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Semiprime R_Γ-Submodules of Multiplication R_Γ-Modules

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Abstract

Let *R* be a *Γ*-ring and *G* be an R_Γ-module. A proper R_Γ-submodule *S* of *G* is said to be semiprime R_Γ-submodule if for any ideal *I* of a *Γ*-ring *R* and for any R_Γ-submodule *A* of *G* such that $(I\Gamma)^2 A \subseteq S$ or $I\Gamma I\Gamma A \subseteq S$ which implies that $I\Gamma A \subseteq S$. The purpose of this paper is to introduce interesting results of semiprime R_Γ-submodule of R_Γ-module which represents a generalization of semiprime submodules.

Keywords: Γ – ring, R_Γ-module, R_Γ-submodule and prime R_Γ-submodule

المقاسات الجزئية شبة الأولية من النمط كاما للمقاسات الجدائية

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الخلاصه

لتكن *R* هي حلقة من النمط *Γ* و *G* هو مقاساً من النمط R_Γ و *S* هو مقاساً جزئي فعلي من النمط R_Γ . نسمي *S* مقاس جزئي شبة أولي من النمط R_Γ اذا كان لكل مثالي *I* في *R* وأي مقاس جزئي *A* في *G* بحيث $(I\Gamma)^2 A \subseteq S$ أو $I\Gamma I\Gamma A \subseteq S$ فأن $I\Gamma A \subseteq S$. الغرض من البحث هو تقديم نظريات و خصائص مثيرة للأهتمام في المقاسات الجزئية شبة الأولية من النمط R_Γ والذي يعتبر هو التعميم للمقاسات الجزئية شبة الأولية.

1. Introduction

'Let *R* and *Γ* be additive abelian groups. We say that *R* is a *Γ*-ring if' there exists a mapping' of $\tau: R\times\Gamma\times R \to R$ 'such that for every' $r, s, g \in R$ and $\alpha, \beta \in \Gamma$, 'the following conditions hold:" $(r+s)\alpha g = r\alpha g + s\alpha g$ ',' $r(\alpha+\beta)g = r\alpha g + r\beta g$ ',' $r\alpha(s+g) = r\alpha s + r\alpha g$,' $(r\alpha s)\beta g = r\alpha(s\beta g)$ [1]. A left R_Γ-module is an additive abelian group *G* 'together with a mapping' $\tau: R\times\Gamma\times G \to G$ 'such that for all' $e, e_1, e_2 \in G$ and $\gamma, \gamma_1, \gamma_2 \in \Gamma$, $r_1, r_2, r_3 \in R$ 'the following conditions hold: $r_3\gamma(e_1+e_2) = r_3\gamma e_1 + r_3\gamma e_2$, $(r_1+r_2)\gamma e = r_1\gamma e + r_2\gamma e$, $r_3(\gamma_1+\gamma_2)e = r_3\gamma_1 e + r_3\gamma_2 e$, $r_1\gamma_1(r_2\gamma_2 e) = (r_1\gamma_1 r_2)\gamma_2 e$. A right R_Γ −module 'is defined in an analogous manner and a non-empty subset *S* of (*G*, +) is said to be' R_Γ-submodule of *G* 'if *S* is a subgroup of *G*' and $R\Gamma S \subseteq S$, where $R\Gamma S = \{g\gamma w : \gamma \in \Gamma, g \in R, w \in S\}$,'that is for all $w, w_1 \in S$ and for all' $\gamma \in \Gamma, g \in R$; $w - w_1 \in S$ and $g\gamma w \in S$. So, In this case we write $S \le G$. Let *K*, *L* be R_Γ-submodule of an R_Γ-module *G*', 'then the R_Γ-residual of *K* by *L* is

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$[K :_{R_\Gamma} L]=\{g \in R \mid g\alpha_i l \in K, \forall \alpha_i \in \Gamma, l \in L\}$ [2]. A proper R_Γ-submodule *S* of *G* is called prime R_Γ-submodule if for any ideal *T* of a *Γ*-ring *R* and for any R_Γ-submodule *H* of *G*, $T\,\Gamma H \subseteq S$, implies $H \subseteq S$ or $T \subseteq [S :_{R_\Gamma} G]$ [3]. Let *G* and G' be arbitrary R_Γ-modules. A mapping $\tau : G \to G'$ is a homomorphism of R_Γ-modules (or R_Γ-homomorphism) if for all $u, v \in G$ and for all $t \in R$, $\gamma \in \Gamma$ we have:-

i. $\tau(u+v)=\tau(u)+\tau(v)$

ii. $\tau(t\gamma u)=t\gamma\,\tau(u)$

A R_Γ-homomorphism τ is R_Γ-epimorphism if τ is onto. We denote the set of all R_Γ-homomorphism from *G* into G' by $Hom_{R_\Gamma}(G,G')$. In particular, if $G = G'$ we denote $Hom_{R_\Gamma}(G,G)$ by $End(G)$ and if $\tau : G \to G'$ is an R_Γ-homomorphism, then $Ker\,\tau = \{u \in G \,;\, \tau(u)=0\}$ and, so, $\operatorname{Im}\tau = \{w \in G' \,; \exists u \in G \,; \tau(u)=w\}$ [2]. An R_Γ-module *G* and $\varphi \neq F \subseteq G$, then the generated R_Γ-submodule of *G*, denoted by $<F>$, is the smallest R_Γ-submodule of *G* containing *F*, i.e., $<F> = \bigcap\{S \mid S \leq G\}$. *F* is called the generator of $<F>$ and $<F>$ is finitely generated if $|F| < \infty$. If $F=\{z_1,z_2,...,z_n\}$ we write $<z_1,z_2,...,z_n>$ instead of $<\{z_1,z_2,...,z_n\}>$. In particular, if $F=\{z\}$ then $<z>$ is called the cyclic submodule of *G*, generated by *z* [2]. An R_Γ-submodule S of an R_Γ-module *G* is called R_Γ-direct summand of *G* if there is R_Γ-submodule *Q* of G such that $S \oplus_\Gamma Q = G$, i.e., if there are R_Γ-homomorphism $\rho : S \to G$ and $i : G \to S$ such that $i \circ \rho = I_S$ [4]. A proper submodule *S* of R-module *G* is said to be prime submodule, if $g\,u \in S$ for $g \in R$ and $u \in G$, implies that either $u \in S$ or $g \in [S : G]$ and *S* is called semiprime submodule of R-module *G*, whenever $g \in R$ and $u \in G$ with $g^2 u \in S$, then $g\,u \in S$ [5]. A proper R_Γ-submodule *S* of *G* is called prime R_Γ-submodule if for any ideal *I* of a *Γ*-ring *R* and for any R_Γ-submodule *K* of *G*, $I\,\Gamma K \subseteq S$ implies $K \subseteq S$ or $I \subseteq [S :_{R_\Gamma} G]$ [3]. In this paper, we provide the definition of semiprime R_Γ-submodule of R_Γ-module and the relation with semiprime R-submodule of R-module, which is a generalization to semiprime R-submodule. Thus, we find the relation of semiprime R_Γ-submodule with multiplication R_Γ-module. As a result, we have come up with an equivalent **Theorem 3.13.** Let *G* be a multiplication R_Γ-module and let *S* be a proper R_Γ-submodule of *G*. Then the following statements are equivalent:-

1. *S* is semiprime R_Γ-submodule of *G*.
2. $x\,\Gamma x \subseteq S$ implies $x \in S$ such that for all $x \in G$.
3. $rad_\Gamma(S)=S$.
4. G/S has no non-zero nilpotent.
5. $K_1\Gamma K_2 \subseteq S$ implies $K_1 \bigcap K_2 \subseteq S$, for every K_1, K_2 are proper R_Γ-submodules of *G*.

