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Numerical Solution of Inverse Coefficient Problem Governing by Heat Equation under Mixed Boundary Conditions and General Integral Extra Data

Mohammed A.J. Al-Shatrah^{1*}, M.S. Hussein², Y.T. Mehraliyev³

¹Ministry of Education, Directorate of Education Thi-Qar, Thi-Qar, Iraq

²Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq

³Baku State University, Baku, Azerbaijan

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Abstract

This study investigates the numerical identification of time-dependent coefficients, numerically in a one-dimensional parabolic partial differential equation subject to nonlocal initial, mixed boundary, and overdetermination conditions of general integral type. The Crank-Nicolson finite difference method (FDM) was employed to solve the direct (forward) problem, whilst the inverse problem (IP) was reshaped as a nonlinear regularized least-square optimization problem and solved iteratively using the lsqnonlin routine from MATLAB's optimization toolbox, given the ill-posed nature of the IP, where small perturbations in input data may cause large deviations in the solution. Therefore, Tikhonov regularization (TR) was applied to ensure stability and improve the reliability of the numerical results.

Keywords: Inverse Problem (IP), Finite Difference Method (FDM), Tikhonov Regularization (TR), Crank-Nicolson (C-N) Scheme, Optimization.

الحل العددي لمسألة تحديد معامل عكسية تحكمها معادلة الحرارة تحت شروط حدية مختلفة ومعطيات تكاملية إضافية عامة

محمد عبد الخضر جابر^{1*}, محمد صباح حسين², ياشار ميهرالييف³

¹وزارة التربية، مديرية تربية ذي قار، ذي قار، العراق

²قسم الرياضيات، كلية العلوم، جامعة بغداد، بغداد، العراق

³جامعة باكو الحكومية، باكو، أذربيجان

الخلاصة

تبحث هذه الدراسة التعيين العددي للمعاملات المتغيرة زمنياً في معادلة تفاضلية جزئية قطع مكافئ أحادية البعد، وذلك تحت شرط ابتدائي، وشروط حدودية مختلفة، إضافة إلى شرط تعيين ذي صيغة تكاملية. ولحل

*Email: mmmsh@sc.uobaghdad.edu.iq



المسألة المباشرة استخدمت طريقة كرانك - نيكلسن للفروق المحدودة (FDM). أما المسألة العكسية (IP) فقد أعيدت صياغتها كمسألة تحسين غير خطية من نمط أقل المربعات مع تنظيم، وحلت بكفاءة باستخدام الروتين (lsqnonlin) من صندوق أدوات التحسين في (MATLAB). ونظرا لكون المسألة سيئة الوضع، حيث تؤدي الاضطرابات الصغيرة في بيانات الإدخال إلى انحرافات كبيرة في الحل، فقد طبق تنظيم تيخونوف (TR) لضمان الاستقرار وتعزيز موثوقية النتائج العددية.

1. Introduction

Partial differential equations (PDEs) are essential tools in describing physical phenomena that depend on several variables and their rates of change. They are widely used to model continuous processes in various disciplines, including physics, engineering, and biology [1]-[6]. Applications range from fluid dynamics and electromagnetism to materials science and financial systems. Due to the complexity of these equations, obtaining closed-form analytical solutions is often impractical or impossible, necessitating numerical approximations. Among numerical techniques for solving PDEs, the finite element method (FEM) is highlighted as a particularly powerful and general approach [7]. Numerous practical applications naturally lead to IPs, in which the objective is to infer unknown causes from observable outcomes. This stands in contrast to direct problems, which focus on predicting the system's response when all influencing coefficients are already specified. Solving IPs is often more difficult than solving their direct problems because of their ill-posed nature. Typically, an IP may lack a solution, have multiple solutions, or exhibit high sensitivity to small changes in input data [8]. In problems of mathematical physics, direct problems aim to determine accurate or approximate functions that characterize physical phenomena, including wave propagation, heat conduction, and electromagnetic fields. These problems generally assume that the properties of the medium and the initial or BCs are known.

Nevertheless, in many practical cases, the properties of the coefficient field are unknown, giving rise to IPs. These problems involve determining the coefficients of the governing equations using available information. Most of these problems are ill-posed. IP techniques are also valuable for identifying the geometry, location, and nature of hidden anomalies such as sources, defects, or inclusions [9]. Inverse boundary value problems find applications across a wide spectrum of disciplines, including geophysical exploration, seismic imaging, biomedical diagnostics, industrial quality assessment, and biological modeling, which makes them an active field of contemporary mathematics and physics [10]. A substantial amount of research has investigated IPs related to parabolic equations governed by nonlocal integral overdetermination conditions, as reported in [11],[12],[13], and for some methods are mesh-based methods such as the finite element method (FEM) [14]-[18], finite difference method (FDM) [19]-[27], while numerical approach associated with Inverse problems for (Elliptic, Hyperbolic and Parabolic) equations have been investigated by many researchers and monographs, in particular, to mention a few, [28]-[33]. This paper's inverse problem for a second-order parabolic equation has been shown by Elvin Azizbayov and Yashar Mehraliyev [34] to have a classical solution. Consequently, the major objective of the current task is a numerical realization of such a problem. This paper is divided into five sections. Section 2 includes a mathematical formulation of the IP under study. In Section 3, a numerical method for solving the forward problem using C-N FDM is presented. In section 4, the IPs mathematical solution is shown with an initial guess. Section 5 will discuss the numerical problem and explain the obtained results.

2. MATHEMATICAL FORMULATION

As a mathematical model, D_T considered to be a rectangular region that is defined by $D_T : \{0 \leq v \leq 1, 0 \leq \tau \leq T\}$ and T be a fixed positive number. The one-dimensional IP of determining a pair of unknown functions $\{w(v, \tau), a(\tau)\}$ for the upcoming parabolic equation,

$$c(\tau)w_\tau(v, \tau) = w_{vv}(v, \tau) + a(\tau)w(v, \tau) + f(v, \tau) \quad (v, \tau) \in D_T, \quad (1)$$

with the nonlocal initial condition (IC),

$$w(v, 0) + \delta w(v, T) + \int_0^T \hat{P}(\tau)w(v, \tau)d\tau = \zeta(v), \quad v \in [0, 1], \quad (2)$$

with boundary condition (BC),

$$w(0, \tau) = w_v(1, \tau) = 0, \quad \tau \in [0, T], \quad (3)$$

and the overdetermination condition

$$\int_0^1 H(v)w(v, \tau)dv = z(\tau), \quad \tau \in [0, T], \quad (4)$$

where, $\delta \geq 0$, is a fixed number, $c(\tau) > 0$, $f(v, \tau)$, $\hat{P}(\tau) \geq 0$, $\zeta(v)$, $H(v)$, and $z(\tau)$, are given functions, $w(v, \tau)$, $a(\tau)$ are the sought functions (v is the space component and τ is the time component).

