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A Class of Harmonic Univalent Functions Defined by Differential Operator and the Generalization

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Abstract
In this paper, a new class of harmonic univalent functions was defined by the differential operator. We obtained some geometric properties, such as the coefficient estimates, convex combination, extreme points, and convolution (Hadamard product), which are required.

فئة من الدوال التوافقية احادية التكافؤ المعرفة بواسطة عامل تفاضلي وتعميمها

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الخلاصه
في هذا البحث فئة جديدة من الدوال احادية التكافؤ التي يحددها المؤثر التفاضلي و نحقق بعض الخواص الهندسية متل تقديرات المعامل,الجمع المحدب,النقاط الخارجية والالتفاف

1. Introduction

A complex - valued continuous harmonic function $f = u + iv$ is harmonic in $\mathcal{D}$ whether both u and v are real harmonic, at any simply connected $\mathbb{B} \subset \mathcal{D}$ which may be written as $f = h + \bar{g}$, where h and g are analytic in $\mathbb{B}$ .We call "h" analytic part and "g" co-analytic part of f . Clunie and Sheil-Small [1] noted that the necessary and sufficient condition for the harmonic functions $f = h + \bar{g}$ a to be locally univalent and sense-preserving in $\mathbb{B}$ is that $|h'(z)| < |g'(z)|$ ($z \in \mathbb{B}$).

Let $\mathbb{S}_H$ denotes the class of harmonic functions $f = h + \bar{g}$, which are univalent and sense-preserving in the unit disk $\Omega = \{ z \in C : |z| = 1 \}$ wherever h and g are analytic in $\mathbb{B}$ for which $f(0) = h(0) = f_z - 1 = 0$. And for $f = h + \bar{g} \in \mathbb{S}_H$, we may express the analytic functions h and g as follows:

$$h(z) = z + \sum_{w=2}^{\infty} a_w z^w , \quad g(z) = \sum_{w=1}^{\infty} b_w z^w , \quad |b_1| < 1 . \tag{1}$$

Notice the $\mathbb{S}_H$ decrease of each normalization functions which are analytic univalent, whether the co-analytic part of f is zero.

And, we symbolize by $\mathbb{S}_{\overline{H}}$ the subclass of $\mathbb{S}_H$ consisting of all the functions $f_k(z) = h(z) + \overline{g_k(z)}$, wherever h and g are given by

$$h(z) = z - \sum_{w=2}^{\infty} |a_w| z^w , \quad g(z) = (-1)^K \sum_{w=1}^{\infty} |b_w| z^w , \quad |b_1| < 1 . \tag{2}$$

In 1984, Clunie and Sheil-Small [1] investigated the class $\mathbb{S}_H$ as well as its geometric subclass and obtained some coefficient bounds. Many authors studied the family of harmonic univalent function [2-7].

In 2016, Makinde [8] introduced the F^k differential operator as follows

$$F^k f(z) = z + \sum_{w=2}^{\infty} c_{wk} z^w, \tag{3}$$

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wherever

$C_{wk} = \frac{w!}{|w-k|!}$ and $F^k f(z) = z^k \left[z^{-(k-1)} + \sum_{w=2}^{\infty} c_{wk} z^w \right], k \in N_0 = N \cup \{0\}$,

and

$F^0 f(z) = f(z)$, $F^1 f(z) = z + \sum_{w=2}^{\infty} c_{w1} z^w$.

Then, this implies that $F^k f(z)$ is identically the same as *f (z)* where k = 0, and it reduces the first differential coefficient of the Salagean differential operator where k = 1.

At $f = h + \bar{g}$, define by (1), Sharma and Ravindar [9] considered the differential operator which is defined by Equation (3) of *f* as

$$F^k f(z) = F^k h(z) + (-1)^k \overline{F^k g(z)}, k \in N_0 = N \cup \{0\}, z \in \mathbb{C}, \quad (4)$$

wherever

$F^k h(z) = z + \sum_{w=2}^{\infty} c_{wk} a_w z^w$, $F^k g(z) = z + \sum_{w=2}^{\infty} c_{wk} b_w z^w$, and $C_{wk} = \frac{w!}{|w-k|!}$

In this paper, motivated by a previous study [9], a new class, $M_H(K, \alpha, \gamma)$ $k \in N_0 = N \cup \{0\}$, $0 \le \gamma \le 1$, $0 \le \alpha < 1$, of harmonic univalent functions in $\Omega = \{ z \in C : |z| = 1 \}$ is presented and studied. Moreover, coefficient conditions, distortion bounds, convex combination, extreme points, and radii of convexity for this class are obtained.