2. Semiprime R_Γ-Submodules of R_Γ-Modules

In this section we illustrate the concept of semiprime R_Γ-submodule and we introduce some basic properties.

Definition 2.1. Let *S* be a proper R_Γ-submodule of R_Γ-module *G*. Then *S* is called semiprime R_Γ-submodule if for any ideal *I* of a *Γ*-ring *R* and for any R_Γ-submodule *A* of *G* such that $(I\Gamma)^2 A \subseteq S$ or $I\,\Gamma I\,\Gamma A \subseteq S$ implies $I\,\Gamma A \subseteq S$.

Theorem 2.2. Let *G* be an R_Γ-module. An R_Γ-submodule *S* of *G* is semiprime R_Γ-submodule if and only if, for each $u \in G$, $g \in R$ such that $<g>\Gamma<g>\Gamma<u> \subseteq S$ implies $g\,\Gamma u \subseteq S$.

Proof: Let *S* be a semiprime R_Γ-submodule of *G* and let $<g>\Gamma<g>\Gamma<u> \subseteq S$, where $u \in G$, $g \in R$. Since *S* is semiprime R_Γ-submodule, then $<g>\Gamma<u> \subseteq S$ and hence $g\,\Gamma u \subseteq S$. Conversely, suppose that $I\,\Gamma I\,\Gamma A \subseteq S$, where *I* is an ideal of a *Γ*-ring *R* and *A* is a R_Γ-submodule of G. Then for any element $g \in R$ and $a \in A$, we have $<g>\Gamma<g>\Gamma<a>\subseteq I\,\Gamma I\,\Gamma A\subseteq S$, then $g\Gamma a\subseteq S$. Thus, $I\,\Gamma A\subseteq S$ and *S* is a semiprime R$_\Gamma$-submodule of *G*.

Theorem 2.3 [3]. Let *G* be an R$_\Gamma$-module. An R$_\Gamma$-submodule *S* of *G* is said to be prime if and only if, for each $u\in G\ ,g\in R$ such that $<g>\Gamma<u>\subseteq S$ implies $u\in S$ or $g\in[S:_{R_\Gamma}G]$.

Lemma 2.4 [3]. Let *G* be an R$_\Gamma$-module. Let *S* be a prime R$_\Gamma$-submodule of *G*. Then $[S:_{R_\Gamma}G]$ is a prime ideal of a *Γ*-ring *R*.

Remarks and Examples 2.5

i. Every semiprime R-submodule is semiprime R$_\Gamma$-submodule but the converse is not true in general, as in the following example:

Let Z_8 be a $Z_{(\bar{2})}-$module, $\Gamma=<\bar{2}>$ and $<\bar{4}>$ be a proper $Z_{<\bar{2}>}-$submodule of Z_8. Then $<\bar{4}>$ is semiprime $Z_{<\bar{2}>}-$submodule, since for any I is an ideal of a *Γ*-ring *Z* and *K* is any $Z_{<\bar{2}>}-$ submodule of Z_8 such that $I<\bar{2}>I<\bar{2}>K\subseteq S$, then $I<\bar{2}>K\subseteq S$. But $<\bar{4}>$ is not semiprime submodule since $2\in Z$, $1\in Z_8$, $k=2$ such that $2^2.1=4\in<\bar{4}>$ but $2.1=2\notin<\bar{4}>$.

ii. Every prime R$_\Gamma$-submodule is semiprime R$_\Gamma$-submodule.

Proof. Let *S* be a prime R$_\Gamma$-submodule of *G*. We have to show that *S* is semiprime R$_\Gamma$-submodule. Let $I\Gamma I\Gamma A\subseteq S$, where *I* is an ideal of a *Γ*-ring *R* and *A* is R$_\Gamma$-submodule of *G*. Since *I* is ideal of a *Γ*-ring *R*, then $I\Gamma A=A\Gamma I$. Since *S* is a prime R$_\Gamma$-submodule of *G*, then either $A\subseteq S$ then $I\Gamma A\subseteq S$ or $I\,\Gamma I\subseteq[S:_{R_\Gamma}G]$ then $I\subseteq[S:_{R_\Gamma}G]$, since $[S:_{R_\Gamma}G]$ is prime by lemma (2.4). Therefore, $I\Gamma A\subseteq I\Gamma G\subseteq S$ and hence $I\Gamma A\subseteq S$. Thus *S* is semiprime R$_\Gamma$-submodule of *G*. The following example explains that the converse is not true in general:

Let $3Z$ be an $Z_{2Z}-$module, $6Z$ be a proper $Z_{2Z}-$submodule of $3Z$. Let $f: Z\times 2Z\times 3Z\rightarrow 3Z$ and $6Z$ is semiprime $Z_{2Z}-$submodule of $3Z$, for any ideal *I* in *Z* and any $Z_{2Z}-$submodules in $3Z$, then $(I\,2Z)^2A\subseteq 6Z$. But $6Z$ is not prime of $Z_{2Z}-$submodule of $3Z$, since $x=3\,,r=2\ ,\gamma=2$, $<3>2Z<2>\subseteq 6Z$ and $3.2.2=12\in 6Z$ but $3\notin 6Z$ and $2\notin[6Z:_{R_\Gamma}3Z]$.

Recall that an ideal *I* in a *Γ*-ring *R* is said to be semiprime ideal of a *Γ*-ring *R* if for any *J* is an ideal in *Γ*-ring *R* such that $J\,\Gamma\,J\subseteq I$ implies $J\subseteq I$ [6].

Proposition 2.6. Let *G* be an R$_\Gamma$-module and *S* be a semiprime R$_\Gamma$-submodule, then $[S:_{R_\Gamma}G]$ is semiprime ideal of a *Γ*-ring R.