2.1. Solvability of the inverse problem

Theorem 1 [31]: Let the assumptions $(\mu 1) - (\mu 3)$ and the condition

$$((E(T) + 2)G(T) + V(T))(E(T) + 2) < 1,$$

where,

$\mu 1) \zeta(v) \in C^2[0, 1]$, $\zeta^{(3)}(v) \in L_2(0, 1)$, $\zeta(0) = \zeta'(1) = \zeta''(0) = 0$;

$\mu 2) f(v, \tau), f_v(v, \tau), f_{vv}(v, \tau) \in C(D_T)$, $f_{vvv}(v, \tau) \in L_2(D_T)$,

$f(0, \tau) = f_v(1, \tau) = f_{vv}(0, \tau) = 0$, $(0 \leq \tau \leq T)$;

$\mu 3) \delta \geq 0$, $0 \leq \hat{P}(\tau) \in C[0, T]$, $H(v) \in C[0, 1]$, $z(\tau) \in C^1[0, T]$, $z(\tau) \neq 0$, $(0 \leq \tau \leq T)$.

$$E_1(T) = \sqrt{6} \|\zeta^{(3)}(v)\|_{L_2(0,1)} + \sqrt{6T} (1 + \delta) \left\| \frac{1}{c(\tau)} \right\|_{C[0,T]} \|f_{vvv}(v, \tau)\|_{L_2(D_T)}, \quad (5)$$

$$E_2(T) = \| [z(\tau)]^{-1} \|_{C[0,T]} \left\{ \left\| c(\tau)z'(\tau) - \int_0^1 H(v)f(v, \tau)dv \right\|_{C[0,T]} + \right. \\ \left. \|H(v)\|_{C[0,1]} \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \left[\|\zeta^{(3)}(v)\|_{L_2(0,1)} + (1 + \delta) \left\| \frac{1}{c(\tau)} \right\|_{C[0,T]} \sqrt{T} \|f_{vvv}(v, \tau)\|_{L_2(D_T)} \right] \right\}, \quad (6)$$

$$V_1(T) = \sqrt{6T} \|\hat{P}(\tau)\|_{C[0,T]}, \quad (7)$$

$$V_2(T) = \| [z(\tau)]^{-1} \|_{C[0,T]} \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \|H(v)\|_{C[0,1]} \|\hat{P}(\tau)\|_{C[0,T]} T, \quad (8)$$

$$G_1(T) = \sqrt{6}(1 + \delta) \left\| \frac{1}{c(\tau)} \right\|_{C[0,T]} T, \quad (9)$$

$$G_2(T) = \| [z(\tau)]^{-1} \|_{C[0,T]} \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \|H(v)\|_{C[0,1]} (1 + \delta) \left\| \frac{1}{c(\tau)} \right\|_{C[0,T]} T, \quad (10)$$

$$\lambda_k = \frac{\pi}{2}(2k - 1), \quad E(T) = E_1(T) + E_2(T), \quad V(T) = V_1(T) + V_2(T) \quad \text{and} \quad G(T) = G_1(T) + G_2(T),$$

holds, then IP (1) – (4), has a unique solution in the ball $K = K_R$, of the space X_T^3 .

3. FDM Scheme for Direct (Forward) Problem

The direct problem will be resolved in this section, i.e., when unknown $a(\tau)$ was assumed to be given. In order to solve this problem, the Crank-Nicolson (C-N) FDM scheme was used to find the solutions of the nonlocal problem given by the equation (1) – (3). Where the

domain D_T was divided into an $M \times N$ mesh with a spatial step size $\Delta v = \frac{1}{M}$ and the time step size $\Delta \tau = \frac{T}{N}$ where M and N are given positive integers. The grid points have been given by

$$v_i = i\Delta v, \quad i = \overline{0, M}. \quad (11)$$

$$\tau_j = j\Delta \tau, \quad j = \overline{0, N}. \quad (12)$$

These quantities' discretized form is given as follows,

$$w(v_i, \tau_j) =: w_{i,j}, \quad a(\tau_j) =: a_j, \quad c(\tau_j) =: c_j, \quad f(v_i, \tau_j) =: f_{i,j}, \quad \text{and} \quad \zeta(v_i) =: \zeta_i$$

Then, the C-N FDM, and the trapezoidal rule quadrature for integration, the equation (1), can be approximated as,

$$\frac{w_{i,j+1} - w_{i,j}}{\Delta \tau} = \frac{1}{2} \left(\tilde{d}_j \frac{w_{i-1,j} - 2w_{i,j} + w_{i+1,j}}{(\Delta v)^2} + \tilde{a}_j w_{i,j} + \tilde{\theta}_j + \tilde{d}_{j+1} \frac{w_{i-1,j+1} - 2w_{i,j+1} + w_{i+1,j+1}}{(\Delta v)^2} + \tilde{a}_j w_{i,j+1} + \tilde{\theta}_{j+1} \right), \quad (13)$$

$$\text{where, } \tilde{d}_j = \frac{1}{c_j}, \quad \tilde{a}_j = \frac{a_j}{c_j}, \quad \tilde{\theta}_j = \frac{f_{i,j}}{c_j}, \quad \tilde{d}_{j+1} = \frac{1}{c_{j+1}}, \quad \tilde{a}_{j+1} = \frac{a_{j+1}}{c_{j+1}}, \quad \text{and, } \tilde{\theta}_{j+1} = \frac{f_{i,j+1}}{c_{j+1}},$$

Simplifying the above equation (13), get the following difference equation,

$$-\hat{E}_{j+1} w_{i-1,j+1} + (1 + \hat{R}_{j+1}) w_{i,j+1} - \hat{E}_{j+1} w_{i+1,j+1} = \hat{E}_j w_{i-1,j} + (1 - \hat{R}_j) w_{i,j} + \hat{E}_j w_{i+1,j} + \frac{\Delta \tau}{2} \hat{F}, \quad (14)$$

$$\text{where, } \hat{E}_j = \frac{\Delta \tau \tilde{d}_j}{2(\Delta v)^2}, \quad \hat{E}_{j+1} = \frac{\Delta \tau \tilde{d}_{j+1}}{2(\Delta v)^2}, \quad \hat{F} = \tilde{\theta}_j + \tilde{\theta}_{j+1}, \quad \hat{R}_{j+1} = (2\hat{E}_{j+1} - \frac{\Delta \tau}{2} \tilde{a}_{j+1}), \quad \hat{R}_j = (2\hat{E}_j - \frac{\Delta \tau}{2} \tilde{a}_j),$$

$$\hat{S}_j = \hat{E}_j - \frac{\Delta \tau}{2} \tilde{a}_j, \quad \text{and} \quad \hat{S}_{j+1} = \hat{E}_{j+1} - \frac{\Delta \tau}{2} \tilde{a}_{j+1}.$$

The C-N FDM discretizes equations (2) – (4) which approximated as,

$$w_{i,0} + \delta w_{i,N} + I(v_i) = \zeta_i, \quad i = \overline{0, M}, \quad (15)$$

where, $I(v_i)$ is integral in the equation (2), which approximated as,

$$I(v_i) = \frac{\Delta \tau}{2} (\hat{P}(\tau_0) w(v_i, \tau_0) + \hat{P}(\tau_N) w(v_i, \tau_N) + 2 \sum_{j=1}^{N-1} \hat{P}(\tau_j) w(v_i, \tau_j)), \quad (16)$$

$$w_{0,j} = 0, \quad i = 0, j = \overline{0, N}, \quad (17)$$

$$\frac{w_{M+1,j} - w_{M-1,j}}{2(\Delta v)} = 0, \quad j = \overline{0, N-1}, \quad (18)$$

