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On a Generalization of Quasi-Paranormal Operator Imposed by Norm Inequality

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Abstract

In Functional Analysis (FA), Operator Theory (OT) is an attractive aspect for the persistent refinement of scientific research. The study is conducted on the employment of norm inequality in the context of Hilbert space to evaluate the generalized quasi-paranormal operator. In this effort, a novel class of (M,n) −quasi-paranormal operators is introduced. Additionally, various analytic and algebraic features of the central outcome, which involves an operator formulated by the inequality principle, are investigated.

Keywords: Hilbert space, paranormal operator, quasi-paranormal operator.

حول تعميم الؤثر شبه فوق السوي باستخدام المتراجحة المعيارية

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الخلاصه

تعد نظرية المؤثرات(OT) عامل مهم في في تطوير البحث العلمي ضمن حقل التحليل الدالي (FA) ، هذه الدراسة تبنى على استخدام دالة المعيار في فضاءات هلبرت لتعميم المؤثر شبه فوق السوي. تقدم هذه الورقة فئة جديدة من المؤثرات شب فوق السوية (M,n) − بالاضافة الى ذلك ينتج البحث العديد من الصفات التحليلية والجبرية كنتيجة اساسية, والتي تتضمن مؤثرا مصاغا وفقا لمبدأ المتراجحات.

1. Introduction

Functional Analysis (FA) is a crucial realm of mathematics focused on vector spaces and linear operators define on these vector spaces. The term "functional" was first coined by the scientist Volterra in 1887, [1]. Integration and differentiation are instances of functionals since they relate one function to another. The most important scientists are Banach, Erwin, Hilbert, Riesz, and others introduced study in the field of functional analysis. It combines numerous diverse areas of mathematics, such as complex analysis and differential equations, and others, see [2-3] An interesting research theme in functional analysis (FA) is to determine characterizations of operators to ensure analytic and algebraic presentations acting on a complex Hilbert space. These involve stipulations in terms of norm or operator inequalities.

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Indeed, Operator Theory (OT) is an attractive area for the development of current research in FA, as substantial interest in OT is on linear and bounded operators in a complex Hilbert space. Operators are posed to investigate their diverse analytical and algebraic advantages. The principle of norm or operator inequalities are effectively employed in developing the structure of new and generalized operators. To view some of the scientific contributions [4-11]. The renowned operator is the paranormality, which played a key role in FA. Meanwhile, the Paranormality operators are a generalized formula of normality and hyponormality operators. Historically, in 1966, Istratescu et al., [12] form first contributed to the formulation of the paranormal class of operators, which was named the class (N). Later, Furuta [13] in 1967 renamed the class (N) to the paranormal class, since then, researchers have paid greater attention to the demeanors and generalizations of paranormal operators. In 1980, Arora and Kumar [14] presented a new formula for generalized paranormal operators, namely M −paranormal operators. They studied some attributes of M −paranormal operators and their relation to paranormal operators. In 1994, Fujii et al., [15] considered a class of operators involving p −paranormality based on norm inequality. Later, in 1998, Furuta et al., [16] investigated a class of operators including absolute−k −paranormality. Afterwards, in 2014, Gao and Li [17] considered a class of k −quasi-paranormality operators and described several of their analytical and algebraic attributes. Subsequently, in 2019, Gunawan et al., [18] provided the expansion of paranormal operators to extend formulas and examined their essential attributes, involving algebra addition, and multiplication. In addition, they yielded the relationship between the different descriptions of operators. After three years, in 2022, Guesba and Mahmoud [19] explored a class of operators named the k-quasi-A-Paranormality on the semi-Hilbertian space. They discussed several attributes concerning characterization, the A-spectral radius, and the approximate spectrum. In 2023, Mohsen [20] posed an operator, called $(k,m)-n$ −paranormal operators, and investigated some related attributes of this operator. In 2024, Hamiti and Makolli [21] presented and examined a remarkable non-normal class of operators, termed (M,k) −quasi paranormal operators. They studied and acquired several interesting attributes, which involve spectral properties and the matrix representation of (M,k) −quasi paranormal operators. Subsequently, numerous delightful studies were conducted on the generalized Paranormality formula and participated in those works [22-25]. Following that, several notable investigations explored the generalized paranormality formula, with active participation from multiple scholars [26-30].