2. Main Results

2.1. The Class $M_H(K, \alpha, \gamma)$

Definition 1 :- Let $f_z = h + \overline{g_k}$ be a harmonic function, where *h (z)* and g (z) are given by (1). Then $f(z) \in M_H(K, \alpha, \gamma)$ and it satisfies

$$\text{Re}\left(\frac{z\left(F^{k+1} f(z)\right)'}{(1-\gamma)z + \gamma\left(F^k f(z)\right)} \right) > \alpha \quad (5)$$

where $k \in N_0 = N \cup \{0\}$, $0 \le \gamma \le 1$, $0 \le \alpha < 1$, $z \in \Omega$, and $F^k f(z)$ defined *by* (4)

Let $M_{\overline{H}}(K, \alpha, \gamma)$ be the subclass of $M_H(K, \alpha, \gamma)$, where $M_{\overline{H}}(K, \alpha, \gamma) = \mathbb{S}_{\overline{H}} \cap M_H(K, \alpha, \gamma)$.

Remark 1. The class $M_{\overline{H}}(K, \alpha, \gamma)$ reduces to the class $B_{\overline{H}}(K, \alpha)$[9] where $\gamma = 1$.

Then , we give enough status for f in $M_H(K, \alpha, \gamma)$,

Theorem (1) : Let $f(z) = h(z) + \overline{g(z)}$ wherever h(z) and g(z) were define by (1) . If

$$\sum_{w=2}^{\infty} \varphi[w, k, \alpha, Y] |a_w| + \sum_{w=1}^{\infty} \wp[w, k, \alpha, Y] |b_w| \le 1, \quad (6)$$

wherever

$\varphi(w, k, \alpha, Y) = \frac{(|w-k|(w) - \alpha\gamma) C_{wk}}{1-\alpha}$

$\wp(w, k, \alpha, Y) = \frac{(|w-k|(w) + \alpha\gamma) C_{wk}}{1-\alpha}$

$\left(k \in N_0 = N \cup \{0\}, 0 \le \gamma \le 1, 0 \le \alpha < 1, w \in \mathbb{N}\right)$.

Consequently, f(z) is harmonic, univalent and sense-preserving in Ω and $f(z) \in M_H(K, \alpha, \gamma)$.

Proof : We begin to prove that $f(z)$ is harmonic and univalent in Ω, let $z_1, z_2 \in \Omega$ for $|z_1| \le |z_2| < 1$. If we have $z_1 \ne z_2$, consequently

$$\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| \ge 1 - \left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| = 1 - \left|\frac{\sum_{w=1}^{\infty} |b_w| (z_1^w - z_2^w)}{(z_1 - z_2) - \sum_{w=1}^{\infty} |a_w| (z_1^w - z_2^w)}\right| \ge 1 - \left|\frac{\sum_{w=1}^{\infty} w|b_w|}{1 - \sum_{w=2}^{\infty} w|a_w|}\right| \ge$$

$$1 - \left|\frac{\sum_{w=1}^{\infty} \frac{(|w-k|\, w - \alpha\gamma)\, C_{wk}}{1-\alpha} |b_w|}{1 - \sum_{w=2}^{\infty} \frac{(|w-k|\, w + \alpha\gamma)\, C_{wk}}{1-\alpha} |a_w|}\right| \ge 0$$

hence f is an univalent function in Ω .