Proof. Let *J* be an ideal in *R* such that $J\,\Gamma\,J\subseteq[S:_{R_\Gamma}G]$, then $J\Gamma J\Gamma G\subseteq S$. Since *S* is semiprime R$_\Gamma$-submodule, then $J\Gamma G\subseteq S$. Therefore, $J\subseteq[S:_{R_\Gamma}G]$ and $[S:_{R_\Gamma}G]$ are semiprime ideals of a *Γ*-ring *R*. To show that the converse is not true in general, the following example is shown:

Let $G=Z\oplus Z$ be a $Z_{<\bar{3}>}$-module and let *S* be an R$_\Gamma$-submodule generated by $<(0,4)>$, then $[S:_{R_\Gamma}G]=\{0\}$ is semiprime ideal of a *Γ*-ring *Z*, but *S* is not semiprime R$_\Gamma$-submodule of G. Let $<\bar{2}>$ be an ideal, *Γ* be a abelian group define by $<\bar{3}>$ and *S* be an R$_\Gamma$-submodule generated by $<(0,4)>$, then $<\bar{2}><\bar{3}><\bar{2}>\subseteq[S:_{R_\Gamma}G]$, then $<\bar{2}><\bar{3}><\bar{2}>=(0)\subseteq[S:_{R_\Gamma}G]$. Then $<(0,g)><(u,0)>=\{(0,0)\}\subseteq[S:_{R_\Gamma}G]$ for all $u\in G\ ,g\in R$ .Thus $[S:_{R_\Gamma}G]=\{0\}$ is semiprime ideal of a *Γ*-ring *Z*, but *S* is not semiprime R$_\Gamma$-submodule of *G*. Let $(0,1)\in G$, $<\bar{3}>\in\Gamma$ such that $<\bar{2}><\bar{3}><\bar{2}><\bar{3}>(0,1)=(0,36)\in S$ and $<\bar{2}><\bar{3}>(0,1)=(0,6)\notin S$.

Theorem 2.7. Let *S* be a proper R$_\Gamma$-submodule of R$_\Gamma$-module *G*, then the following statements are equivalents:

1. *S* is semiprime R$_\Gamma$-submodule of *G*.

2. The ideal $[S :_{R_\Gamma} K]$ is semiprime in *Γ*-ring *R*, for all *K* is a proper R_Γ-submodule of *G* such that $S \subset K$.
3. The ideal $[S :_{R_\Gamma} <u>]$ is semiprime in *Γ*-ring *R*, for all $u \in G$ and $u \notin S$.

Proof. $(1 \to 2)$

Let *K* be a proper R_Γ-submodule of *G*. Let *I* be an ideal of *Γ*-ring *R* such that $I \Gamma I \subseteq [S :_{R_\Gamma} K]$, then $I \Gamma I \Gamma K \subseteq S$. Since *S* is semiprime R_Γ-submodule of *G*, then $I \Gamma K \subseteq S$ and hence $I \subseteq [S :_{R_\Gamma} K]$. Thus, $[S :_{R_\Gamma} K]$ is semiprime ideal in *Γ*-ring *R*.

$(2 \to 3)$

Let $u \in G$, $u \notin S$, let *J* be an ideal of *Γ*-ring *R* such that $J \Gamma J \subseteq [S :_{R_\Gamma} <u>]$ and let $g \in R$ such that $J = <g>$, then $<g> \Gamma <g> \subseteq [S :_{R_\Gamma} <u>]$. We have to show that $<g> \subseteq [S :_{R_\Gamma} <u>]$. Since $u \in G$ and $u \notin S$, then $<u>$ is R_Γ-submodule of *G*, $<u> \subseteq S + <u>$, then $[S :_{R_\Gamma} <u>] \subseteq [S :_{R_\Gamma} S + <u>]$ and $S \subseteq S + <u>$. By hypothesis (2), $[S :_{R_\Gamma} S + <u>]$ is semiprime ideal of *R*, then $J \subseteq [S :_{R_\Gamma} S + <u>]$ and $J \Gamma <u> \subseteq S$. Thus, $J \subseteq [S :_{R_\Gamma} <u>]$ and $[S :_{R_\Gamma} <u>]$ is semiprime ideal in *Γ*-ring *R*.

$(3 \to 1)$

Let *I* be an ideal in *Γ*-ring *R* and $u \in G$, then $<u>$ is R_Γ-submodule of *G*. Let $I \Gamma I \Gamma <u> \subseteq S$, to show that $I \Gamma <u> \subseteq S$. Since $[S :_{R_\Gamma} <u>]$ is semiprime ideal in *R*, then $I \subseteq [S :_{R_\Gamma} <u>]$ and, hence, $I \Gamma <u> \subseteq S$. Thus *S* is semiprime R_Γ-submodule of *G* by Proposition (2.6).

Proposition 2.8. Let *S* be a proper R_Γ-submodule of R_Γ-module *G*, if *S* is prime R_Γ-submodule of *G* and $S = \bigcap_{i \in \Lambda} S_i$ where each S_i is prime R_Γ-submodule of *G*, then *S* is semiprime R_Γ-submodule of *G*.

Proof. Let *K* be a proper R_Γ-submodule of *G* and let *I* be an ideal of a *Γ*-ring *R* such that $I \Gamma I \Gamma K \subseteq S$. We have to show that $I \Gamma K \subseteq S$. Since S_i is a prime R_Γ-submodule of *G*, then S_i is semiprime R_Γ-submodule of *G* by Remark ((2.5), ii). Then $I \Gamma K \subseteq S_i$ for all $i \in \Lambda$,which implies that $I \Gamma K \subseteq \bigcap_{i \in \Lambda} S_i = S$. Thus, *S* is semiprime R_Γ-submodule of *G*.

Proposition 2.9. Let *G* be R_Γ-module and let S be a proper R_Γ-submodule of *G*. If *S* is semiprime R_Γ-submodule of *G* and *L* is a proper R_Γ-submodule of *G* such that $L \not\subset S$, then $L \bigcap S$ is semiprime R_Γ-submodule of *G*.

Proof. Let $w \in L$, $g \in R$ such that $<g> \Gamma <g> \Gamma <w> \subseteq S \bigcap L$, then $<g> \Gamma <g> \Gamma <w> \subseteq S$ and $<g> \Gamma <g> \Gamma <w> \subseteq L$. But $w \in L$, hence $g \Gamma w \subseteq L$. As *S* is semiprime R_Γ-submodule of *G*, then $g \Gamma w \subseteq S$, hence $g \Gamma w \subseteq S \bigcap L$ which implies that $L \bigcap S$ is semiprime R_Γ-submodule of *G*.

Proposition 2.10. Let *G* be R_Γ-module and S_α be a family semiprime R_Γ-submodule of *G*, for each $\alpha \in \Lambda$, then $\bigcap_{\alpha \in \Lambda} S_\alpha$ is semiprime R_Γ-submodule of *G*.

Proof. Let *I* be an ideal of *Γ*-ring *R* and *H* be a proper R_Γ-submodule of *G* such that $I \Gamma I \Gamma H \subseteq \bigcap_{\alpha \in \Lambda} S_\alpha$, to show that $I \Gamma H \subseteq \bigcap_{\alpha \in \Lambda} S_\alpha$. Then $I \Gamma I \Gamma H \subseteq S_\alpha$ for all $\alpha \in \Lambda$, since S_α is semiprime R_Γ-submodule of *G*, then $I \Gamma H \subseteq S_\alpha$ for all $\alpha \in \Lambda$. Thus $I \Gamma H \subseteq \bigcap_{\alpha \in \Lambda} S_\alpha$ for all $\alpha \in \Lambda$. Hence $\bigcap_{\alpha \in \Lambda} S_\alpha$ is semiprime R_Γ-submodule of *G*.