Continuing this approach to study, this paper presents a new extended paranormal operator called the (M,n) −quasi-paranormal operator, where n and m are any positive integers. Then, a characteristic of this operator is discussed. The proposed various analytic and algebraic attributes are based on the employment of the characteristic of the considered operator.

2- The paranormal method

This section reviews and assembles the essential concepts underlying the paranormal type operators in the Hilbert field. Let H be the Hilbert space.

Definition 2.1. [13] Let $\varphi: H \to H$ be a linear operator if $\|\varphi_x\|^2 \leq \|\varphi_x{}^2\|$ is named paranormal operator for every unit vector $x \in H$.

Recalling the generalization of a paranormal operator in the following definition.

Definitions 2.2.

i) $\varphi: H \to H$ is named M-paranormal operator if $\|\varphi_x\|^2 \leq M\|\varphi_x{}^2\|$for any $x \in H$, [14].

ii) $\varphi: H \to H$ is named M*-paranormal operator if $\|\varphi_x{}^*\|^2 \leq M\left\|\varphi^2{}_x\right\|$ for any $x \in H$, [14].

iii) $\varphi: H \to H$ is named K-paranormal operator if $\|\varphi_x\|^{k+1} \leq \left\|\varphi^{k+1}{}_x\right\|\|x\|^n$, $\forall x \in H, k \geq 1$, [21].

iv) $\varphi: H \to H$ is named an (m,n)-paranormal operator if $\|\varphi_x\|^{n+1} \leq m\left\|\varphi^{n+1}{}_x\right\|\|x\|^n$, $\forall x \in H$,[23].

v) $\varphi: H \to H$ is named a (m,n)*-paranormal operator if $\left\|\varphi^*{}_x\right\|^{n+1} \leq m\left\|\varphi^{n+1}{}_x\right\|\|x\|^n$, $\forall x \in H$, [23].

vi) $\varphi: H \to H$ is named *-r-paranormal operator if $\left\|\varphi^*{}_x\right\|^{r+1} \leq \left\|\varphi^{r+1}{}_x\right\|\|x\|^r$, $\forall x \in H$, [22].

vii) $\varphi: H \to H$ is named quasi-paranormal operator if $\left\|\varphi^2{}_x\right\|^2 \leq \left\|\varphi^3{}_x\right\|\|\varphi_x\|$, $\forall x \in H$, [20].

viii) $\varphi: H \to H$ is named quasi-*-n-paranormal operator if satisfy $\|\varphi^*\varphi_x\|^{n+1} \leq \left\|\varphi^{n+2}{}_x\right\|\|\varphi_x\|^n$, $\forall x \in H$, [21].

3. Results

A new class of non-normal operators in terms of norm inequality, and for any positive integers n and M, namely (M, n)-quasi paranormal operators, have been introduced in this section. Moreover, the characteristic attributed for this proposed operator is studied, as well as several analytical and algebraic attributes.

Definition 3.1. Let $\varphi: H \to H$ be linear and bounded operator define on Hilbert space H, then φ is called (M,n) −quasi-paranormal operator if each positive integer n and M, satisfy $\left\|\varphi^2{}_x\right\|^{n+1} \leq M\left\|\varphi^{n+2}{}_x\right\|\|\varphi_x\|^n$, $\forall x \in H$.

The following remark presents the relationship between (M,n) −quasi-paranormal operators and various subclasses of paranormal operators:

Remarks 3.2.

i) If φ is an isometry operator, and $M = n = 1$, then φ stands as a paranormal operator.

ii) If φ is an isometry operator, with $n = 1$, so φ is considered type M −paranormal operator.

iii) If φ is an isometry and also a self-adjoint operator with $n = 1$, so φ is considered with the M*-paranormal operator.

v) If φ is an isometry operator and $M = 1$, then φ is considered type n-paranormal operator.

vi) If φ is an isometry and also self-adjoint operator with $M = n = 1$, so φ is *-paranormal operator.

vii) If $n = M = 1$, then φ is considered a quasi-paranormal operator.

viii) If φ is a self-adjoint with $M = n = 1$, so φ is considered with the quasi-*-paranormal operator.

x) If φ is an isometry and also self-adjoint operator, then φ is an (m,n) −paranormal and (m,n) $*$ −paranormal operator correspondingly

Theorem 3.3. For any positive integers n and M the operator $\varphi: H \to H$ should be considered (M,n)-quasi-paranormal in the case that, $M^{\frac{2}{n+1}}\varphi^{*(n+2)}\varphi^{(n+2)} - ג^n(1+n)\varphi^{*2}\varphi^2 + M^{\frac{2}{n+1}} n\, ג^{n+1}\varphi^*\varphi \geq 0, \ldots 1$, for each $ג > 0$.