Observe that f is sense-preserving in Ω . This is because

$$|h'(z)| \ge 1 - \sum_{w=2}^{\infty} w |a_w| |z|^{w-1} > 1 - \sum_{w=2}^{\infty} w |a_w| \ge$$

$$1 - \sum_{w=2}^{\infty} \frac{(|w-k|\, w - \alpha\gamma)\, C_{wk}}{1-\alpha} |a_w|$$

$$\ge \sum_{w=1}^{\infty} \frac{(|w-k\,|\, w + \alpha\gamma)\, C_{wk}}{1-\alpha} |b_w| \ge$$

$$\sum_{w=1}^{\infty} w|b_w| \ge \sum_{w=1}^{\infty} w |b_w| |z|^{w-1} \ge |g'(z)| .$$

By the condition of Equation (5), we prove that if Equation (6) holds, consequently

$$\text{Re}\left(\frac{z\left(F^{k+1}f(z)\right)'}{(1-\gamma)z+\gamma\ (F^{k}f(z))}\right)=\ \text{Re}\ \left(\frac{A(z)}{B(z)}\right)\ \geq\alpha$$

Observe that

A(z) = $z\left(F^{k+1}f(z)\right)'$

and

B(z) = $(1-\gamma)z+\gamma\ (F^{k}f(z))$

Using Re (V) $>\ \alpha$ if and only if $|V-(1+\alpha)|\ \leq|V+(1-\alpha)|$, it is enough to show that $|A(z)-(1+\alpha)B(z)|-|A(z)+(1-\alpha)B(z)|\ \leq 0.$

We compensate for A(z) and B(z) in $|A(z)-(1+\alpha)B(z)|$, then we get

$|A(z)-(1+\alpha)B(z)|=$

$\left|z\left(F^{k+1}f(z)\right)'-(1+\alpha)[(1-\gamma)z+\gamma\,(F^{k}f(z))]\right|=$

$$\left|\begin{array}{c}\left[z+\sum_{w=2}^{\infty}wc_{w(k+1)}a_w z^w+(-1)^{k+1}\sum_{w=1}^{\infty}wc_{w(k+1)}b_w\bar{z}^w\right]\\ -(1+\alpha)\left[(1-\gamma)z+\gamma z+\gamma\sum_{w=2}^{\infty}c_{w(k)}a_w z^w+\gamma(-1)^{k+1}\sum_{w=1}^{\infty}wc_{w(k)}b_w\bar{z}^w\right]\end{array}\right| \quad (8)$$

$\leq\ \alpha|z|+\sum_{w=2}^{\infty}|[(\gamma(1+\alpha))-|w-k|\,w)]\ |C_{wk}|a_w||z|^w+\sum_{w=1}^{\infty}|[(\gamma(1+\alpha))+|w-k|\,w)]|\ C_{wk}|b_w||\bar{z}|^w$

Now, we compensate for A(z) and B(z) in $|A(z)+(1-\alpha)B(z)|$, then we get

$|A(z)+(1-\alpha)B(z)|=$

$\left|z\left(F^{k+1}f(z)\right)'+(1-\alpha)[(1-\gamma)z+\gamma\,(F^{k}f(z))]\right|=$

$$\left|\begin{array}{c}\left[z+\sum_{n=2}^{\infty}w\,c_{w(k+1)}a_w z^w+(-1)^{k+1}\sum_{w=1}^{\infty}wc_{w(k+1)}b_w\bar{z}^w\right]+\\ (1-\alpha)\left[(1-\gamma)z+\gamma z+\gamma\sum_{w=2}^{\infty}c_{w(k)}a_w z^w+\gamma(-1)^{k}\sum_{w=1}^{\infty}c_{w(k)}b_w\bar{z}^w\right]\end{array}\right| \quad (9)$$

$\geq(2-\alpha)|z|-\sum_{w=2}^{\infty}|[\gamma(\alpha-1)-|w-k|w]|\ C_{wk}|a_w||z|^w-\sum_{w=1}^{\infty}|[|w-k|\text{w}-\gamma(1-\alpha)]\ |C_{wk}|b_w||\bar{z}|^w$.

Then we compensate for Equations (8) and (9), and we get

$|A(z)-(1+\alpha)B(z)|$ - $|A(z)+(1-\alpha)B(z)|$

$=\ \alpha|z|+\sum_{w=2}^{\infty}|[(\gamma(1+\alpha))-|w-k|\,w)]\ |C_{wk}|a_w||z|^w$

$+\sum_{w=1}^{\infty}|[(\gamma(1+\alpha))+|w-k|\,w)]|\ C_{wk}|b_w|\,|\bar{z}|^w$

$+(\alpha-2)|z|+\sum_{w=2}^{\infty}|[\gamma(\alpha-1)-|w-k|w]|\ C_{wk}|a_w||z|^w$