Recall that *T* is an ideal of *Γ*-ring *R*. The radical of *T*, denoted by $rad_\Gamma(T)$, is defined to be the intersection of all prime ideals containing *T* [3].

Recall that *G* is an R_Γ-module and *S* is an R_Γ-submodule of *G* that is said to be primary if for any R_Γ-submodule *V* of *G* and for any ideal *I* of a *Γ*-ring *R*, $I\Gamma V \subseteq S$ and $V \not\subset S$ implies $I \subseteq rad_\Gamma[S :_{R_\Gamma} G]$ [3].

Proposition 2.11. Let *G* be R_Γ-module and *S* be a proper R_Γ-submodule of *G*. If *S* is primary R_Γ-submodule of *G*, then $[S :_{R_\Gamma} G]$ is semiprime ideal of a *Γ*-ring *R* if and only if *S* is semiprime R_Γ-submodule of *G*.

Proof. Suppose that $[S :_{R_\Gamma} G]$ is semiprime ideal of *R*. Let *I* be an ideal of *Γ*-ring *R* and *K* be a proper R_Γ-submodule of *G* such that $I\Gamma I\Gamma K \subseteq S$. We have to show that $I\Gamma K \subseteq S$. Since $[S :_{R_\Gamma} G]$ is semiprime ideal of *R*, then $I\Gamma I \subseteq [S :_{R_\Gamma} G]$. Since $I\Gamma I\Gamma K \subseteq S$ then $I\Gamma I \subseteq [S :_{R_\Gamma} K] \subseteq [S :_{R_\Gamma} G]$, and as $[S :_{R_\Gamma} G]$ is an ideal of a *Γ*-ring *R*, we obtain $[S :_{R_\Gamma} G]\Gamma K \subseteq S$ and $I \subseteq [S :_{R_\Gamma} G]$. Thus $I\Gamma K \subseteq S$ and *S* is semiprime R_Γ-submodule of *G*. The converse is true, by Proposition (2.6).

Proposition 2.12. - Let G, G' be R_Γ-modules and let $\varphi : G \to G'$ be an R_Γ-epimorphism, then :

1) If *S* is semiprime R_Γ-submodule of *G* and $Ker\,\varphi \subseteq S$, then $\varphi(S)$ is semiprime R_Γ-submodule of G'.

2) If S' is semiprime R_Γ-submodule of G', then $\varphi^{-1}(S')$ is semiprime R_Γ-submodule of *G*.

Proof.

1) Let $h \in R$, $u' \in G'$ such that $(h\Gamma)^2 u' \subseteq \varphi(S)$, $(h\gamma)^2 u' \in \varphi(S)$ for all $\gamma \in \Gamma$. Since φ is epimorphism, then there exists $u \in G$ such that $u' = \varphi(u)$.
$(h\gamma)^2 \varphi(u) \in \varphi(S)$, then $\varphi((h\gamma)^2 u) \in \varphi(S)$. Since φ is R_Γ-homomorphism and there exists $v \in S$ such that $\varphi((h\gamma)^2 u) = \varphi(v)$, then $v - (h\gamma)^2 u \in Ker\,\varphi \subseteq S$ and $(h\gamma)^2 u \in S$. Since *S* is semiprime R_Γ-submodule of *G*, then $h\gamma u \in S$ and $\varphi(h\gamma u) \in \varphi(S)$. Thus, $h\gamma u' \in \varphi(S)$ and, hence, $\varphi(S)$ is semiprime R_Γ-submodule of G'.

2) Let $h \in R$, $u \in G$ such that $(h\Gamma)^2 u \subseteq \varphi^{-1}(S')$, $u = \varphi^{-1}(u')$, $u' \in G'$ for all $\gamma \in \Gamma$. $(h\gamma)^2 u \in \varphi^{-1}(S')$, then $\varphi((h\gamma)^2 u) \in S'$ and $(h\gamma)^2 \varphi(u) \in S'$. Since S' is semiprime R_Γ-submodule of G', then $h\gamma\varphi(u) \in S'$ and $h\gamma u \in \varphi^{-1}(S')$. Hence, $\varphi^{-1}(S')$ is semiprime R_Γ-submodule of *G*.

Corollary 2.13. Let *S* be a proper R_Γ-submodule of R_Γ-module *G* and let *H* be any proper R_Γ-submodule of *G* such that $H \subseteq S$, then *S* is semiprime R_Γ-submodule of *G* if and only if S/H is a semiprime R_Γ-submodule of G/H.

3. Semiprime R_Γ-Submodules of Multiplication R_Γ-Modules

Notice that *G* is multiplication R_Γ-module, if for any *S* be a proper R_Γ-submodule of *G*, there exists an ideal *I* of a *Γ*-ring *R* such that $S = I\Gamma G$ [3, 7] .

Proposition 3.1. Let *G* be multiplication R_Γ-module and *S* be a proper R_Γ-submodule of *G*, then *S* is semiprime R_Γ-submodule of *G* if and only if $[S :_{R_\Gamma} G]$ is semiprime ideal of Γ-ring R.

Proof. The first side is clear.

Conversely, suppose that $[S :_{R_\Gamma} G]$ is semiprime ideal of R. Let $g \in R$, $w \in G$; $w \notin S$, then $<g>$ is an ideal in *Γ*-ring *R* and $<w>$ is R_Γ-submodule of *G* such that $<g>\Gamma<g>\Gamma<w> \subseteq S$, to show that $<g>\Gamma<w> \subseteq S$. Since *G* is multiplication R_Γ-module, then $S = [<w> :_{R_\Gamma} G]\Gamma G$ where $<w>$ is R_Γ-submodule of *G* generated by *w* and $[<w> :_{R_\Gamma} G]$ is an ideal in R. $<w> = [<w> :_{R_\Gamma} G]\Gamma G$ and $w = v_1\gamma_1 k_1 + v_2\gamma_2 k_2 + \ldots + v_n\gamma_n k_n$, where $k_i \in [<w> :_{R_\Gamma} G]$, $\gamma \in \Gamma$ and $v_i \in G$, for all $i = 1,2,3,\ldots, n$. $g\gamma k_i \in [<g\gamma w> :_{R_\Gamma} G]$ and $g\gamma w \in S$. Then $[<g\gamma w> :_{R_\Gamma} G] \subseteq [S :_{R_\Gamma} G]$ and $g\gamma k_i \in [S :_{R_\Gamma} G]$,

$w = g\gamma k_1\gamma_1 v_1 + g\gamma k_2\gamma_2 v_2 + ... + g\gamma k_n\gamma_n v_n \in S$. Then $g\Gamma w \subseteq S$ and, hence, S is semiprime R_Γ-submodule of *G*.