Proof: For any $x \in H$, and since φ is of type (M,n) −quasi-paranormal operator, it's so satisfy $\left\|\varphi^2{}_x\right\|^{n+1} \leq M\left\|\varphi^{n+2}{}_x\right\| \|\varphi_x\|^n$, which gives us $\left\|\varphi^2{}_x\right\|^2 \leq M^{\frac{2}{n+1}}\left\|\varphi^{n+1}{}_x\right\|^{\frac{2}{n+1}}\|\varphi_x\|^{\frac{2n}{n+1}}$ for any $x \in H$. Applying the generalized arithmetic-geometric mean inequality, [17] allows us to advance the discussion

$$\frac{ג^{-n}M^{\frac{2}{n+1}}}{n+1}\left\|\varphi^{n+2}{}_x\right\|^2 + \frac{n ג M^{\frac{2}{n+1}}}{n+1}\|\varphi_x\|^2 \geq M^{\frac{2}{n+1}}\left(ג^{-n}\left\|\varphi^{n+2}{}_x\right\|^2\right)^{\frac{1}{n+1}}(ג\|\varphi_x\|^2)^{\frac{n}{n+1}}$$

$$=M^{\frac{2}{n+1}}\left(\left\|\varphi^{n+2}{}_x\right\|^2\right)^{\frac{1}{n+1}}(\|\varphi_x\|^2)^{\frac{n}{n+1}} \geq \left\|\varphi^2{}_x\right\|^2.$$

It follows that: $M^{\frac{2}{n+1}}\varphi^{*n+2}\varphi^{n+2} - (n+1)\lambda^n\varphi^{*2}\varphi^2 + M^{\frac{2}{n+1}}n\lambda^{n+1}\varphi^*\varphi \geq 0.$ ... (1)

By assuming, the Equation (1) is satisfying. So, in case $x \in H$ with $\left\|\varphi^{n+2}{}_x\right\| = 0$, then when product via λ^{-n} by letting $\lambda \to 0$, we have, $M\lambda^{-n}\varphi^{*n+2}\varphi^{n+2} - (1+n)\varphi^{*2}\varphi^2 + Mn\lambda\varphi^*\varphi$ and $\lambda \to 0$. Which is leading to $\left\|\varphi^2{}_x\right\|^{n+1} = 0$, and for that reason $\left\|\varphi^2{}_x\right\|^{n+1} \leq \left\|\varphi^{n+2}{}_x\right\|\|\varphi_x\|^n$. If $x \in H$ with $\left\|\varphi^{n+2}{}_x\right\| > 0$ and by letting $\lambda = (\frac{\|\varphi^{n+2}{}_x\|}{\|\varphi_x\|})^{\frac{2}{n+1}}$ in Equation (1), we get $\left\|\varphi^2{}_x\right\|^{n+1} \leq \left\|\varphi^{n+2}{}_x\right\|\|\varphi_x\|^n$. Therefore, φ is an (M,n) −quasi-paranormal.

Proposition 3.4. Assume $\varphi: H \to H$ is an (M,n) −quasi-paranormal operator, then

i) $\varphi_{|M}$ is an operator type (M,n) −quasi-paranormal.

ii) $\beta\varphi$ is an operator type (M,n) −quasi-paranormal, where $\beta \in F$.

iii) φ^k is an operator type (M,n) −quasi-paranormal.

Proof:

i) Let M be a closed invariant subspace of H, and suppose that z belongs to M. $\left\|\left(\varphi_{|M}\right)^2{}_z\right\|^{n+1} = \left\|\varphi^2{}_z\right\|^{n+1} \leq M\left\|\varphi^{n+2}{}_z\right\|\|\varphi_z\|^n$, consequently, $\varphi_{|M}$ is an (M,n) −quasi-paranormal.

