semimodule over an additively-idempotent semiring has an injective hull. For more details on an injective hull of semimodules over semiring, see for example information described previously [13].

Lemma 3.2. Let R be a semisubtractive semiring, and let M and N be cancellative R-semimodules. If $x \in M$ and $y \in N$ with ann(Rx) =ann(Ry), then f: $Rx \longrightarrow Ry$ defined by : $rx \mapsto ry$ is well-defined R-homomorphism.

Proof: Assume $rx = \acute{r}x$, then either $r = \acute{r} + s$, for some $s \in R$. Hence $(\acute{r} + s)x = \acute{r}x$, $\rightarrow \acute{r}x + sx = \acute{r}x$, $\rightarrow sx = 0$, $\rightarrow s \in$ann(Rx), $\rightarrow s \in$ann(Ry), $\rightarrow sy = 0$, $\rightarrow ry = (\acute{r} + s)y = \acute{r}y + sy = \acute{r}y$. Or $r + s = \acute{r}$, by similar process $rx = \acute{r}x$, $\rightarrow ry = \acute{r}y$, and then f is well-defined. On the other hand, it is clear that f is R-homomorphism.

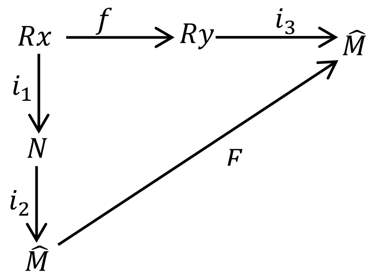
Note that it is considered in this work that all semiring R is a semisubtractive and all R-semimodules are cancellative. The following proposition gives another characterization of prime semimodules, which is analogous for modules [4].

Proposition 3.3. Let M be a non-zero R-semimodule having an injective hull $\widehat{M}$, then the following statements are equivalent:

1. M is a prime semimodule.
2. M is contained in every non-zero invariant subsemimodule of $\widehat{M}$.

Proof: (1)$\Rightarrow$(2) Let N be a non-zero invariant subsemimodule of $\widehat{M}$. We want to prove that $M \subseteq N$. Since $\widehat{M}$ is an essential extension of M, then $M \cap N \neq 0$. Thus $0 \neq x \in M \cap N$. Since M is prime, then

$\forall 0 \neq y \in M$, ann(Rx) =ann(Ry). We define f: Rx ⟶ Ry as follows: f(rx) = ry, ∀r ∈ R. By Lemma 3.2 we have that f is a well-defined R-hhomomorphism. Since $\widehat{M}$ is injective R-semimodule then f can be extended to F: $\widehat{M} \longrightarrow \widehat{M}$, as in the following diagram.



where i_1, i_2 and i_3 are the inclusion R-homomorphisms. Since N is an invariant subsemimodule of $\widehat{M}$, then $F(Rx) \subseteq N$, but $f(Rx) = Ry$, then $y \in N$, hence $M \subseteq N$.

(2) ⇒(1). Let N be a non-zero subsemimodule of M. Since ann(M) ⊆ann(N), we want to show that ann(N) ⊆ann(M). Assume that $\exists r \in R$ such that $r \in$ann(N), and $\exists x \in M$ with $rx \neq 0$. Since $0 \neq N$, $\exists 0 \neq y \in N$. Now, $\pi(Ry, \widehat{M}) = \sum_\phi \phi(Ry)$, $\phi \in Hom(Ry, \widehat{M})$. Since $Ry \subseteq M \subseteq \widehat{M}$, so $\pi(Ry, \widehat{M})$ is a non-zero submodule of $\widehat{M}$, and it is easy to check that $\pi(Ry, \widehat{M})$ is an invariant nonzero submodule of $\widehat{M}$. Thus by assumption $M \subseteq \pi(Ry, \widehat{M})$. Then $\exists r_1, r_2, \cdots, r_n \in R$, and $\exists \phi_1, \phi_2, \cdots, \phi_n \in Hom(Ry, \widehat{M})$ such that, $x = \sum_{i=1}^{n} \phi_i(r_i y)$. Thus, $rx = \sum_{i=1}^{n} r\phi_i(r_i y) = \sum_{i=1}^{n} \phi_i(rr_i y) = 0$, which is a contradiction. Then ann(N) ⊆ann(M), and hence M is a prime semimodule.