Theorem 3.2. Let *G* be a multiplication R_Γ-module and let *S* be a semiprime R_Γ-submodule of *G* such that $K_1 \cap K_2 \subseteq S$, where K_1, K_2 are R_Γ-submodules of *G*, then $K_1 \subseteq S$ or $K_2 \subseteq S$.

Proof.

Let *S* be a semiprime R_Γ-submodule of G and $K_1 \cap K_2 \subseteq S$. Then $[K_1 \cap K_2 :_{R_\Gamma} G] \subseteq [S :_{R_\Gamma} G]$ and $[K_1 :_{R_\Gamma} G] \cap [K_2 :_{R_\Gamma} G] \subseteq [S :_{R_\Gamma} G]$. Since $[S :_{R_\Gamma} G]$ is semiprime ideal of a *Γ*-ring *R* by Proposition (2.6), then $[K_1 :_{R_\Gamma} G] \subseteq [S :_{R_\Gamma} G]$ or $[K_2 :_{R_\Gamma} G] \subseteq [S :_{R_\Gamma} G]$, then $[K_1 :_{R_\Gamma} G]\Gamma G \subseteq [S :_{R_\Gamma} G]\Gamma G$ or $[K_2 :_{R_\Gamma} G]\Gamma G \subseteq [S :_{R_\Gamma} G]\Gamma G$. Since *G* is multiplication R_Γ-module and by Proposition (3.1), hence $K_1 \subseteq S$ or $K_2 \subseteq S$.

Recall that *G* is an R_Γ-module that is called irreducible or (simple), if $G\Gamma R \neq 0$ and it has only the trivial R_Γ-submodules $\{0\}$ and *G* itself [4].

Proposition 3.3. Let *G* be an R_Γ-module. If *S* is irreducible R_Γ-submodule of *G*, then *S* is semiprime R_Γ-submodule of *G* if and only if *S* is a prime R_Γ-submodule of *G*.

Proof. The first side is clear. Conversely, suppose that *S* is not prime R_Γ-submodule of *G*. Let $h \in R$, $h \notin [S :_{R_\Gamma} G]$, $u \in G$, $u \notin S$ and $\alpha \in \Gamma$ such that $h\alpha u \in S$. Since $h \notin [S :_{R_\Gamma} G]$, there exists $v \in G$ such that $h\alpha v \notin S$. We claim that $K_1 \cap K_2 = S$. Let $w \in K_1 \cap K_2$ and $K_1 = S + \langle u \rangle$, $K_2 = S + \langle h\alpha v \rangle$. Let $s_1, s_2 \in S$ and $t_1, t_2 \in R$ such that $w = s_1 + t_1\alpha u = s_2 + t_2\alpha h\alpha v$, then $w = s_1 - s_2 + t_1\alpha u = t_2\alpha h\alpha v$. By multiplying this equation by $h_1 \in R$, we obtain $h_1\gamma s_1 - h_1\gamma s_2 + h_1\gamma t_1\alpha u = h_1\gamma t_2\alpha h\alpha v$ where $\gamma \in \Gamma$. $h_1\gamma s_1 - h_1\gamma s_2 + h_1\gamma t_1\alpha u = h_1\gamma t_2\alpha h\alpha v \in S$. Since *S* is semiprime R_Γ-submodule of *G*, then $t_2\alpha h\alpha v \in S$ and $h_2\alpha v \in S$ such that $t_2\alpha h = h_2$, also $s_2 + t_2\alpha h\alpha v = w \in S$. Hence, $K_1 \cap K_2 = S$, which is a contradiction, since *S* is irreducible. Thus *S* is prime R_Γ-submodule of *G*.

Recall that an R_Γ-module *G* is called R_Γ-faithful if its R_Γ-annihilator $l_R(G) = 0$ [4].

Definition 3.4. Let *G* be an R_Γ-module. If *J* is a maximal ideal of *Γ*-ring *R*, then we define $T_{J\Gamma}(G) = \{u \in G, \alpha \in \Gamma ; (1-j)\alpha u = 0\}$ for some $j \in J$. Clearly, $T_{J\Gamma}(G)$ is R_Γ-submodule of *G*.

Definition 3.5. Let *G* be an R_Γ-module and *J* is a maximal ideal of a *Γ*-ring *R*. We say that *G* is *J*-cyclic if there exist $j \in J$, $u \in G$ and $\alpha \in \Gamma$ such that $(1-j)\Gamma G \subseteq R\Gamma u$.

Theorem 3.6. Let *R* be a commutative *Γ*-ring with identity. Then an R_Γ-module *G* is a multiplication R_Γ-module if and only if, for every maximal ideal *J* of *Γ*-ring *R*, either $G = T_{J\Gamma}(G)$ or *G* is *J*-cyclic.

Proof.

Suppose that *G* is a multiplication R_Γ-module. Let *J* be a maximal ideal of a *Γ*-ring *R*. Suppose that $G = J\Gamma G$, and let $u \in G$. Then $J\Gamma u = I\Gamma G$ for some *I* is an ideal of a *Γ*-ring *R* and, hence, $R\Gamma u = I\Gamma G = I\Gamma J\Gamma G = J\Gamma I\Gamma G = J\Gamma u$ and $1\alpha u = j\alpha u$ such that $1 \in R$, $j \in J$, $\alpha \in \Gamma$. Thus, $(1-j)\alpha u = 0$ and $u \in T_{J\Gamma}(G)$. It follows that $G = T_{J\Gamma}(G)$. Now suppose that $G \neq J\Gamma G$, there exist $w \in G$ and $w \notin J\Gamma G$. There exists an ideal *B* of *Γ*-ring *R* such that $R\Gamma w = B\Gamma G$. Clearly, $B \not\subset J$ and, hence, $1 - t \in B$ for some $t \in J$. Clearly, $(1-t)\Gamma G \subseteq R\Gamma w$ and *G* is *J*-cyclic. Conversely, suppose that, for each maximal ideal *J* of a *Γ*-ring *R*, either $G = T_{J\Gamma}(G)$ or *G* is *J*-cyclic. Let *S* be a R_Γ-submodule of *G* and $K = ann_{R_\Gamma}(G/S)$. Clearly, $K\Gamma G \subseteq S$. Let $y \in S$ and $H = \{h \in R ; h\gamma y \in K\Gamma G\}$. Suppose that $H \neq R$, then there exists a maximal ideal *Q* of a *Γ*-ring *R* such that $H \subseteq Q$. If $G = T_{Q\Gamma}(G)$, then $(1-s)\gamma y = 0$ for some $s \in Q, \gamma \in \Gamma$ and, hence,

$(1-s)\in H\subseteq Q$, which is a contradiction. Thus, by hypothesis, there exist $s_1\in Q$, $z\in G$ such that $(1-s_1)\Gamma G\subseteq R\Gamma z$. It follows that $(1-s_1)\Gamma S$ is a R_Γ-submodule of $R\Gamma z$ and hence $(1-s_1)\Gamma S=F\Gamma z$ where *F* is an ideal such that $F=\{h\in R\ ;\ h\gamma z\in(1-s_1)\Gamma S\}$ of a *Γ*-ring *R*. $(1-s_1)\Gamma F\Gamma G=F\Gamma(1-s_1)\Gamma G\subseteq F\Gamma z\subseteq S$ and hence $(1-s_1)\Gamma F\subseteq K$. It follows that $(1-s_1)\gamma(1-s_1)\gamma y\in(1-s_1)\Gamma(1-s_1)\Gamma S=(1-s_1)\Gamma F\Gamma z\subseteq K\Gamma G$. But this gives a contradiction of $(1-s_1)\gamma(1-s_1)\in H\subseteq Q$. Thus, $H=R$ and $y\in K\Gamma G$. It follows that $S=K\Gamma G$ and *G* is multiplication R_Γ-module.