From equation (14), with the nonlocal initial conditions $w(v, 0) = \zeta(v_i)$, (assuming $\hat{P}(\tau) = \delta = 0$), and using equations (15) – (18), with each time step τ_j , it can be described in matrix form as

$$LW_{j+1} = L1W_j + \frac{\Delta\tau}{2}\hat{F}, \quad i = \overline{1, M}, j = \overline{0, N-1}, \quad (19)$$

$$L = \begin{bmatrix} 1 + \hat{R}_{j+1} & -\hat{E}_{j+1} & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ -\hat{E}_{j+1} & 1 + \hat{R}_{j+1} & -\hat{E}_{j+1} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & -\hat{E}_{j+1} & 1 + \hat{R}_{j+1} & -\hat{E}_{j+1} & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \ddots & \ddots & \ddots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & -\hat{E}_{j+1} & 1 + \hat{R}_{j+1} & -\hat{E}_{j+1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & \dots & -\hat{E}_{j+1} & 1 + \hat{R}_{j+1} & -\hat{E}_{j+1} \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & \dots & 0 & -\hat{E}_{j+1} & 1 + \hat{S}_{j+1} \end{bmatrix}_{(M) \times (M)}$$

with,

$$L1 = \begin{bmatrix} 1 - \hat{R}_j & \hat{E}_j & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ \hat{E}_j & 1 - \hat{R}_j & \hat{E}_j & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & \hat{E}_j & 1 - \hat{R}_j & \hat{E}_j & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \ddots & \ddots & \ddots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & \hat{E}_j & 1 - \hat{R}_j & \hat{E}_j & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & \dots & \hat{E}_j & 1 - \hat{R}_j & \hat{E}_j \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & \dots & 0 & \hat{E}_j & 1 - \hat{S}_j \end{bmatrix}_{(M) \times (M)},$$

and,

$$\hat{F} = \begin{bmatrix} \left(\frac{f_{1,j}}{c_j} + \frac{f_{1,j+1}}{c_{j+1}} \right) \\ \left(\frac{f_{2,j}}{c_j} + \frac{f_{2,j+1}}{c_{j+1}} \right) \\ \left(\frac{f_{3,j}}{c_j} + \frac{f_{3,j+1}}{c_{j+1}} \right) \\ \vdots \\ \left(\frac{f_{M-1,j}}{c_j} + \frac{f_{M-1,j+1}}{c_{j+1}} \right) \\ \left(\frac{f_{M,j}}{c_j} + \frac{f_{M,j+1}}{c_{j+1}} \right) \end{bmatrix}, W_{j+1} = [w_{1,j+1}, w_{2,j+1}, \dots, w_{M+1,j+1}, w_{M,j+1}]^T$$

$$W_j = [w_{1,j}, w_{2,j}, \dots, w_{M-1,j}, w_{M,j}]^T, \quad j = \overline{0, N}$$

In equation (4), the desired output data will be approximated using the trapezoidal rule as

$$\begin{aligned} z(\tau_j) &= \int_0^1 H(v)w(v, \tau_j)dv \\ &= \frac{\Delta\tau}{2} (H(v_0)w(v_0, \tau_j) + H(v_M)w(v_M, \tau_j) + \\ &2 \sum_{k=1}^{M-1} H(v_k)w(v_k, \tau_j)), \end{aligned} \quad (20)$$

3.1 Stability and Convergence Analysis

The adopted C-N finite difference scheme for solving the direct problem is unconditionally stable and second-order accurate in both spatial and temporal discretization. A brief von Neumann stability analysis shows that the amplification factor satisfies $|G| = 1$, for all mesh ratios, guaranteeing the unconditional stability of the scheme. The local truncation error is of order $\mathcal{O}((\Delta v)^2, (\Delta\tau)^2)$ confirming that the method provides second-order accuracy and reliable convergence toward the analytical solution as the mesh is refined. Hence, the C-N formulation ensures both numerical stability and accuracy for the present problem.

3.2 Example for the direct problem

3.2.1. Example 1: Assume an illustration for the direct problem given by (1) – (3) with $D_T : \{ 0 \leq v \leq 1, 0 \leq \tau \leq 1 \}$, and the input data are

$$f(v, \tau) = \frac{e^{-4.1+4\tau}}{2} (12v^2 - 40v^3 + 30v^4) - 1206e^{-4.1+4\tau}(v^4 - 2v^5 + v^6) - \frac{e^{-4.1+4\tau}(v^4 - 2v^5 + v^6)}{2(1+\tau^2)}, \zeta(v) = \frac{-(v^6 - 2v^5 + v^4)e^{-4.1}}{2}, c(\tau) = 603, z(\tau) = \frac{-1}{210}e^{-4.1+4\tau},$$

$$H(v) = 1, \quad \tau \in [0,1], v \in [0,1],$$

$$\text{and the analytical solution is } w(v, \tau) = \frac{-(v^6 - 2v^5 + v^4)e^{-4.1+4\tau}}{2}, a(\tau) = \frac{-1}{1+\tau^2}, \quad v \in [0,1].$$

The accuracy of the solution was investigated for diverse mesh grid sizes, say $M = N \in \{20, 40, 80, 160\}$. In Figure 1(a) – (d). It can explicitly observe an exceptional match between the exact solution and numerical approximation, and that accurate results are found. As the number of meshes expands, the accuracy of the acquired solution increases, revealing the freedom of mesh selection. Moreover, the absolute error graph at each mesh exhibits that the maximum magnitude of error didn't exceed (10^{-6}) as depicted in the third column in Figure 1(a) – (d).