$+\ \sum_{w=1}^{\infty}|[|w-k|\text{w}-\gamma(1-\alpha)]\ |C_{wk}|b_w||\bar{z}|^w$

$=\ 2\ \sum_{w=2}^{\infty}|\,[|w-k|w-\alpha\gamma]\ \ |C_{wk}|a_w|+2\sum_{w=1}^{\infty}\left|\left[|w-k|w+\alpha\gamma\right]\right.\ |C_{wk}|b_w|\ -2\left(1-\alpha\right)\leq 0$

then we get

$$\sum_{n=2}^{\infty}|\,[|w-k|w-\alpha\gamma]\ \ |C_{wk}|a_w|+\sum_{w=1}^{\infty}\left|\left[|w-k|w+\alpha\gamma\right]\right.\ |C_{wk}\,|b_w|$$

$\leq\left(1-\alpha\right)$.

This completes the proof of Theorem 1.

The function are harmonic univalent

$$f(z)=\ z\ +\ \sum_{w=2}^{\infty}\frac{1}{\varphi(w,k,\alpha,\Upsilon)}\ X_w\ z^w\ +\ \sum_{w=1}^{\infty}\frac{1}{\wp\,(w,k,\alpha,\Upsilon)}\overline{Y_w z^w}, \quad (10)$$

wherever $k\in N_0$ and $\sum_{w=2}^{\infty}|X_w|+\sum_{w=1}^{\infty}|Y_w|=1$, indicating that the coefficient bound define by (6) is true. Because

$\sum_{w=2}^{\infty}\varphi(w,k,\alpha,\Upsilon)|a_w|+\sum_{w=1}^{\infty}\wp\,(w,k,\alpha,\Upsilon)|b_w|\ \leq 1$,

$\sum_{w=2}^{\infty}\varphi\,(w,k,\alpha,\Upsilon)\ \frac{1}{\varphi(w,k,\alpha,\Upsilon)}\ |X_w|+\sum_{w=1}^{\infty}\wp\,(w,k,\alpha,\Upsilon)\ \frac{1}{\wp\,(w,k,\alpha,\Upsilon)}|Y_w|\ =$

$=\ \sum_{w=2}^{\infty}|X_w|+\sum_{w=1}^{\infty}|Y_w|=1$.

Here, we need to show that the condition of (6) is as well necessary for functions $f_k=h+\bar{g}_k$, wherever h and g_w are define by (6).

Theorem (2). Let $f_k=h+\bar{g}_k$ be given by (6). Consequently, $f_k(z)\in M_{\overline{H}}$ (K, α , γ) if and only if the coefficient in condition of (6) holds .

Proof:- We want to prove the "only if " part of the theorem since $M_{\overline{H}}(\mathrm{K},\alpha,\gamma) \subset M_H(\mathrm{K},\alpha,\gamma)$. Consequently, by (5), we get

$$\mathrm{Re}\left(\frac{z\left(F^{k+1} f(z)\right)'}{(1-\gamma) z+\gamma\ (F^k f(z))}\right) > \alpha$$

Or, equally

$$\mathrm{Re}\left[\frac{\begin{array}{c}[z+\sum_{w=2}^{\infty} w\ c_{w\ (k+1)}\, a_w\, z^w+(-1)^{2k+1} \sum_{w=1}^{\infty} w\, c_{w(k+1)} b_w\, \bar{z}^w]\\ -\alpha\,[(1-\gamma)z+\gamma z+\gamma \sum_{w=2}^{\infty} c_{w(k)} a_w\, z^w+\gamma(-1)^{2k} \sum_{w=1}^{\infty} c_{w(k)} b_w\, \bar{z}^w]\end{array}}{[(1-\gamma)z+\gamma z+\gamma \sum_{w=2}^{\infty} c_{w(k)} a_w\, z^w+\gamma(-1)^{2k} \sum_{w=1}^{\infty} c_{w(k)} b_w\ \bar{z}^w]}\right] \geq 0 \qquad (11)$$

We note that the above-required of (11) must be for each z in Ω. If we choose z be real and $z \to 1^-$,then we have

$$\frac{(1-\alpha)-\sum_{w=2}^{\infty} (|w-k|w-\alpha\gamma) C_{wk}|a_w|+\sum_{w=1}^{\infty}(|w-k|w+\alpha\gamma)\, C_{wk}|b_w|}{1-\gamma\sum_{w=2}^{\infty} C_{wk}|a_w|\, z^{w-1}+\gamma\ \sum_{w=1}^{\infty} c_{wk}|b_w|\, z^{w-1}} \geq 0 \qquad (12)$$

Whether the condition (6) does not hold then the numerator in (1.2) is negative for r sufficiently close to 1. Thus, there exists $z_0 = r_0$ in (0,1) for which the quotient in Equation (12) is negative, and this contrasts with $f_k \in M_{\overline{H}}$ (K, α, γ).