Theorem 3.7. Let *R* be a commutative *Γ*-ring with identity and *G* be an R_Γ-faithful R_Γ-module. Then *G* is a multiplication R_Γ-module if and only if

i. $\bigcap_{\lambda\in\Lambda}(I_\lambda\Gamma G)=(\bigcap_{\lambda\in\Lambda}I_\lambda)\Gamma G)$ for any non-empty collection of ideals $I_\lambda(\lambda\in\Lambda)$ of a *Γ*-ring *R*.

ii. For any R_Γ-submodule *S* of *G* and an ideal *A* of a *Γ*-ring *R*, such that $S\subset A\Gamma G$, there exists an ideal *B* with $B\subset A$ and $S\subseteq B\Gamma G$.

Proof.

To prove (i), suppose that *G* is a multiplication R_Γ-module. Let $I_\lambda(\lambda\in\Lambda)$ be any non-empty collection of ideals of a *Γ*-ring *R* and let $I=\bigcap_{\lambda\in\Lambda}I_\lambda$. Clearly, $I\Gamma G\subseteq\bigcap_{\lambda\in\Lambda}(I_\lambda\Gamma G)$. Let $x\in\bigcap_{\lambda\in\Lambda}(I_\lambda\Gamma G)$ and let $J=\{g\in R\ ;\ g\gamma x\in I\Gamma G\}$. Suppose that $J\neq R$, then there exists a maximal ideal *P* of *R* such that $J\subseteq P$. Clearly, $x\notin T_{P\Gamma}(G)$ and hence *G* is *P*-cyclic by Theorem 3.6. There exist $t\in P$ and $m\in G$ such that $(1-t)\Gamma G\subseteq R\Gamma m$. Then $(1-t)\beta x\in\bigcap_{\lambda\in\Lambda}(I_\lambda\Gamma m)$ for each $\beta\in\Gamma$. There exists $a_\lambda\in I_\lambda$ such that $(1-t)\beta x=a_\lambda\beta m$. Choose $\alpha\in\Lambda$ for each $\lambda\in\Lambda$, $a_\alpha\beta m=a_\lambda\beta m$ and, so, $(a_\alpha-a_\lambda)\beta m=0$. Now, $(1-t)\Gamma(a_\alpha-a_\lambda)\Gamma G=(a_\alpha-a_\lambda)\Gamma(1-t)\Gamma G\subseteq(a_\alpha-a_\lambda)\Gamma R\Gamma m=0$ implies $(1-t)\Gamma(a_\alpha-a_\lambda)=0$. Therefore, $(1-t)\Gamma a_\alpha=(1-t)\Gamma a_\lambda\in I_\lambda$, $(\lambda\in\Lambda)$ and, hence, $(1-t)\Gamma a_\alpha\in I$. Thus $(1-t)\Gamma(1-t)\Gamma x=(1-t)\Gamma a_\alpha\Gamma m\in I\Gamma G$. It follows that $(1-t)\Gamma(1-t)\in J\subseteq P$, which is a contradiction and, hence, $J=R$ and $x\in I\Gamma G$. Thus $\bigcap_{\lambda\in\Lambda}(I_\lambda\Gamma G)\subseteq I\Gamma G$.

Now to prove that (ii), let *S* be a R_Γ-submodule of *G* and *A* be an ideal of a *Γ*-ring *R* such that $S\subseteq A\Gamma G$. There exists an ideal *C* of a *Γ*-ring *R* such that $S\subseteq C\Gamma G$. Let $B=A\bigcap C$. Clearly, $B\subset A$ and $S=A\Gamma G\bigcap C\Gamma G=(A\bigcap C)\Gamma G=B\Gamma G$, by (i).

Conversely, suppose that (i) and (ii) hold. Let *S* be a R_Γ-submodule of *G* and let $S=\{I\,|\,I$ be an ideal of a *Γ*-ring *R* and $S\subseteq I\Gamma G\}$, let $I_\lambda(\lambda\in\Lambda)$ be any non-empty collection of ideals in *S*. By (i) , $\bigcap_{\lambda\in\Lambda}I_\lambda\in S$. By Zorn's Lemma, *S* has a minimal member *A*, then $S\subseteq A\Gamma G$. Suppose that $S\neq A\Gamma G$, by (ii) there exists an ideal *B* with $B\subset A$ and $S\subseteq B\Gamma G$. In this case, $B\subset S$, contradicting the choice of *A*, and, thus, $S=A\Gamma G$. It follows that *G* is a multiplication R_Γ-module.

Lemma 3.8. Let *P* be a prime ideal of a *Γ*-ring *R* and *G* a R_Γ-faithful multiplication R_Γ-module. Let $h\in R$, $\alpha\in\Gamma$ and $u\in G$, satisfying that $h\alpha u\in P\Gamma G$. Then $h\in P$ or $\alpha u\in P\Gamma G$.

Proof

Suppose that $h \notin P$ and let $J = \{s \in R \; ; s\gamma u \in P\,\Gamma G\}$. Suppose that $J \neq R$, then there exists a maximal ideal *Q* of a *Γ*-ring *R* such that $J \subseteq Q$. Clearly, $u \notin T_{Q\Gamma}(G)$. By Theorem 3.6., *G* is *Q*-cyclic, that is there exist $m \in G$, $q \in Q$ such that $(1-q)\Gamma G \subseteq R\,\Gamma m$. In particular, $(1-q)\alpha u = h\,\alpha\, m$ and $(1-q)\alpha\, h\beta u = p\,\alpha\, m$ for some $\beta \in \Gamma$, $p \in P$ and $s \in R$, thus $(h\,\gamma s - p)\gamma\, m = 0$; $\gamma \in \Gamma$. Now, $[(1-q)\Gamma ann_{R_\Gamma}(G)]\Gamma G = 0$ implies $(1-q)\Gamma ann_{R_\Gamma}(G) = 0$, because *G* is R_Γ-faithful, and, hence, $(1-q)\alpha\, h\beta s = (1-q)\alpha\, p \in P$. But $P \subseteq J \subseteq Q$ so that $s \in P$ and $(1-q)\alpha u = s\,\alpha\, m \in P\,\Gamma G$. Hence, $(1-q) \in J \subseteq Q$, which is a contradiction. Thus $J = R$ and $\alpha u \in P\,\Gamma G$.