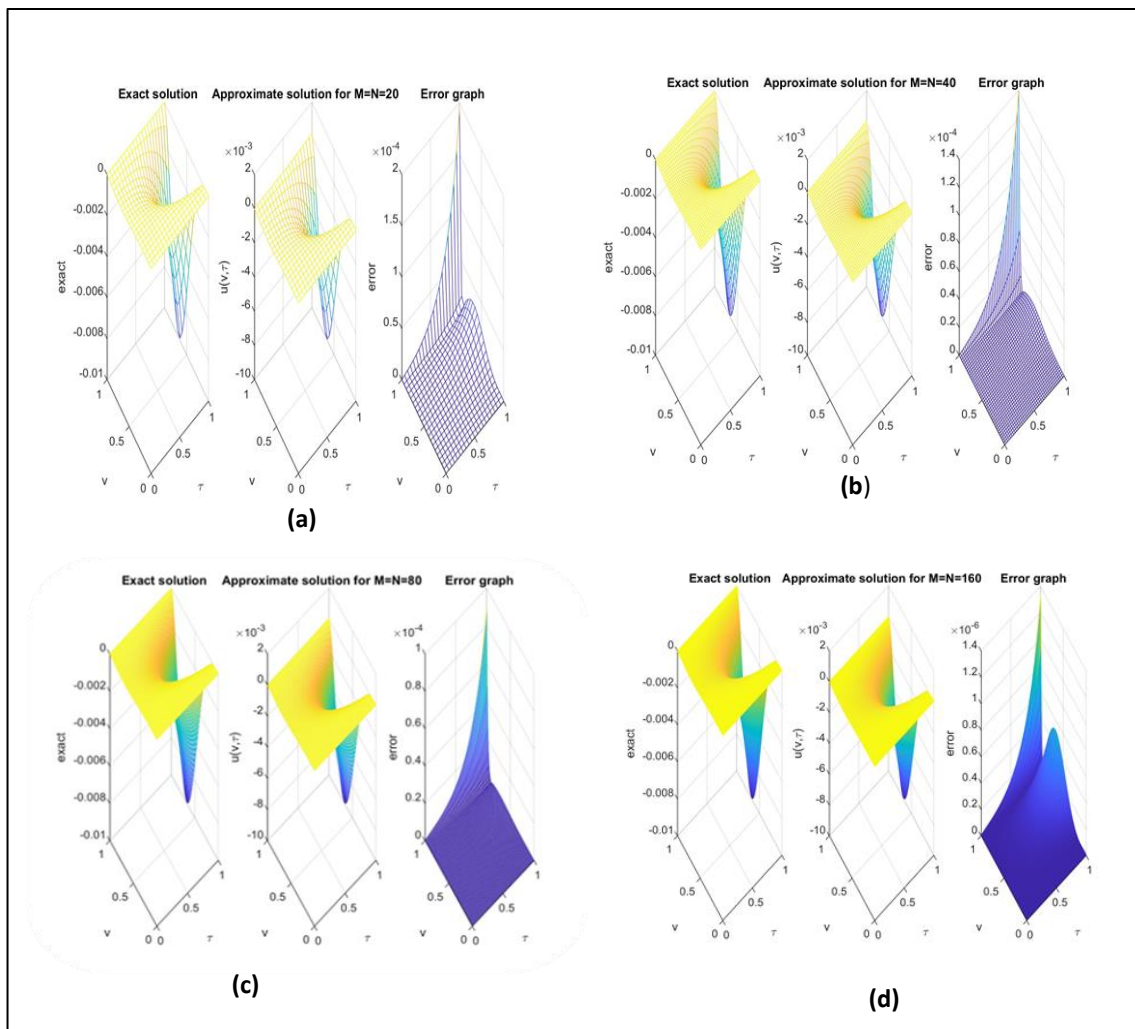


Figure 1: The graphs showing analytical and computational distribution with absolute error for the direct problem (1) – (4), for example, 1(a) – (d), represent the size of mesh $N = M = \{20, 40, 80, 160\}$, respectively.

Whilst Figure 2 shows the numerical outcomes for the required information, $z(\tau)$ evaluated at various mesh sizes, it is observed that a strong agreement has been obtained.

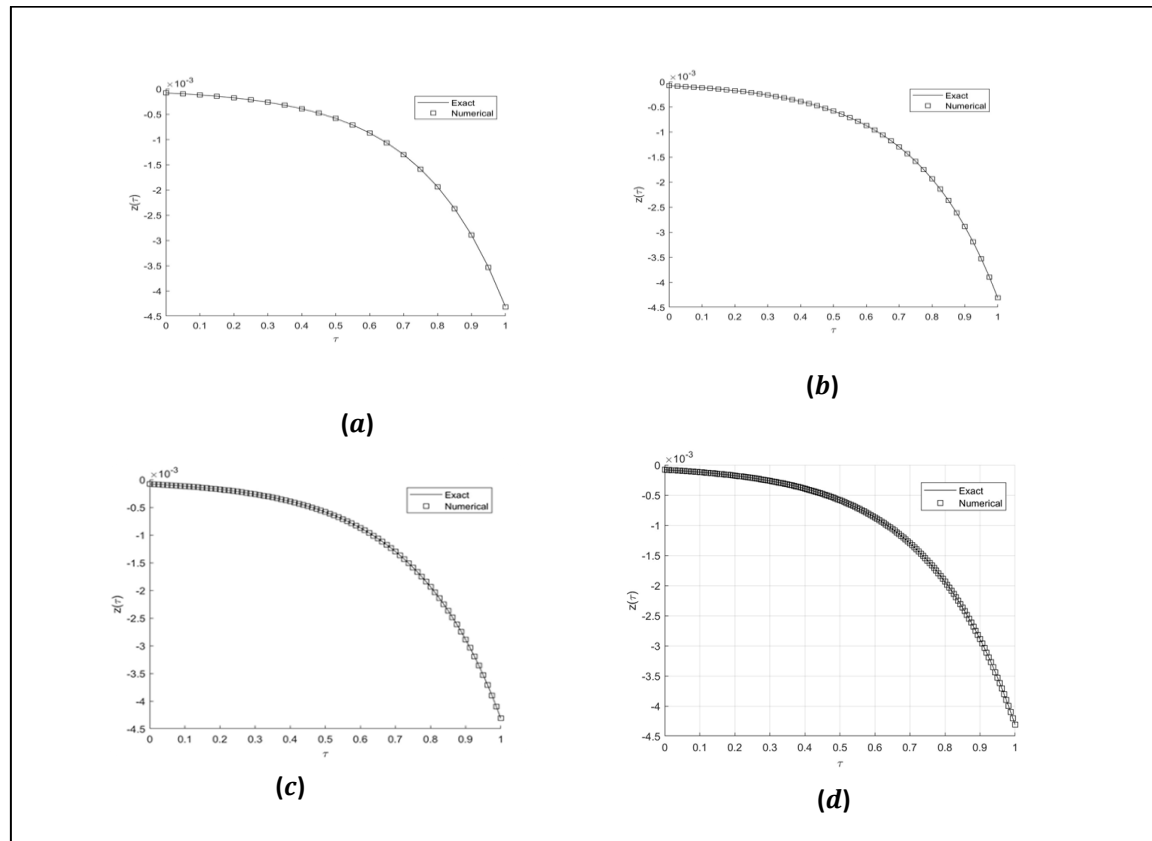


Figure 2: The graphs illustrate a comparison between the computed results and the exact solution of the target output $z(\tau)$ equation (4), for example 1 (a) – (d), with the size of mesh $N = M = \{20, 40, 80, 160\}$, respectively.

4. Solution Technique for the Inverse Problem

For the numerical solution of the nonlinear inverse problem (IP) defined by equations (1) – (4). The problem is reformulated as a nonlinear least-squares minimization task. Due to the ill-posedness of such problems, particularly when noisy data was included, Tikhonov Regularization (TR) was employed to stabilize the solution. The regularization parameter $\beta \geq 0$ was carefully selected to balance the trade-off between the measured and numerically computed solutions. For this purpose, MATLAB routine *lsqnonlin* was employed with the Trust-Region-Reflective (TRR) algorithm [35]-[36], to minimize the Tikhonov Regularized objective functional. The procedure begins with a suitably selected initial guess. The TR functional was derived based on the condition (4), and the functional error is incorporated as follows:

$$F(a) = \left\| \int_0^1 H(v)w(v, \tau)dv - z(\tau) \right\|_{L^2[0, \tau]}^2 + \beta \|a(\tau)\|_{L^2[0, \tau]}^2 \quad (21)$$

where $\beta > 0$ is a regularization parameter. The discretization form of (21) is

$$F(\underline{a}) = \sum_{j=1}^N [\int_0^1 H(v)w(v, \tau_j)dv - z(\tau_j)]^2 + \beta \sum_{j=1}^N a_j^2 \quad (22)$$

The unregularized case, i.e., $\beta = 0$, produces a regularized nonlinear least-squares functional that is inherently unstable when dealing with noisy data. The MATLAB routine

lsqnonlin is used to minimize F subject to physical constraints. The following parameters were used for the subroutine:

- (MaxIter) maximum number of iterations = $10^6 \times N$.
- Solution and Objective function tolerance = 10^{-10}
- The lower and upper bounds on the component of the vector $\underline{a}$ are -10^{-2} and 10^2 , respectively.