From Proposition 2.4, we have that every π semimodule is a prime semimodule. Thus we have the following corollary.

Corollary 3.4. Let M be a semimodule having an injective hull $\widehat{M}$. If M is a π semimodule then M is contained in every non-zero invariant submodule of $\widehat{M}$.

Proposition 3.5. Let M be a non-zero semimodule having an injective hull $\widehat{M}$. If M is invariant subsemimodule of $\widehat{M}$ then the following statements are equivalent:

1. M is a prime semimodule.
2. M has no non-trivial invariant subsemimodules.

Proof: (1) ⇒ (2). Let N be a non-zero invariant subsemimodule of M. Because M is an invariant subsemimodule of $\widehat{M}$, so it can easily seen that N is also invariant subsemimodule of $\widehat{M}$. Thus, by Proposition 3.3 we have $M \subseteq N$, and hence $M = N$.

(2)⇒ (1). Let K be a non-zero invariant subsemimodule of $\widehat{M}$. By Proposition 3.3, it is enough to show that $M \subseteq K$. Since $\widehat{M}$ is an essential extension of M, hence $M \cap K \neq (0)$. Now we claim that $M \cap K$ is an invariant subsemimodule of M. If this is proved, then by assumption M has no non-trivial invariant subsemimodules and thus $M \cap K = M$, which implies that $M \subseteq K$.

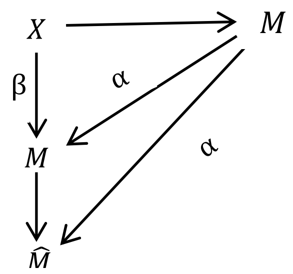
To prove the claim, consider f any homomorphism in Hom(M, M). Since $f(M \cap K) \subseteq f(M) \cap f(K)$, and since $f(M) \subseteq M$, so it is enough to show that $f(K) \subseteq K$. Because $\widehat{M}$ is an injective semimodule, then f can be extended to $F \in Hom(\widehat{M}, \widehat{M})$, but K is an invariant subsemimodule of $\widehat{M}$. Thus $f(K) = F(K) \subseteq K$, hence $M \cap K$ is an invariant subsemimodule of M.

Now, as in the modules [7, page 22], we say that an R-semimodule M is said to be **quasi-injective** if each homomorphism from any subsemimodule N into M can be extended to a homomorphism of M to M. Note that any simple semimodule, and any injective semimodule, is quasi-injective. However, a quasi-injective semimodule needs not to be injective. For example, for each prime number p, $\mathbb{N}_{p^n}$ is considered as a ℕ-semimodule which is quasi-injective. In verity, the only non-zero subsemimodules of $\mathbb{N}_{p^n}$ are $\mathbb{N}_{p^k}$, $1 \leq k \leq n$. Then, for each $f \in Hom(\mathbb{N}_{p^k}, \mathbb{N}_{p^n})$,and all $x \in \mathbb{N}_{p^k}$, the order of f(x) is less than or equal to p^k, hence $f(\mathbb{N}_{p^k}) \subseteq \mathbb{N}_{p^k}$. It is clear that f can be extended to a homomorphism in $Hom(\mathbb{N}_{p^n}, \mathbb{N}_{p^n})$. Whereas, $\mathbb{N}_{p^n}$ is not injective.

The following theorem gives the relation between invariant and quasi-injective Semimodules.

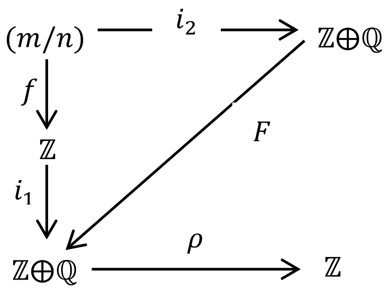
Theorem 3.6.et M be a semimodule having an injective hull $\widehat{M}$. If M is an invariant subsemimodule of $\widehat{M}$ then M is a quasi-injective.