Corollary 3.9. The following statements are equivalent for a proper R_Γ-submodule *S* of a multiplication R_Γ-module G :-

i. *S* is prime R_Γ-submodule of *G*.

ii. $ann_{R_\Gamma}(G/S)$ is a prime ideal of a *Γ*-ring *R*.

iii. $S = P\,\Gamma G$ for some prime ideal *P* of a *Γ*-ring *R* with $ann_{R_\Gamma}(G) \subseteq P$.

Proof. $(1 \to 2)$

Let *I* and *J* be ideals of a *Γ*-ring *R* such that $I\,\Gamma J \subseteq ann_{R_\Gamma}(G/S)$. Then, $G\,\Gamma I\,\Gamma J \subseteq S$. Since *S* is a prime R_Γ-submodule of *G*, $G\,\Gamma I \subseteq S$ or $J \subseteq ann_{R_\Gamma}(G/S)$. Therefore, $I \subseteq ann_{R_\Gamma}(G/S)$ or $J \subseteq ann_{R_\Gamma}(G/S)$.

$(2 \to 3)$

Let *S* be R_Γ-submodule of G. Then $S = I\,\Gamma G$ for some *I* is an ideal of a *Γ*-ring *R*, therefore $I \subseteq ann_{R_\Gamma}(G/S) \subseteq P$. Then, $S = I\,\Gamma G \subseteq P\Gamma G \subseteq S$. Consequently, $S = P\,\Gamma G$.

$(3 \to 1)$

Suppose that *P* is a prime ideal *P* of *R* such that $ann_{R_\Gamma}(G) \subseteq P$. Let *K* be a R_Γ-submodule of *G* such that $K \not\subset S$ and let *I* be an ideal of a *Γ*-ring *R*, $I \not\subset ann_{R_\Gamma}(G/S)$. But $K\,\Gamma I \subseteq S$, where *K* is a R_Γ-submodule of *G*. Since *G* is multiplication R_Γ-module, then $G\,\Gamma J \subseteq K$ where *J* is an ideal of a *Γ*-ring *R*. Then $K\,\Gamma I = G\,\Gamma J\,\Gamma I$ and, so, $J\,\Gamma I \subseteq ann_{R_\Gamma}(G/S)$ by (ii), and $I \not\subset ann_{R_\Gamma}(G/S)$, $J \subseteq ann_{R_\Gamma}(G/S)$. Therefore $K = G\Gamma J \subseteq S$. This is a contradiction.

Theorem 3.10. Let *G* be a multiplication R_Γ-module and let *S* be a proper R_Γ-submodule of *G*, then $G - rad_\Gamma(S) = \sqrt{A}\,\Gamma G$, where $A = ann_{R_\Gamma}(G/S)$.

Proof. Let *P* denotes the collection of all prime ideals of a *Γ*-ring *R* such that $A \subseteq P$. If $B = \sqrt{A}$ then $B = \bigcap_{i\in\Lambda} P$ and, hence by Theorem 3.7, $B\Gamma G = \bigcap_{i\in\Lambda} P\,\Gamma G$. Let $G = P\,\Gamma G$ then $G - rad_\Gamma(S) \subseteq P\,\Gamma G$. If $G \neq P\,\Gamma G$ then $S = A\,\Gamma G \subseteq P\,\Gamma G$ implies $G - rad_\Gamma(S) \subseteq P\,\Gamma G$ by Corollary 3.9. It follows that $G - rad_\Gamma(S) \subseteq B\,\Gamma G$.

Conversely, suppose that *K* is a prime R_Γ-submodule of *G* containing *S*. By Corollary (3.9), there exists a prime ideal *Q* of *R* such that $A \subseteq Q$ and by Lemma (3.8) and hence $B \subseteq Q$, thus $B\,\Gamma G \subseteq K$ .It follows that $B\,\Gamma G \subseteq G - rad_\Gamma(S)$ and, therefore, $B\,\Gamma G = G - rad_\Gamma(S)$.

Theorem 3.11. Let *G* be a multiplication R_Γ-module and *S* be a proper R_Γ-submodule of *G*, then $rad_\Gamma(S) = \{u \in G \; ; (u\,\Gamma)^n \subseteq S \text{ for some } n \geq 0\}$.

Proof.
Let $K=\left\{u\in G\ ;\ (u\,\Gamma)^n\subseteq S\ for\ some\ \ n\geq 0\right\}$, to show that K is R_Γ-submodule of G. Let $x,y\in K$ and I, J be ideals, respectively, of x, y. Then, $(x\Gamma)^s=(I\,\Gamma)^s$ and $(y\Gamma)^r=(J\,\Gamma)^r$ such that $(I\,\Gamma)^s\subseteq S$ and $(J\,\Gamma)^r\subseteq S$ for some $s,r>0$. Let $k=\max\{s,r\}$, then $(x-y)^k=(I\,\Gamma-J\,\Gamma)^k=((I-J)\Gamma G)^k$, that is $x-y\in K$. Also, for $x\in K$ and $h\in R$, we have $(x\,\Gamma r)^s\subseteq S$ since $(x\,\Gamma)^s\subseteq S$. Thus K is R_Γ-submodule of G. Suppose that $u\in K$ and B is presentation of u. Then $(u\Gamma)^n=B^n\Gamma G\subseteq S$ for some $n>0$ and, hence by Theorem (3.10), we have $G-rad_\Gamma((u\Gamma)^n)=\sqrt{B^n\Gamma G}=\sqrt{B}\Gamma G\subseteq G-rad_\Gamma(S)$.
Thus $G-rad_\Gamma((u\Gamma)^n)=\sqrt{B^n\Gamma G}=\sqrt{B}\Gamma G\subseteq G-rad_\Gamma(S)$,which this implies that $K\subseteq G-rad_\Gamma(S)$. Conversely, let $u\in G-rad_\Gamma(S)=\sqrt{I}\Gamma G$, where $I=ann_{R_\Gamma}(G/S)$. Then $u=\sum_{i=1}^{n}h_i\alpha_i u_i$ for $h_i\in\sqrt{I}$,$\alpha_i\in\Gamma$ and $u_i\in G$. Thus, $h_i^{n_i}\in I$ for some $n_i>0$. Thus, for a sufficiently large n, we have $(u\Gamma)^n\subseteq I\,\Gamma G=S$ and, hence, $G-rad_\Gamma(S)\subseteq K$. Therefore, $G-rad_\Gamma(S)=K$.

Lemma 3.12. Let G be a multiplication R_Γ-module, S be a R_Γ-submodule of G, and $\varphi:G\to G/S$ is a natural R_Γ-homomorphism. Then, every R_Γ-submodules S_1 and S_2 of G, $S_1\Gamma S_2\subseteq S$ if and only if $\overline{S_1}\Gamma\overline{S_2}=\overline{0}$.