The IP (1) – (4) were resolved via both precise and noisy measurements (4). By including a random error, the noisy data was numerically simulated,

$$z^\varepsilon(\tau_j) = z(\tau_j) + \varepsilon_j \quad j = \overline{0, N}. \quad (23)$$

where ε is a random Gaussian normal distribution vector, with mean zero, and standard deviations σ , given by

$$\sigma = \rho \times \max_{\tau \in [0, T]} |z(\tau)| \quad (24)$$

where ρ is the percentage of noise. The MATLAB function "normrnd", was used as:

$$\underline{\varepsilon} = \text{normrnd}(0, \sigma, N), \quad (25)$$

to generate the random variables, $\underline{\varepsilon} = (\varepsilon_j)$ and, $j = \overline{0, N}$.

5. Numerical outputs and discussion

To evaluate the precision and stability of the numerical methods, a set of numerical experiments was conducted. These experiments are designed to assess the accuracy of the computed outcomes by simulating realistic measurement conditions that introduce noise into the input data. To quantitatively evaluate the discrepancy between the exact and numerically calculated solutions, the root-mean-square error (RMSE) is used as follows.

$$\text{RMSE}(a) = \sqrt{\frac{1}{N} \sum_{i=1}^N (a_i^{\text{exact}} - a_i^{\text{numerical}})^2}, \quad (26)$$

For simplicity, $T = 1$ was taken as an example below:

5.1. Test Example for IP:

5.1.1 Example 2: Consider the IP (1) – (4) with the following input data,

$$f(v, \tau) = \frac{1}{4} (\pi^2 (3 + \tau^2) - 4(3 + 4\tau + \tau^2)) \sin\left(\frac{\pi v}{2}\right), \quad \zeta(v) = 3 \sin\left(\frac{\pi v}{2}\right), \quad z(\tau) = \frac{2(3 + \tau^2)}{\pi}, \quad c(\tau) = 1 + \tau^2, \quad H(v) = 2 \cos\left(\frac{\pi v}{2}\right),$$

where the analytical solution for this IP is provided as follows,

$$w(v, \tau) = (3 + \tau^2) \sin\left(\frac{\pi v}{2}\right), \quad a(\tau) = 1 + 2\tau, \quad \tau \in [0, T], \quad (v, \tau) \in D_T,$$

The time-dependent coefficient $a(\tau)$, was then reconstructed (in case, $M = N = \{20, 40, 80, 160\}$, is taken), and consider the case of noise-free in the measurements, this mean $p = 0$, in (22) and, no regularization applied, Figure 5, shows the objective function (22) without regularization, i.e., ($\beta = 0$), while, in Figure 6, no noise with regularization $\beta \in \{10^{-2}, \dots, 10^{-8}\}$.

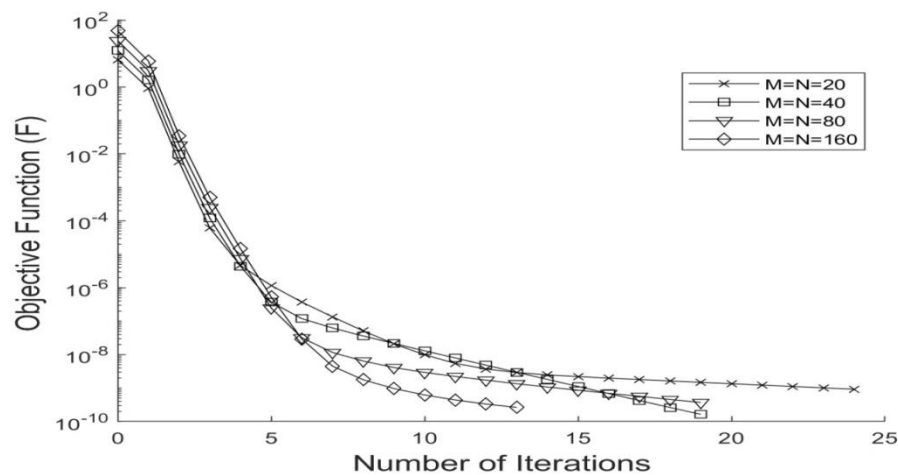


Figure 5: The graph showing the objective function when no noise and no regularization for the IP (1) – (4), in example 2, when the size of the mesh $N = M = \{20, 40, 80, 160\}$, respectively.

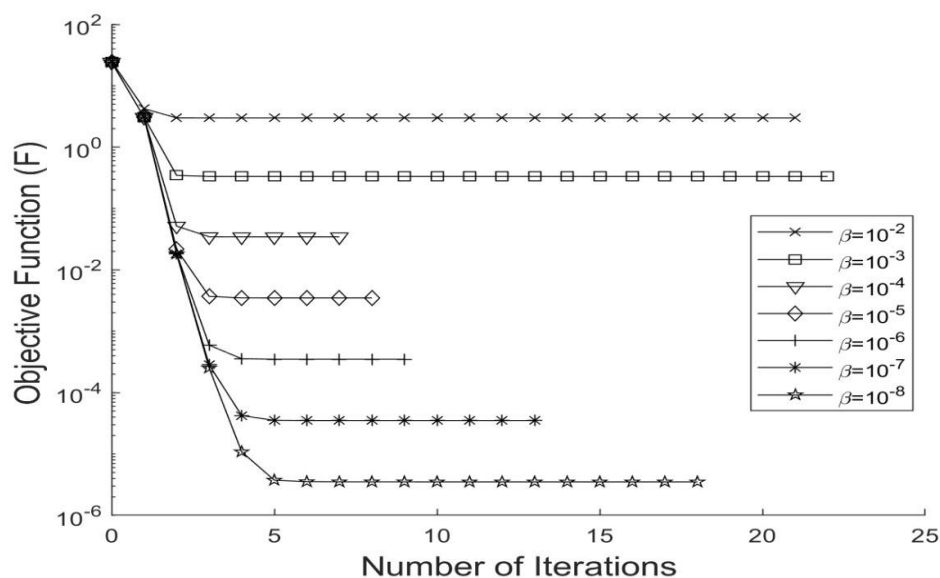


Figure 6: The graph showing the objective function when no noise with regularization, $\beta \in \{10^{-2}, \dots, 10^{-8}\}$, for the IP (1) – (4), in example 2, when the size of mesh $N = M = 80$.

From this figure, it can be seen that a rapid decline in the convergence of the objective function. On the other hand, Figures 7 and 9 show the results when noise is present (i.e., $p = 1\%$). In Figure 6, the best choice for the regularization parameters is 10^{-8} , and in Figure 9, the best choice for the regularization parameters is 10^{-3} , Table 2, which results in the lowest RMSE values.

