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A Hyponormality Operator Formulated by Draizn Inverse Operators Combining the Fuzzy Soft

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Abstract:
This study aims to investigate the realm of operator theory (OT) related to Fuzzy Soft Theory (FST). Based on a fuzzy soft Hilbert space, a novel formula of the fuzzy soft hyponormal operator, namely the fuzzy soft power Drazin inverse hyponormal operator. Moreover, some analytical traits are revealed for this imposed operator. In addition, the direct sum and tensor product are discussed.

Keywords: Hilbert space, Soft set, Fuzzy soft set, Fuzzy soft Hilbert space, Fuzzy soft Drazin invertible, Hyponormal operator.

تقنية المجموعة الضبابية الناعمة لبنى جديدة من المؤثرات ذات القوى معكوس درازين فوق السوية

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الخلاصه:
تهدف هذه الدراسة الى دراسة نطاق نظرية المؤثر (OT) المتعلقة بالنظرية الضبابية الناعمة (FST) وبالاعتماد على فضاءات هلبرت الضبابي الناعم. ووضعت صيغة جديدة لمؤثر فوق السوي الضبابي الناعم هو مؤثر قوى معكوس درازين فوق السوي الضبابي الناعم علاوة على ذلك تم اكتشاف العديد من الصفات التحليلية لهذا المؤثر , بالإضافة الى الجمع المباشر والضرب التنسوري نوقشت.

1.Introduction

Operator theory (OT) is a significant realm of mathematics, mainly functional analysis. It is an intrinsic resource that numerous engineers and scientists rely on. This theory studies operators, their merits, and behaviors on complex Hilbert spaces. Typically, a bounded linear operator $\mathcal{T}$ is defined on a complex Hilbert space H. The extended normal operator, namely a hyponormal operator, is one of the most interesting operators. In 1962, Stampfli [1] formulated the principle of hyponormality under the diverse name, subnormality. Furthermore, several merits of hyponormal operators are demonstrated. Indeed, a hyponormal operator based on the concept of adjoint, for instance, the unilateral shift operator. Over the past decades, numerous scholars have investigated the generalized hyponormal operators. For instance, in 2019, Okelo [2] proposed a precise characterization of compact hyponormal

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operators that achieved absolutely norm-attaining. The author established necessary and sufficient situations based on a bounded compact linear operator on a complex Hilbert space that possesses this merit. In 2019, Mahmoud and Ahmed [3] introduced the classes of (n,m)-power D-normal and (n,m)-power D-quasi-normal operators. They presented a comprehensive investigation into the development and analysis of the properties of these classes, extending the classical notions of normal and quasi-normal operators by employing the composite power (n,m) and a specific operator D. In 2021, D. Bekai et al., [4] proposed that powers of closed range hyponormal operators continues to have closed range. In 2021, P. Dharmarha and S. Ram [5] studied the class (m, n)-paranormal operators and discussed Spectral mapping theorem and Weyl's theorem of this class under the necessary conditions. In 2020, Dana and Yousefi [6] proposed the generalized class of D-hyponormal operators, where the concept of D-normality is extended to encompass a broader range of cases within Hilbert spaces. In the same year, Dharmarha and Ram [7] introduced and studied two generalized classes of bounded linear operators on Hilbert spaces, namely the (m,n)-paranormal and (m,n)$*$-paranormal operators. In 2021, Mortad [8] provided three counterexamples illustrating that some properties commonly attributed to paranormal operators do not hold when these operators are unbounded. In 2022, Mesbah and Messaoudene [9] investigated and presented certain new classes, which represent the D-hyponormal and the D-quasi-hyponormal operators. They examined several key features. In 2024, Mohsen [10] introduced a completely new class of quasi operators, named (n,D)-quasi operators on the Hilbert spaces using Drazin inverse,also studied the scalar and power of this concept and proved of the Tensor product and direct product for this class. Moreover, Shen et al., [11] introduced a class of p-quasi-n-hyponormal operators and demonstrated their structural properties via the Hansen inequality and Lowner-Heinz inequality. Afterwards, in 2025, Dana et al., [12] likewise provided some findings on the inequalities and generalized outcomes of the D-hyponormal operator. In addition, Al-Shammari [13] formulated the structure of a p-tuple of quasi (n, m)-Drazin operators, known as (n, m)-$\mathcal{D}$-normal operators. It serves as a generalization of the class of p-tuple n-normal operators. They also extended these operators to the single-variable case, namely n-$\mathcal{D}$-normal and (n, m)-$\mathcal{D}$-normal operators and explored several of their fundamental merits. In 2020, Kai. Yan and Q. Zeng [14] introduced new proposals of Cline's formula Jacobson's lemma and for Drazin inverses, generalized Drazin inverses. Whereas, in 2004, Caradus [15] provided and discussed the Drazin inverse operator, as a generalization of a hyponormal operator, which is frequently utilized in Cauchy problems, differential equations, and analyzing Markov chains, [16,17]. Several researchers, such as Chen and Sheibani [18], Qin and Lu [19], and Duggal and Kim [20] have designed new frontiers for the structure of Drazin-type inverse operators and their advantages. The studies aim to extend renowned spectral outcomes and generalize key concepts, such as Weyl's theorem and spectral mapping theorem. Besides, they analyze the algebraic structure of the Drazin inverse and its implementations in Hilbert spaces and Banach algebras. In this context, Zadeh [21] in 1965 introduced a significant mathematical concept called fuzzy sets. Subsequently, Molodsov [22] in 1999 established Soft Set Theory (SS-T) as an elegant mathematical tool for treating uncertainty and assorted complex problems. In 2001, Maji [23] combined the principles of fuzzy and soft into an interesting idea, which is named Fuzzy Soft Set (FSS). Fuzzy Soft Set Theory (FSs-T) is a substantial theme. Many investigators have studied numerous elegant implementations of FSs-T in varied disciplines and contributed to the expansion and highlighting of considerable new characteristics on this topic. Among their pivotal contributions are, for instance, [24- 33]. In 2020, Faried et al., [34] imposed various gorgeous and fruitful outcomes in FSs-T. Vital generalizations of FS-N, called fuzzy soft inner product (FS-IP) provided on fuzzy soft vector spaces (FS-V) and fuzzy soft Hilbert spaces (FS-H) are constructed. They also attempted to evolve Fuzzy soft operator theory (FS-OT) by studying certain fuzzy soft linear operators on FS-H, for instance, fuzzy soft left and right shift