ii) Let $\beta \neq 0, n, M \geq 1$,

$$M^{\frac{2}{n+1}}(\beta\varphi)^{*n+2}(\beta\varphi)^{n+2} - (n+1)\lambda^n(\beta\varphi)^{*2}(\beta\varphi)^2 + M^{\frac{2}{n+2}}n\lambda^{n+1}(\beta\varphi)^*(\beta\varphi)$$
$$= M^{\frac{2}{n+1}}(|\beta|^2)\varphi^{*n+2}\varphi^{n+2} - (|\beta|^2)^2(n+1)\lambda^n\varphi^{*2}\varphi^2 + (|\beta|^2)M^{\frac{2}{n+1}}n\lambda^{n+1}\varphi^*\varphi$$
$$= (|\beta|^2)^{n+2}\left(M^{\frac{2}{n+1}}\varphi^{*n+2}\varphi^{n+2} - (n+1)\left(\frac{\lambda}{|\beta|^2}\right)^n\varphi^{*2}\varphi^2 + M^{\frac{2}{n+1}}(\frac{\lambda}{|\beta|^2})^{n+1}\varphi^*\varphi\right) \geq 0.$$

Therefore, $\beta\varphi$ is an (M,n) −quasi-paranormal.

iii) By applying method of induction, we show that φ^{k} is an (M,n) −quasi-paranormal. Thus, we obtain the following conclusion:

1- The result is true for $k = 1$, that is $\left\|\varphi^2{}_x\right\|^{n+1} \leq \left\|\varphi^{n+2}{}_x\right\|\|\varphi_x\|^n$.

2- Assume the result is valid for a fixed integer $k = m$,

$\left(\left\|\varphi^2{}_x\right\|^{n+1}\right)^m \leq \left(\left\|\varphi^{n+2}{}_x\right\|\|\varphi_x\|^n\right)^m$.

3-Proving for $k = m+1$, that

$$\left(\left\|\varphi^2{}_x\right\|^{n+1}\right)^{m+1} = \left(\left\|\varphi^2{}_x\right\|^{n+1}\right)^m \left\|\varphi^2{}_x\right\|^{n+1}$$
$$\leq \left(\left\|\varphi^{n+2}{}_x\right\|\|\varphi_x\|^n\right)^m\left(\left\|\varphi^{n+1}{}_x\right\|\|\varphi_x\|^n\right)$$

by (1) and (2)

$$= \left(\left\|\varphi^{n+1}{}_x\right\|\|\varphi_x\|^n\right)^{m+1}.$$

Then,

$\left(\left\|\varphi^2{}_x\right\|^{n+1}\right)^{m+1} \leq \left(\left\|\varphi^{n+2}{}_x\right\|\|\varphi_x\|^n\right)^{m+1}$. Hence, φ^k is an (M,n)-quasi-paranormal.

Remark 3.5. The following example is offered that φ, S be two operators type (M,n) −quasi-paranormal operators, but $\varphi + S$ is not necessarily considered as an (M,n) −quasi-paranormal operator.

Example 3.6. Let $\varphi = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$, then by Theorem 3.3, φ and S are (1,1)-quasi-paranormal operator, for all $\lambda > 0$. Observe that $\varphi^{*3}\varphi^3 - 2\lambda\varphi^{*2}\varphi^2 + \lambda^2\varphi^*\varphi = \begin{pmatrix} 64 - 32\lambda + 4\lambda^2 & 0 \\ 0 & 64 - 32\lambda + 4\lambda^2 \end{pmatrix}$, be zero or positive for each $\lambda > 0$, accordingly, φ is type (1,1)-quasi-paranormal operator. As $S^{*3}S^3 - 2\lambda S^{*2}S^2 + \lambda^2 S^*S = \begin{pmatrix} \lambda^2 & 0 \\ 0 & 0 \end{pmatrix}$, which is positive for each $\lambda > 0$. This shows that S satisfies the condition of being a (1,1)-quasi-paranormal operator. But, consider the operator $h = \varphi + S = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$, in that case:

$h^{*3}h^3 - 2\lambda h^{*2}h^2 + \lambda^2 h^*h = \begin{pmatrix} 4 - 8\lambda + \lambda^2 & 0 \\ 0 & 16 - 8\lambda + 4\lambda^2 \end{pmatrix}$, since the expression is not positive when $\lambda = 1$, it follows that h fails to be a (1,1)-quasi-paranormal.

Now, Theorem 3.7 presents the condition which make Remark 3.5 true.

Theorem 3.7. Let $\varphi, S: H \to H$ are (M,n) −quasi-paranormal operators in domain of a Hilbert space H and S to be a self-adjoint. If $\varphi^*S = S^*\varphi = S^2 = 0$, then $\varphi + S$ is (M,n)-quasi-paranormal operator.