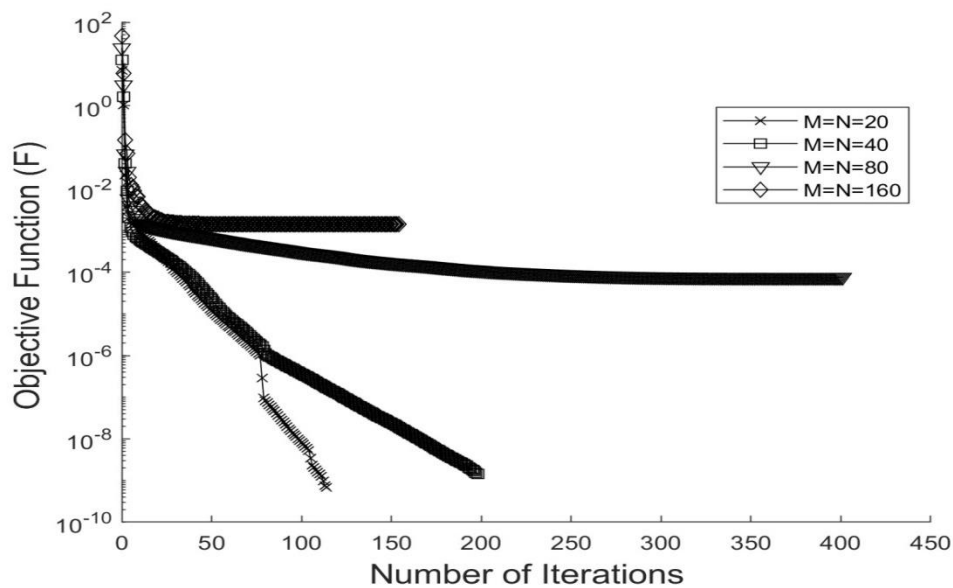


Figure 7: The graph showing the objective function when $p = 1\%$, no regularization for the IP (1) – (4), in example 2, when the size of the mesh $N = M = \{20, 40, 80, 160\}$, respectively.

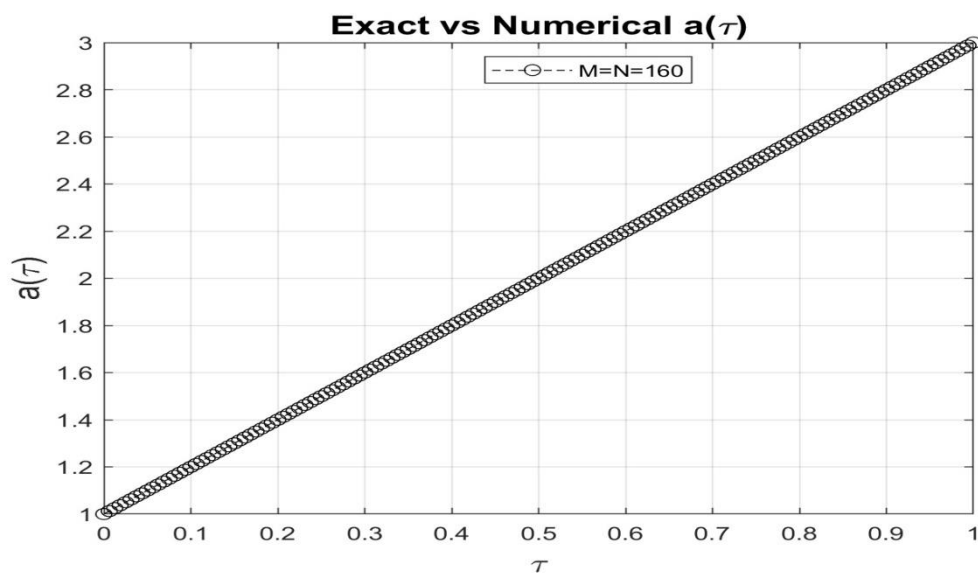


Figure 8: The graph showing the exact and (unstable) numerical solution, when $p = 1\%$, no regularization for the IP (1) – (4), in example 2, when the size of the mesh is $N = M = 160$.

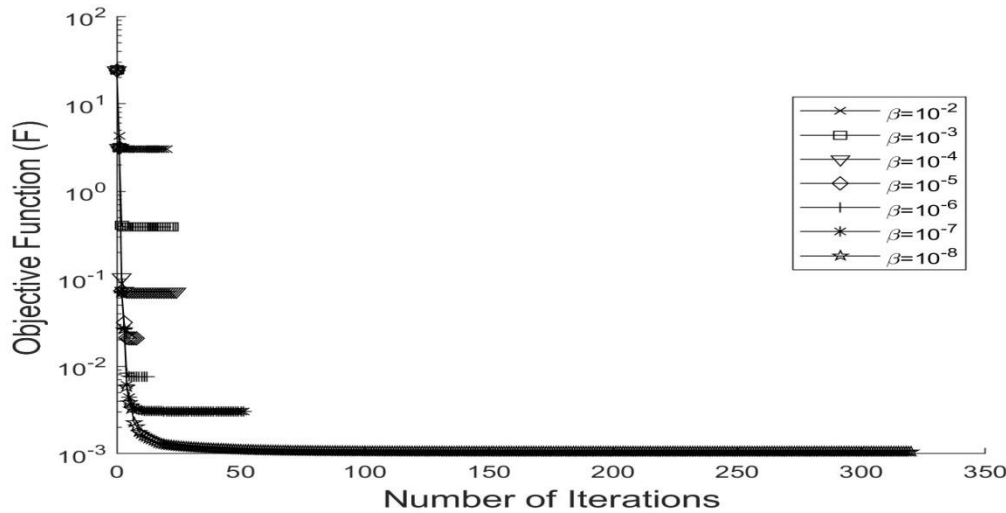


Figure 9: The graph showing the objective function when $p = 1\%$, with regularization, $\beta \in \{10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}, 10^{-7}, 10^{-8}\}$, for the IP (1) – (4), in example 2, when the size of mesh $N = M = 80$.

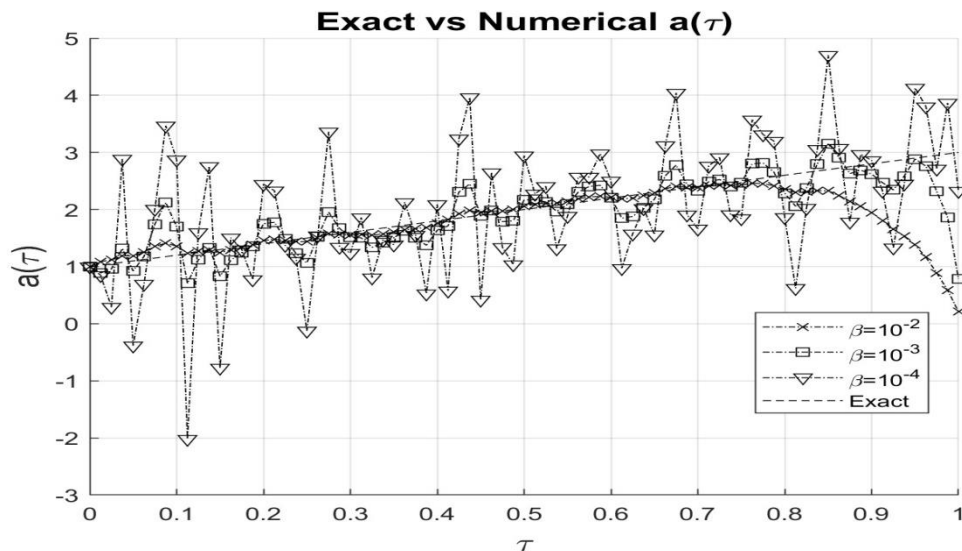


Figure 10: The graph showing the exact and (stable) numerical solution when $p = 1\%$, and regularization, $\beta \in \{10^{-2}, \dots, 10^{-8}\}$, for the IP (1) – (4), in example 2, when the size of mesh $N = M = 80$.