2.2 Extreme points

Herein, we determine the extreme points of the closed convex hull of $M_{\overline{H}}(\mathrm{K},\alpha,\gamma)$ given by class $M_{\overline{H}}(\mathrm{K},\alpha,\gamma)$.

Theorem (3). Suppose that f_k is defined by (1.2). Consequently, $f_k \in M$ (K, α,γ) if and only if $f_k(z) = \sum_{w=1}^{\infty} (X_w\, h_w(z) + Y_w\, g_{k_w}(z))$.

wherever

$h_1(z) = z,\ h_w(Z) = z - \left(\frac{1}{\varphi(w,k,\alpha,\mathrm{Y})}\right) z^n$, w = 2,3…..

$g_{k_w}(z) = = z + (-1)^k \left(\frac{1}{\wp(w,k,\alpha,\mathrm{Y})}\right) \bar{z}^w$, w = 1,2,…..

$X_w \geq 0\ Y_w \geq 0$ and $X_w = 1 - \sum_{w=1}^{\infty} (X_w + Y_w) \geq 0$

In particular, the extreme points of $M_{\overline{H}}$ (K, α,γ) are $\{h_w\}$, consequently $\{g_{k_w}\}$.

Proof. Let

$f_k(z) = \sum_{w=1}^{\infty} (X_w\, h_w + Y_w\, g_{k_w})$

$= \sum_{w=1}^{\infty} [X_w\, h_w + Y_w\, g_{k_w}]\ z - \sum_{w=2}^{\infty} \frac{1}{\varphi(w,k,\alpha,\mathrm{Y})} X_w z^w + (-1)^k \sum_{w=1}^{\infty} \frac{1}{\wp(w,k,\alpha,\mathrm{Y})} \overline{Y_w z^w}$,

$= z - \sum_{n=2}^{\infty} \frac{1}{\varphi(w,k,\alpha,\mathrm{Y})} X_w z^w + (-1)^{k-1} \sum_{w=1}^{\infty} \frac{1}{\wp(w,k,\alpha,\mathrm{Y})} \overline{Y_w z^w}$

Therefore

$\sum_{w=2}^{\infty} \varphi(w,k,\alpha,\mathrm{Y})|a_w| + \sum_{w=1}^{\infty} \wp(w,k,\alpha,\mathrm{Y})|b_w|$

$= \sum_{w=2}^{\infty} \varphi(w,k,\alpha,\mathrm{Y}) \left(\frac{1}{\phi(w,k,\alpha,\mathrm{Y})} X_w\right) + \sum_{w=1}^{\infty} \wp(w,k,\alpha,\mathrm{Y}) \left(\frac{1}{\wp(w,k,\alpha,\mathrm{Y})} Y_w\right) =$

$= \sum_{w=2}^{\infty} |X_w| + \sum_{w=1}^{\infty} |Y_w| = 1 - X_1 \leq 1$.

Hence $f_k \in M_{\overline{H}}$ (K, α,γ) ,

Conversely, whether $f_k \in A_{\overline{H}}$ (K, α,γ),thus

Set $X_w = \varphi(w,k,\alpha,\mathrm{Y})|a_w|$, $(w = 2,3,\ldots)$ and $Y_w = \wp(w,k,\alpha,\mathrm{Y})$ $(w = 1,2,\ldots)$

$X_1 = 1 - \sum_{w=2}^{\infty} X_w - \sum_{w=1}^{\infty} Y_w$.