Proof.
Let $S_1=I_1\Gamma G$, $S_2=I_2\Gamma G$ and $S=J\,\Gamma G$ for some ideals I_1,I_2 and J of a Γ-ring R. Obviously, G/S is multiplication R_Γ-module. Then $\overline{S_1}\Gamma\overline{S_2}=\overline{0}$ if and only if $(I_1+J)\Gamma(I_2+J)\Gamma\, G/S=S$, which is equivalent with $(I_1+J)\Gamma(I_2+J)\Gamma G\subseteq S$. But $S=J\,\Gamma G$, therefore $(I_1+J)\Gamma(I_2+J)\Gamma G\subseteq S$ if and only if $S_1\Gamma S_2=(I_1\Gamma I_2)\Gamma G\subseteq S$.

Theorem 3.13. Let G be a multiplication R_Γ-module and S be a proper R_Γ-submodule of G. Then the following statements are equivalent:

6. S is semiprime R_Γ-submodule of G.
7. $x\,\Gamma x\subseteq S$ implies $x\in S$ such that for all $x\in G$.
8. $rad_\Gamma(S)=S$.
9. G/S has no non-zero nilpotent.
10. $K_1\Gamma K_2\subseteq S$ implies $K_1\cap K_2\subseteq S$, for every K_1,K_2 are proper R_Γ-submodules of G.

Proof. $(1\to 2)$

Let $x\,\Gamma x\subseteq S$ for some $x\in G$. Let I be an ideal in R; $I\,\Gamma x=R\,\Gamma x$. Since S is semiprime R_Γ-submodule of G, then $(I\,\Gamma)^2G\subseteq S$ and, hence, $x\in R\,\Gamma x=I\,\Gamma x\subseteq S$. Thus, $x\in S$.

$(2\to 3)$

It is clear that $S\subseteq rad_\Gamma(S)$. Let $m\in rad_\Gamma(S)$ by Theorem (3.11), then.

i. If n is even, $n=2k$; $0<k<n$, then $\left((m\Gamma)^k\right)^2=(m\Gamma)^n\subseteq S$. Let $(m\Gamma)^k=m_0\Gamma$ then $m_0\Gamma\subseteq S$ and so $(m\Gamma)^k\subseteq S$, which is a contradiction.

ii. If n is odd, $n = 2k+1$; $0 < k < n$, then $\left((m\Gamma)^{k+1}\right)^2 = (m\Gamma)^{n+1} \subseteq (m\Gamma)^n \subseteq S$. Let $(m\Gamma)^{k+1} = m_0\Gamma$ then $m_0\Gamma \subseteq S$ and, so, $(m\Gamma)^{k+1} \subseteq S$, which is a contradiction. Then, $n = 1$ and, thus, $rad_\Gamma(S) = S$.

$(3 \to 4)$

Let $m + S \in G/S$. Suppose that G/S is nilpotent, then $(m+S)^n = S$ for some $n \geq 0$. By Lemma (3.12), $m^n \subseteq S$, and by Theorem (3.11), $m \in rad_\Gamma(S) = S$, then $m + S = S$, which is a contradiction. Thus, $\frac{G}{S}$ has no non zero nilpotent.

$(4 \to 5)$

Let $K_1\Gamma K_2 \subseteq S$, for some K_1, K_2 are proper R_Γ-submodules of *G*. Let $w \in K_1 \cap K_2$, then $w \in K_1$ and $w \in K_2$ and, so, $w\Gamma w \subseteq K_1\Gamma K_2 \subseteq S$. Then by Lemma (3.12), $(w+S)^2 = (w+S)\Gamma(w+S) = S$. Since G/S has no non zero nilpotent, hence $w + S = S$. Thus, $w \in S$.

$(5 \to 1)$

Let $I\Gamma I\Gamma G \subseteq S$ for some *I* is an ideal in *Γ*-ring *R*, then $(I\Gamma G)(I\Gamma G) = (I\Gamma G)^2 \subseteq S$ by (5), then $I\Gamma G \subseteq S$. Thus, *S* is semiprime R_Γ-submodule of *G*.

Definition 3.14. Let *G* be an R_Γ-module and *S* be a proper R_Γ-submodule of *G* that is called R_Γ-injective envelope of *S* in *G*, denoted by $E_{G\Gamma}(S) = \{h = g\gamma m \; ; g \in R, m \in G \text{ suchthat } g\gamma g\gamma m \in S\}$

Proposition 3.15. Let *G* be an R_Γ-module and *S* be a proper R_Γ-submodule of *G*, then *S* is semiprime if and only if $E_{G\Gamma}(S) = S$.

Proof: Suppose that *S* is semiprime R_Γ-submodule of *G*, to show that $E_{G\Gamma}(S) = S$.

Clearly, $S \subseteq E_{G\Gamma}(S)$. Let $h = g\gamma m \in E_{\Gamma G}(S)$, where $g \in R, m \in G$ such that $g\gamma g\gamma m \in S$. But *S* is semiprime R_Γ-submodule of *G*, then $h = g\gamma m \in S$, thus $E_{\Gamma G}(S) = S$.

Conversely let $g \in R, m \in G$ such that $g\gamma g\gamma m \in S$, then $g\gamma m \in E_{\Gamma G}(S) = S$. Thus, *S* is semiprime R_Γ-submodule of G [8-10].

Refrences

1. Nobusawa, N. **1964**. "On a Generalization of the Ring Theory". *Osaka Journal of Mathematics*, **1**(1): 81-89.
2. Ameri, R. and Sadeghi, R. **2010**. "Gamma Modules". *Ratio mathematica*, 2010. **20**(1): 127-147.
3. Sengu, U.T.U. **2005**. "On Prime ΓM-Submodules of ΓM-Modules". *International journal of Pure and Applied Mathematics*, **19**: 123-128.
4. Abbas, H.A. **2018**. "Projective Gamma Modules and Some Related Concepts", in department of Mathematics, Al Mustansiryah University: Baghdad, Iraq.
5. Athab, E.A. **1996**. "Prime and Semiprime Modules", in Department of Mathematics ,College of Science, University of Baghdad: Baghdad, Iraq.
6. Nobusawa, N. **1964**. On a generalization of the ring theory. *Osaka Journal of Mathematics*, **1**(1): 81-89.
7. Abbas, M.S. **2018**. ΓR-Multiplication and ΓR-Projective Gamma Modules. *International Journal of Contemporary Mathematical Sciences*, **13**(2): 87-94.
   Nekooei, M.E.a.R., "On Generalizations of Prime Ideals". *Communications in Algebra*, 2012. **40** (4).
8. Estaji, A.A., Khorasani, A.A.S. and Baghdari, S. **2014**. "On Multiplication Γ-Modules". *Ratio Mathematica*, 2014. **26**: 21-38.

9. Al-Mothafar, N.S. and Athab, I.A**. 2017**. "J- Semiprime Submodules". *International Journal of Science and Research* (IJSR), July 2017. **6**(7).
10. Kasch, F. **1982**. "*Modules and Rings*", London: Academic Press I ns