The time-dependent coefficient $a(\tau)$ was investigated under both exact and perturbed measurements, as demonstrated in Figures 5 – 9. The associated results are unstable, as illustrated in Figure 5. This reveals that the problem is not properly posed; a noticeable instability appears in the results, which is anticipated due to the ill-posed nature of the investigated IP. The unregularized solutions exhibit noticeable instability even in the noise-free case, demonstrating the problem's sensitivity to minute perturbations in the supplied input data. The TR approach is used by incorporating the penalty term $\beta \|a(\tau)\|_{L^2[0,\tau]}^2$ into the classical least-squares formulation, as presented in equation (22). In Figure 6, the noise-free strategy is employed to enhance the solution's robustness with regularization parameters, $\beta \in \{10^{-2}, \dots, 10^{-8}\}$, yielding numerically stable regularized reconstructions of $a(\tau)$, as shown in

Table 1. These results indicate that smaller values of β are effective when the data is not contaminated.

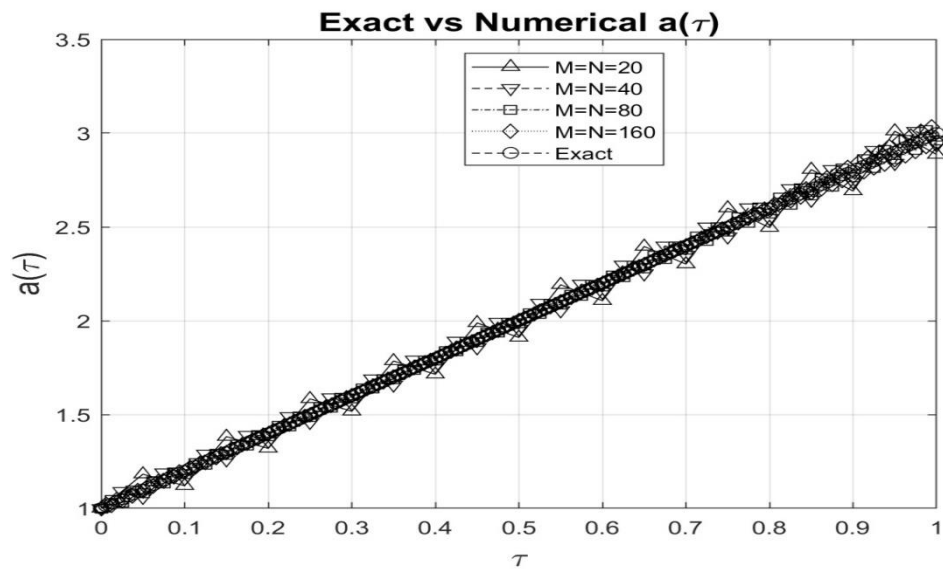


Figure 11: The graph showing the exact value of $a(\tau)$, when no noise, no regularization for the IP (1) – (4), in example 2, when the size of the mesh $N = M = \{20, 40, 80, 160\}$, respectively.

Next, will study the proportion of noise contamination with a noise ratio of $p = 1\%$ in (23) via (24); the associated results are unstable, as illustrated. This reveals that the problem is not properly posed, and that small errors in the input data lead to large errors in the output solution; thus, the problem needs to be regularized. As a consequence, for the situation of $p = 1\%$ noise, the best selection for the regularization parameters is 10^{-3} as in Table 1, which results in the lowest RMSE values.

Table 1: Numerical information for various no regularization $\beta = 0$ and noise $p = 0$

N	20	40	80	160
RMSE	0.0937	0.0475	0.0223	0.0151
Computational time	38Sec	191Sec	1458Sec	8703Sec
No. of Iterations	25	20	20	14
Obj. Function	9.2251×10^{-10}	1.6385×10^{-10}	3.7151×10^{-10}	2.6420×10^{-10}

Table 2: Numerical information for various regularization parameters and noise = 1%, ($N = 80$)

$\beta = 10^{-2}$		$\beta = 10^{-3}$		$\beta = 10^{-4}$	
RMSE	Computational time	RMSE	Computational time	RMSE	Computational time
0.6140	1501Sec	0.4001	1622Sec	1.0287	1780Sec
$\beta = 10^{-5}$		$\beta = 10^{-6}$		$\beta = 10^{-7}$	
2.8931	682Sec	5.6015	1012Sec	12.6326	3649Sec
				$\beta = 10^{-8}$	
				27.0585	22609Sec

6. Conclusions

In this study, the recovery of the unknown coefficient $a(\tau)$ A one-dimensional parabolic PDE is investigated. The problem is formulated from a numerically evaluated nonlocal integral subjected to initial and boundary conditions (BCs). The direct problem is solved using the

Crank-Nicolson scheme, while the IP is transformed into a nonlinear least squares optimization problem using the lsqnonlin routine in MATLAB. A test case is employed in numerical experiments to evaluate the accuracy and stability of the proposed method. According to the results, using TR significantly enhances the solution's stability. When the root-mean square error (RMSE) numbers are at their lowest, the most effective approach is identified. Furthermore, numerical results have been provided to illustrate the accuracy and stability of the numerical method. It found that applying the TR method stabilizes the solution. A numerical result exists with noise $p = 1\%$ (with and without regularization). In the first case, when no noise and various regularizations are used as in Figure 6, the results are stable and accurate. On the contrary, in other cases, when noise is introduced with no regularization, highly oscillated results for the reconstructed coefficient $a(\tau)$, which are (unstable) numerical solutions. As for the remaining cases, when including noise $p = 1\%$, and some regularizations, we observe that the results are stable and steady when the regularization values are 10^{-3} . In addition, will address in future work extending the finite difference approach to multi-dimensional domains for higher-order parabolic, hyperbolic, and Elliptic PDEs, and develop numerical approaches for third and fourth-order PDEs with mixed, integral, or nonlocal boundary conditions.

References

- [1] M. Ravi, " Partial Differential Equations and Some Of The Applications", *ShodhKosh: Journal of Visual and Performing Arts*, vol.5, no. 7, pp. 1098-1106, 2024.
- [2] K. Belakroum, A. Ashyralyev, and A. Guezne-Lakoud, "A Note on the Nonlocal Boundary Value Problem for a Third Order Partial Differential Equation", *Filomat*, vol. 32, no. 3, pp. 801-808, 2018.
- [3] C.L. Gardner, *Applied Numerical Methods for Partial Differential Equations*, Text in Applied Mathematics, vol. 78, Springer Nature Switzerland AG, 2024.
- [4] M. Giga, Y. Giga, and J. Saal, "Nonlinear Partial Differential Equations: Asymptotic Behavior of solutions and Self-Similar Solutions", *Progress in nonlinear Differential Equations and Their Applications*, vol. 79, 2010.
- [5] V. Isakov, *Inverse Problems for Partial Differential Equations*, Third Edition, Springer Nature Switzerland, 2017.
- [6] K. Sankara Rao, *Introduction to Partial Differential Equations*, 3rd ed., PHI Learning Private Limited, New Delhi, 2011.
- [7] E. Suli, *Lecture Notes On Finite Element Methods For Partial Differential Equations*, Oxford, UK: Mathematical Institute, University of Oxford, 11, 2020.
- [8] D. Lesnic, *Inverse Problems With Applications In Science And Engineering*, 2nd ed. Boca Raton, FL: CRC. Press, 2022.
- [9] S.I. Kabanikhim, "Definitions and Examples of Inverse and Ill-Posed Problems ", *Journal of Inverse and Ill-posed Problems*, vol.16, pp. 317-357, 2008.
- [10] M.J. Huntul and T.-E. Oussaeif, " Solvability of the Nonlocal Inverse Parabolic Problem and Numerical Results ". *Computer Systems Science and Engineering*, vol. 40, no. 3, pp. 1109-1126, 2022.
- [11] M. Sadybekov, G. Dildabek, and M. Ivanova, " Direct and Inverse Problems for Nonlocal Heat Equation with Boundary Conditions of Periodic Type ". *Boundary Value Problems*, vol. 2022, no. 53, 2022.
- [12] Y.T. Mehraliyev, S.R. Shafi, A.T. Ramazanova, "Two-Dimensional Inverse Boundary Value Problem for a Third-Order Pseudo-Hyperbolic Equation with an Additional Integral Condition", *European Journal of Pure and Applied Mathematics*, vol. 16, no. 2, pp. 670-686, 2023.
- [13] I. Baglana, F. Kanca and V.N. Mishra, "Solution of Inverse Euler-Bernoulli Problem with Integral Overdetermination and Periodic Boundary Conditions", *Iranian Journal of Mathematical Sciences and Informatics*, vol. 17, no. 1, pp. 191-206, 2022.