The required representation is obtained as

$f_k(z) = z - \sum_{n=2}^{\infty}|a_w|\, z^w + (-1)^k \sum_{w=1}^{\infty}|b_w|\, \bar{z}^w$

$= z - \sum_{w=2}^{\infty} \frac{1}{\varphi(w,k,\alpha,\mathrm{Y})} X_w z^w + (-1)^{k-1} \sum_{w=1}^{\infty} \frac{1}{\wp(w,k,\alpha,\mathrm{Y})} \overline{Y_w z^w}$

$= z - \sum_{w=2}^{\infty}[z - h_w(z)]\, X_w + \sum_{w=1}^{\infty}[z - g_{k_w}(z)] Y_w$

$= [1 - \sum_{w=2}^{\infty} X_w - \sum_{w=1}^{\infty} Y_w]\, z + \sum_{w=2}^{\infty} h_w(z)]\, X_w + \sum_{w=1}^{\infty} g_{k_w}(z) Y_w$

$$= \sum_{w=2}^{\infty} X_w h_w + \sum_{w=1}^{\infty} Y_w g_{k_w}$$

2.3 Convex Combination

Now, we want to show that the class $M_{\overline{H}}(\mathrm{K}, \alpha, \gamma)$ is closed under the convex combination of its members. Suppose that the function $f_{\mathrm{k,i}}$ (z) is given, where i = 1 , 2 , … , m, by

$f_{k,i}(z) = z - \sum_{w=2}^{\infty} |a_{w,i}| z^w + (-1)^k \sum_{w=1}^{\infty} |b_{w,i}| \overline{z}^w$ (13)

Theorem (4): Suppose that $f_{k,i}(z)$ is given by (13) in $M_{\overline{H}}(\mathrm{K}, \alpha, \gamma)$, for each i = 1,2,…,m. Therefore c_i(z) is given by

$c_i(\mathrm{z}) = \sum_{i=1}^{\infty} t_i f_{k,i}(z).\ 0 \le t_i \le 1$

and in $M_{\overline{H}}(\mathrm{K}, \alpha, \gamma)$, wherever $\sum_{i=1}^{\infty} t_i = 1.$

Proof: By the definition of c_i(z), we get

$c_i(\mathrm{z}) = z - \sum_{w=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{w,i}| \right) z^n + (-1)^k \sum_{w=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{w,i}| \right) \overline{z}^w$

Furthermore, because $f_{k,i}(z)$ in $M_{\overline{H}}(\mathrm{K}, \alpha, \gamma)$, per i = 1,2,…, m, then by Theorem 2, we get

$\sum_{k=2}^{\infty} \varphi(w,k,\alpha,\Upsilon) \left(\sum_{i=1}^{\infty} t_i |a_{n,i}| \right) + \sum_{k=1}^{\infty} \wp(w,k,\alpha,\Upsilon) \left(\sum_{i=1}^{\infty} t_i |b_{w,i}| \right)$

$\sum_{i=1}^{\infty} t_i \left(\sum_{k=2}^{\infty} \varphi(w,k,\alpha,\Upsilon) |a_{w,i}| + \sum_{k=1}^{\infty} \wp(w,k,\alpha,\Upsilon) |b_{w,i}| \right) \le \sum_{i=1}^{\infty} t_i = 1.$

This completes the proof of the theorem 4.

2.4: Convolution (Hadamard product) Property

Herein, we need to prove that $M_{\overline{H}}(\mathrm{K}, \alpha, \gamma)$ is closed under convolution (Hadamard product) property. The involution of two harmonic functions

$$f_k(z) = z - \sum_{w=2}^{\infty} |a_w| z^w + (-1)^k \sum_{w=1}^{\infty} |b_w| \overline{z}^w . \quad (14)$$

and

$$\theta_w(z) = z - \sum_{w=2}^{\infty} |L_w| z^w + (-1)^k \sum_{w=1}^{\infty} |A_w| \overline{z}^w \quad (15)$$

is given as

$$(f_w * \theta_w)(Z) = f_w(z) * \theta_w(z) = z - \sum_{w=2}^{\infty} |a_w L_w| z^w + (-1)^k \sum_{w=1}^{\infty} |b_w A_w| \overline{z}^w \quad (16)$$

By (12) - (14), we prove the following theorem

Theorem (5). Let $f(z) \in M_{\overline{H}}(\mathrm{K}, \alpha, \gamma)$ and $\theta_w \in M_{\overline{H}}(\mathrm{K}, \mu, \gamma)$. Then

$f_w * \theta_w \in M_{\overline{H}}(\mathrm{K}, \alpha, \gamma) \subset M_{\overline{H}}(\mathrm{K}, \mu, \gamma)$. Where $0 \le \mu \le \alpha < 1, k \in N_0 = N \cup \{0\}$.