operators, and to discuss numerous concerning merits of fuzzy soft spectral theory (FS-ST). In 2021, Faried et al., [35] investigated the fuzzy soft Hilbert space principle alongside its attributes and distinct new outcomes. In 2020, Faried et al., [36] presented a fuzzy soft hermitian operator within the framework of FS-H. Furthermore, related outcomes are imposed. They also investigated examples in favor of and against this operator. In 2022, Mohsen [37] formulated an advanced class within fuzzy soft Hilbert space called fuzzy soft quasi normal operator and discussed some properties of this concept. Several researchers have also conducted studies in semi Hilbertian space, introduced new classes and discussed some of the fundamental properties of these classes [38,39]. In 2024, Radharamani and Nagajothi [40] studied and presented a fuzzy soft paranormal operator. They also examined the diverse key merits of this operator in FS-H. Several notions relevant to this operator are also discussed in FS-H. In this study, a new class of fuzzy soft hyponormal operators by using the Drazin inverse is introduced, namely the fuzzy soft power Drazin inverse hyponormal operator. The analytical traits of the thin new operator are investigated. The direct sum and tensor product are also examined.

2. Basic concepts

This section introduces the key concepts in the study of FS-H, which will be employed to get the main findings.

Definition 2.1. [21] The set of order pair $\widetilde{\mathcal{G}}$= {(Y, $\mu_{\tilde{\mathcal{G}}}$(Y))| Y ∈ X, $\mathcal{A}$(Y) ∈ $\mathcal{I}$} named fuzzy set on X with a membership function $\mu_{\tilde{\mathcal{G}}}$:X ⟶ Γ, where Γ = [0,1]. A set $\widetilde{\mathcal{G}}$ is also sometimes be written as $\widetilde{\mathcal{G}}$=$\left\{\frac{\mu_{\tilde{\mathcal{G}}}(Y)}{Y}\ Y \in X\right\}$. The real number, $\mu_{\tilde{\mathcal{G}}}$(Y) , namely the membership of Y in $\widetilde{\mathcal{G}}$.

Definition 2.2. [22] The set $\mathfrak{f}_{\mathcal{G}}$= {$\mathfrak{f}$ (ω) ∈ (X): ω ∈ $\mathcal{G}$} named soft set over X, with a set of parameters Σ and (X) the set of all subsets $\mathcal{X}$ such that $\mathcal{G}$⊆ Σ, $\mathfrak{f}$: $\mathcal{G}$⟶ X , represented by $\mathfrak{f}_{\mathcal{G}}$.