- [14] E.D. Priore and L. Lampani, "A methodology for Applying Isogeometric Inverse Finite Element Method to the Shape Sensing of Stiffened thin-Shell Structures", *Thin-Walled Structures*, vol. 199, 111837, 2024.
- [15] M. Srati, A. Oulmelk, L. Afraites and A. Hadri, "An Inverse Problem of Identifying Two Coefficients in a Time-Fractional Reaction Diffusion System", *Discrete and Continuous Dynamical Systems-Series S*, vol.18, no. 1, pp. 113-147, 2025.
- [16] I. Khalid, Z.A. Qureshi, S. Oterkus, and E. Oterkus, "Progress in Inverse Finite Element Method for Aerospace Structural Health Monitoring Applications", *Progress in Aerospace Sciences*, vol. 157, pp. 101132, 2025.
- [17] Y. Gu, J. Lei, C.-M. Fan, and X.-Q.He, "The Generalized Finite Difference Method for an Inverse Time-Dependent Source Problem Associated with Three-Dimensional Heat Equation", *Engineering Analysis with Boundary Elements*, vol. 91, pp.73-81, 2018.
- [18] Z. Li, C.V. Pao, and Z. Qiao, "A Finite Difference Method and Analysis for 2D Nonlinear Poisson-Boltzmann Equations", *Journal of Scientific Computing*, vol. 30, no. 1, pp. 61-81, 2007.
- [19] U. D. Durdiev, "Inverse Problem of Determining the Kernel in the Integro-Differential Beam Vibration Equation", *Eurasian Journal of Mathematical and Computer Applications*, vol. 13, no. 1, pp. 50-63, 2025.
- [20] M.N. Koleva and L.G. Vulkov, "Numerical Solution of the Retrospective Inverse Parabolic Problem on Disjoint Intervals", *Computation*, vol. 11, no. 10, Oct. 2023.
- [21] G. Delo, R. Roy, K. Worden, and C. Surace, "Using the Inverse Finite-Element Method to Harmonise Classical Modal Analysis with Fibre-optic Strain Data for Robust Population-Based Structural Health Monitoring", *Strain*, vol. 2024, e12481, 2024.
- [22] A. Yernazar, I. Baglan and E. Aslan, "Analytical and Computational Solutions for Nonlinear Hyperbolic Equation with Inverse Coefficient under Periodic Boundary Conditions", in *Proc. Kocaeli Science Congress (KOSC-2024)*, Kocaeli, Türkiye, Oct. 2-4, 2024, pp.42-48.
- [23] G. Zaki, A.J. Akbarfam, and S. Irandoust-Pakchin, "On the Inverse Problem of Time Fractional heat Equation using Crank-Nicholson Type Method and Genetic Algorithm", *University of Nis.*, vol. 38, no. 19, pp. 6849-2024.
- [24] D. Arjmand and M. Ashyraliyev, "Efficient Low Rank Approximations for Parabolic Control Problems with Unknown Heat Source", *Journal of Computational and Applied Mathematics*, vol. 450, no. 115959, 2024.
- [25] O. Nosko, "An Inverse Algorithm for Contact Heat Conduction Problems with an Interfacial Heat Source Based on a First-Order Thermocouple Model", *International Communications in Heat and Mass Transfer*, vol. 158, Art.no. 107889, 2024.
- [26] R. Djeflal, D. Lalmi, S.M.E. Bekkouche, T. Bechouat, and Z. Younsi, "New Method for Solving the Inverse Thermal Conduction Problem (θ –Scheme Combined with CG Method under Strong Wolfe Line Search)", *Buoldings*, vol. 13, no. 1, no. 243, 2023.
- [27] G. Bakanov, S. Chandragiri, S. Kabanikhin, and M. Shishlenin, "Comparative Analysis of Numerical Methods for Solving 3D Continuation Problem for Wave Equation", *Mathematics*, vol. 13, 2979, 2025.
- [28] O. Kaytmaz and M. Yildiz, "Solvability of an Inverse Problem for an Elliptic-Type Equation", *Cumhuriyet Science Journal*, vol. 45, no. 1, pp. 130-134, 2024.
- [29] C. Ashyralyev and A. Gonenc, "Numerical Solution of the Nonlocal Reverse Parabolic Problem with Second Kind Boundary and Integral Conditions", *AIP Conference Proceedings.*, vol. 2325, 02005. 2021.
- [30] A.V. Velisevich and A.S. Lyubanova, "On the Stability of the Solutions of Inverse Problems for Elliptic Equations", *Journal of Siberian Federal University. Mathematics and Physics*, vol. 17, no. 3, pp. 398-407, 2024.
- [31] A.I. Kozhanov, U.U. Abulkayirov, and G.R. Ashurova, "Inverse Problems of Determining Coefficients of Time Type in a Degenerate Parabolic Equation", *Bulletin of the Karaganda University*, vol. 106, no. 2, pp. 128-142, 2022.
- [32] M. Cui, W.-W. Duan, and X.-W. Gao, "A New Inverse Analysis Method Based On A Relaxation Factor Optimization Technique For Solving Transient Nonlinear Inverse Heat Conduction Problems", *International Journal of Heat and Mass Transfer*, vol. 90, pp. 491-498, 2015.

- [33] A. Ashyralyev and C. Ashyralyev, "Numerical Solution of Time-Nonlocal Problem for ParabolicEquation", *AIP Conference Proceedings*, vol. 2483, 060001, 2022.
- [34] E. Azizbayov, and Y.T. Mehraliyev, "On a Nonlocal Inverse Problem With the Integral Overdetermination Condition For a Parabolic Equation of Second Order", *International Journal of Pure and Applied Mathematics*, vol. 116, no. 3, pp. 601- 616, Oct. 2017.
- [35] T.F. Coleman, and Y. Li, "An Interior Trust Region Approach for Nonlinear Minimization Subject to Bounds", *SIAM Journal on Optimization*, vol. 6, no. 2, pp. 418-445,1996.
- [36] MathWorks, Optimization Toolbox Documentation. [Online]. Available: <https://www.mathworks.com> [Accessed: Jul.1, 2025].