Proof: By Theorem 3.3 we have:

$$M^{\frac{2}{n+1}}(\varphi+S)^{*n+2}(\varphi+S)^{n+2} - (n+1)\lambda\lambda^n(\varphi+S)^{*2}(\varphi+S)^2 + M^{\frac{2}{n+1}}n\lambda^{n+1}(\varphi+S)^*(\varphi+S)$$
$$= M^{\frac{2}{n+1}}\big((\varphi^*+S^*)(\varphi+S)\big)^{n+2} - (n+1)\lambda^n\big((\varphi^*+S^*)(\varphi+S)\big)^2 + M^{\frac{2}{n+1}}(\varphi^*+S^*)(\varphi+S)$$
$$= M^{\frac{2}{n+1}}(\varphi^*\varphi + \varphi^*S + S^*\varphi + S^*S)^{n+2} - (n+1)\lambda^n(\varphi^*\varphi + \varphi^*S + S^*\varphi + S^*S)^2 + M^{\frac{2}{n+1}}n\lambda^{n+1}(\varphi^*\varphi + \varphi^*S + S^*\varphi + S^*S)$$
$$= M^{\frac{2}{n+1}}\varphi^{*n+2}\varphi^{n+2} - (n+1)\lambda^n\varphi^{*2}\varphi^2 + M^{\frac{2}{n+1}}n\lambda^{n+1}\varphi^*\varphi \geq 0.$$

Therefore, $\varphi + S$ is an (M,n) −quasi-paranormal.

Remark 3.8. If φ, S are two (M,n) −quasi-paranormal operators then φS is not necessarily be an (M,n) −quasi-paranormal. To illustrate that, we present the following example.

Example 3.9. Let $\varphi = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ then by means of Theorem 3.3, φ and S are (1,1)-quasi-paranormal operators, for each $\lambda > 0$. Now, consider the following:

$\varphi^{*3}\varphi^3 - 2\lambda\varphi^{*2}\varphi^2 + \lambda^2\varphi^*\varphi = \begin{pmatrix} 64 - 32\lambda + 4\lambda^2 & 0 \\ 0 & 1 - 2\lambda + \lambda^2 \end{pmatrix}$.

Which is zero or positive on each $\lambda > 0$, consequence to that, φ be a (1,1)-quasi-paranormal, and by Example 3.6, we get S to be a (1,1)-quasi-paranormal.

Next, consider the operator $h = \varphi S = \begin{pmatrix} 0 & 4 \\ 2 & 0 \end{pmatrix}$, for each $\lambda > 0$, and then consider the following:

$h^{*3}h^3 - 2\lambda h^{*2}h^2 + \lambda^2 h^*h = \begin{pmatrix} 256 - 128\lambda + 4\lambda^2 & 0 \\ 0 & 1024 - 128\lambda + 16\lambda^2 \end{pmatrix}$.

Which is zero or positive when $\lambda = 7$, therefore, $h = \varphi S$ is not a (1,1)-quasi-paranormal.

The next theorem presenting the condition that makes Remark 3.6 true.

Theorem 3.10. If φ is an operator of the type (M,n) −quasi-paranormal operator with S is isomorphic operator. Then, φS is an (M,n) −quasi-paranormal operator if it provided the following requirement conditions:
i) $\varphi S = S\varphi$.
ii) $\varphi S^* = S^*\varphi$.
Proof: We have $S^*S = I$. By using Theorem 3.3, we are able to get:
$M^{\frac{2}{n+1}}(\varphi S)^{*n+2}(\varphi S)^{n+2} - (1+n)ג^n(\varphi S)^{*2}(\varphi S)^2 + M^{\frac{2}{n+1}}nג^{n+1}(\varphi S)^*(\varphi S)$
$= M^{\frac{2}{n+1}}(S\varphi)^{*n+2}(\varphi S)^{n+2} - (1+n)ג^n(S\varphi)^{*2}(\varphi S)^2 + M^{\frac{2}{n+1}}nג^{n+1}(S\varphi)^*(\varphi S)$
$= M^{\frac{2}{n+1}}\big((\varphi^*S^*)(\varphi S)\big)^{n+2} - (1+n)ג^n((\varphi^*S^*)(\varphi S))^2 + M^{\frac{2}{n+1}}nג^{n+1}(\varphi^*S^*)(\varphi S)$
$= M^{\frac{2}{n+1}}(\varphi^*(\varphi S^*)S)^{n+2} - (1+n)ג^n(\varphi^*(\varphi S^*)S)^2 + M^{\frac{2}{n+1}}nג^{1+n}(\varphi^*(\varphi S^*)S)$
$= M^{\frac{2}{n+1}}(\varphi^*\varphi)^{n+2}(S^*S)^{n+2} - (1+n)ג^n(\varphi^*\varphi)^{n+2}(S^*S)^{n+2} + M^{\frac{2}{n+1}}nג^{1+n}(\varphi^*\varphi)(S^*S)$
$= M^{\frac{2}{n+1}}\varphi^{*n+2}\varphi^{n+2} - (1+n)ג^n\varphi^{*2}\varphi^2 + M^{\frac{2}{n+1}}nג^{n+1}\varphi^*\varphi \geq 0$. Consequently, the operator φS is of the type (M,n) −quasi-paranormal operator.