Proof: Assume that M is a non-zero invariant subsemimodule having an injective hull, and $\alpha \in Hom(\widehat{M}, \widehat{M})$. Since $\widehat{M}$ is injective, it is enough to consider that $\alpha \in Hom(M, \widehat{M})$. Let $X \hookrightarrow M$ and $\beta : X \longrightarrow M$ be a homomorphism. Since $\widehat{M}$ is injective, β can be extended to $\alpha : M \longrightarrow \widehat{M}$. By assumption, $\alpha(M) \subseteq M$, and hence $\alpha : M \longrightarrow M$ extends β. Therefore M is Quasi-injective. See the diagram below.



Remark 3.7. We showed in Example 2.6 that $M = \mathbb{Z}\oplus\mathbb{Q}$ is considered as a $\mathbb{Z}$-semimodule which is a prime semimodule, and we proved that M is not π semimodule. We show now that M is not a quasi-injective semimodule.

Proof: Let $N = (m/n)$ be a cyclic subsemimodule of $\mathbb{Q}$ generated by the non-zero element m/n, where g.c.d $(m, n) = 1$. We define $f : (m/n) \longrightarrow \mathbb{Z}$ as follows : $f(r \cdot m/n) = rm, \forall r \in \mathbb{Z}$.
It is clear that f is a well-defined $\mathbb{Z}$-semimodule. Consider the diagram.



where i_1 is the inclusion into the first factor and i_2 is the inclusion in the second factor. Suppose that f can be extended to $F \in Hom(\mathbb{Z}\oplus\mathbb{Q}, \mathbb{Z}\oplus\mathbb{Q})$. Let $\rho : \mathbb{Z}\oplus\mathbb{Q} \longrightarrow \mathbb{Z}$ be the natural projection, and let $f_1 = \rho\, F|_{\mathbb{Q}}$. It is easily seen that f_1 is a non-zero element in $Hom(\mathbb{Q}, \mathbb{Z})$. But $Hom(\mathbb{Q}, \mathbb{Z}) = (0)$, which is a contradiction. This completes the proof.

We conclude that $M = \mathbb{Z}\oplus\mathbb{Q}$ is not an invariant subsemimodule of its injective hull $\widehat{M} = \mathbb{Q}\oplus\mathbb{Q}$. *Thus we arrive at the following main theorem.*

Theorem 3.8. Let M be any prime semimodule having an injective hull $\widehat{\text{M}}$. If M is an invariant subsemimodule of $\widehat{\text{M}}$, then M is a π semimodule.

Proof: We use the characterization of π semimodules given in Proposition 2.11. So let N be a non-zero stable subsemimodule of M, then we have to show that M is a contained in N. From the definition of stability, it is easy to see that N is invariant subsemimodule of M. By assumption, M is an invariant and prime semimodule and, using Proposition 3.5, M has no non-trivial invariant subsemimodule. Therefore, M ⊆ N. This completes the proof.

The following corollary is immediate from Corollary 3.4 and Theorem 3.8.

Corollary 3.9. Let M be a semimodule having an injective hull $\widehat{M}$, if $M$is an invariant subsemimodule of $\widehat{M}$. Then M is a π semimodule iff M is a prime semimodule.

Next, similar to the case in the modules [20], we can say that an R-semimodule M is called **compressible** if every non-zero subsemimodule of M contains an isomorphic copy of M. As a trivial example:

- Every simple R-semimodule is compressible.
- ℕ as a ℕ-semimodule is compressible.

- $\mathbb{Q}^+$as a $\mathbb{N}$-semimodule is not compressible since ${\mathbb{Q}^+}^* = Hom(\mathbb{Q}^+, \mathbb{N}) = (0)$.

The following shows that the class of prime semimodules contains the class of compressible semi-modules.

Theorem 3.10. Every compressible R-semimodule is a prime R-semimodule.

Proof: Let M be a compressible R-semimodule, and let $0 \neq N \hookrightarrow M$. Now, we show that ann$(N)$ =ann
(M), since ann(M) ⊆ann(N). So it is enough to prove that ann(N)⊆ann(M). Since M is compressible, then ∃ a monomorphism $\alpha: M \longrightarrow N$. Hence, $\forall\ r$ ∈ann(N), $r\alpha(\ M) = (0)$, thus $\alpha(rM) = (0)$, which implies that $rM = (0)$, and $r \in ann(M)$, thus ann(N) ⊆ann(M). This completes the proof.

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