Proof: Suppose that

$f_k(z) = z - \sum_{w=2}^{\infty} |a_w| z^w + (-1)^k \sum_{w=1}^{\infty} |b_w| \overline{z}^w$

be in the class $M_{\overline{H}}(\mathrm{K}, \alpha, \gamma)$ and

$\theta_w(z) = z - \sum_{w=2}^{\infty} |L_w| z^w + (-1)^k \sum_{w=1}^{\infty} |A_w| \overline{z}^w$

be in $M_{\overline{H}}(\mathrm{K}, \mu, \gamma)$.

Therefore, the convolution $f_w * \theta_w$ is defined by (16). We need to prove that the coefficients of $f_w * \theta_w$ satisfy the condition of Theorem 1.

For $\theta_w \in M_{\overline{H}}(\mathrm{K}, \mu, \gamma)$, we note that $|L_w| < 1$ *and* $|A_w| < 1$. Now consider the convolution functions $f_w * \theta_w$ as follows

$\sum_{k=2}^{\infty} \varphi(w,k,\alpha,\Upsilon) |a_w||L_w| + \sum_{k=1}^{\infty} \wp(w,k,\alpha,\Upsilon) |b_w| |A_w|,$

$\le \sum_{k=2}^{\infty} \varphi(w,k,\alpha,\Upsilon) |a_w| + \sum_{k=1}^{\infty} \wp(w,k,\alpha,\Upsilon) |b_w| \le 1,$

Because $0 \le \mu \le \alpha < 1$, and $f_w \in M_{\overline{H}}(\mathrm{K}, \alpha, \gamma)$ therefore $f_w * \theta_w \in M_{\overline{H}}(\mathrm{K}, \alpha, \gamma) \subset M_{\overline{H}}(\mathrm{K}, \mu, \gamma)$.

2.5 Integral Operator

Herein , we check the closure quality of the class $A_{\overline{H}}(\mathrm{K}, \alpha, \gamma)$ by circular Bernardi-Libera-Livingston integral $\mathrm{T}_u(f)$ [10, 11] that is given by

$$\mathrm{T}_u(f) = \frac{u+1}{z^u} \int_0^2 t^{u-1} f(t) dt, u > -1 \quad (17)$$

Theorem 6. Suppose that $f_k \in M_{\overline{H}}(\mathrm{K}, \alpha, \gamma)$. Therefore

$\mathrm{T}_u(f_k(z)) \in M_{\overline{H}}(\mathrm{K}, \alpha, \gamma)$

Proof. By the definition of $\mathrm{T}_u(f_k(z))$ defined by (17) , as follows :

$\mathrm{T}_u(f_k(z)) = \frac{u+1}{z^u} \int_0^2 t^{u-1} \left(t - \sum_{w=2}^{\infty} |a_w| t^w + (-1)^k \sum_{w=1}^{\infty} |b_w| \overline{t}^w \right) dt,$

$= z - \sum_{w=2}^{\infty} \frac{u+1}{u+w} |a_w| z^w + (-1)^k \sum_{w=1}^{\infty} \frac{u+1}{u+w} |b_w| \overline{z}^w$

$= z - \sum_{n=2}^{\infty} D_w z^w + (-1)^k \sum_{w=1}^{\infty} L_w \overline{z}^w$

Wherever
$D_w = \frac{u+1}{u+w}|a_w|$ and $L_w = \frac{u+1}{u+w}|b_w|$
Therefore
$\sum_{k=2}^{\infty} \varphi(w,k,\alpha,\Upsilon)\frac{u+1}{u+w}|a_w| + \sum_{k=1}^{\infty} \wp(w,k,\alpha,\Upsilon)\frac{u+1}{u+w}|b_w|$,

$$\leq \sum_{w=2}^{\infty} \varphi(w,k,\alpha,\Upsilon)|a_w| + \sum_{w=1}^{\infty} \wp(w,k,\alpha,\Upsilon)|b_w| \leq 1,$$

from Theorem 2.
Hence, we have $T_u(f_k(z)) \in M_{\overline{H}}(K,\alpha,\gamma)$.

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