Remark 3.11. The adjoint of an (M,n) −quasi-paranormal operator is not necessarily be an (M,n) −quasi-paranormal, and this can be demonstrated through the following example.

Example 3.12. Let $\varphi = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ be a (1,1)-quasi-paranormal operator, by Theorem 3.3, and for each $ג > 0$, consider what follows:
$\varphi^{*3}\varphi^3 - 2ג\varphi^{*2}\varphi^2 + ג^2\varphi^*\varphi = \begin{pmatrix} 2ג^2 & 0 \\ 0 & 8 - 8ג + 2ג^2 \end{pmatrix}$.
Which is positive for each $ג > 0$, i.e., φ is a (1,1)-quasi-paranormal.
On the other hand, considering the operator $S = \varphi^* = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, by Theorem 3.3, for each $ג > 0$, and consider $S^{*3}S^3 - 2ג S^{*2}S^2 + ג^2 S^*S = \begin{pmatrix} 4 - 8ג + 2ג^2 & 4 \\ -4 & 4 + 2ג^2 \end{pmatrix}$.
Which is not positive. Therefore, φ^* is not a (1,1)-quasi-paranormal.

Theorem 3.13. Assuming that $\varphi: H \to H$ is an (M,n) −quasi-paranormal operator, it follows that both $\varphi \otimes I$ and $I \otimes \varphi$ inherit the (M,n) −quasi-paranormality.
Proof: By using Theorem 3.3, we get
$M^{\frac{2}{n+1}}(\varphi\otimes I)^{*n+2}(\varphi\otimes I)^{n+2} + M^{\frac{2}{n+1}}nג^{n+1}(\varphi\otimes I)^*(\varphi\otimes I)$
$= M^{\frac{2}{n+1}}\big(\varphi^{*n+2}\varphi^{n+2}\otimes I\big) + M^{\frac{2}{n+1}}nג^{1+n}(\varphi^*\varphi\otimes I)$
$= \Big(M^{\frac{2}{n+1}}\varphi^{*n+2}\varphi^{n+2} + M^{\frac{2}{n+1}}nג^{1+n}\varphi^*\varphi\Big)\otimes I$
$\geq (n+1)ג^n\big(\varphi^{*2}\varphi^2\big)\otimes I$
$= (n+1)ג^n\big(\varphi^{*2}\otimes I\big)(\varphi^2\otimes I)$
$= (n+1)ג^n(\varphi^*\otimes I)^2(\varphi\otimes I)^2$. Thus, we have
$M^{\frac{n}{n+1}}(\varphi\otimes I)^{*2}(\varphi\otimes I)^{n+2} - (n+1)ג^n(\varphi^*\otimes I)^2(\varphi\otimes I)^2 + M^{\frac{2}{n+1}}nג^{n+1}(\varphi\otimes I)^*(\varphi\otimes I) \geq 0$.
Therefore, $\varphi \otimes I$ is an (M,n) −quasi-paranormal. Similarly, we can show that $I \otimes \varphi$ is also an (M,n) −quasi-paranormal operator.

4. Conclusions

In terms of complex Hilbert space, determining the non-normal operator is vital in FA. This paper aims to formulate and analyze a new class of paranormal operators, namely (M,n)-quasi-

paranormal operator, driven by distributed order of the positive integers n and M. This approache enhance typical paranormal operator model by adding distributed orders positive integers n and M. Then, successfully investigated the characteristic attribute for this considered operator alongside some analytic and algebraic attributes

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