Definition 2.3. [23] The soft set $\mathfrak{f}_{\mathcal{G}}$ is named fuzzy soft set (FSs) over $\mathcal{X}$, where $\mathfrak{f}$: $\mathcal{G}$ ⟶Γ^X, has the range {(ω) ∈Γ^X: ω ∈ $\mathcal{G}$}, and the class of all (FSs), indicated by FSs($\tilde{X}$).

Definition 2.4. [35] The fuzzy soft set (G,B) ∈ FSS (ʉ) named a fuzzy soft point over ʉ symbolized by ($\tilde{v}_{f_{G(e)}}$, B), (briefly denoted by $\tilde{V}_{f_{G(e)}}$), if for the element e ∈B and u∈ʉ ,α ∈ (0,1] is the value of the membership degree, $f_{G(e)}(v) =$
$\begin{cases}\alpha\,, if\ u = u_0 \in v \text{ and } e = e_0 \in B;\\ 0, if\ u \in v\ - \{u_0\} \text{ or } e \epsilon B - \{e_0\}.\end{cases}$

Definition2.5. [35] The set fuzzy soft vector is FS-vector space, according with the following two operations below:

i. $\tilde{v}^1_{f_1{}_{G(e_1)}} + \tilde{v}^2_{f_2{}_{G(e_2)}} = \widetilde{(v^1 + v^2)}_{\left(f_1{}_{G(e_1)} + f_1{}_{G(e_2)}\right)}$ for all $\tilde{v}^1_{f_1{}_{G(e_1)}}, \tilde{v}^2_{f_2{}_{G(e_2)}}$ ∈ FS-V.

ii. $\tilde{r}\tilde{v}_{f_{G(e)}} = \widetilde{(rv)}_{f_{G(e)}}$ for all $\tilde{v}_{f_{G(e)}}$ ∈ FS-V and ∀$\tilde{r}$ ∈ $\tilde{R}_B$.

Definition 2.6. [35] Assume $\widetilde{H}$ be FS-V and $\mathcal{R}(\mathcal{A})$ be fuzzy soft real set. A mapping $\|\widetilde{.}\|$: $\widetilde{H}$ → $\mathcal{R}(\mathcal{A})$ namely fuzzy soft norm (FS-N) on $\widetilde{H}$ if $\|\widetilde{.}\|$ achieves the following:

i. for all $\tilde{v}_{f_{G(e)}} \in \widetilde{H}$ $\left\|\widetilde{\tilde{v}_{f_{G(e)}}}\right\| \gtrsim \tilde{0}$;

ii. $\left\|\tilde{v}_{f_{G(e)}}\right\| \cong \widetilde{0}$ if and only if $\tilde{v}_{f_{G(e)}} \cong \tilde{0}$;

iii. for all $\tilde{v}_{f_{G(e)}} \in \widetilde{H}$, and $\tilde{r} \in \mathcal{C}(\mathcal{A})$ $\left\|\widetilde{\tilde{r}\tilde{v}_{f_{G(e)}}}\right\| \cong |\tilde{r}| \left\|\widetilde{\tilde{v}_{f_{G(e)}}}\right\|$;

iv. for all$\tilde{v}_{f_{G(e_1)}}$, $\tilde{u}_{f_{G(e_2)}} \in \widetilde{H}$ obtain $\left\|\widetilde{\tilde{v}_{f_{G(e_1)}} + \tilde{u}_{f_{G(e_2)}}}\right\| \gtrsim \left\|\widetilde{\tilde{v}_{f_{G(e_1)}}}\right\| + \left\|\widetilde{\tilde{u}_{f_{G(e_2)}}}\right\|$.

The FS-V $\widetilde{H}$ with $\|\widetilde{.}\|$ indicated by ($\widetilde{H}$, $\|\widetilde{.}\|$),namely fuzzy soft normed space (FS-N space).

Definition 2.7. [34] Assume $\widetilde{H}$ be FS-V and $(\mathcal{A})$ be fuzzy soft complex set. Then, the mapping $\widetilde{\langle .,. \rangle}$:$\widetilde{H} \times \widetilde{H} \to (\mathcal{A})$, called fuzzy soft inner product (FS-IP) on $\widetilde{H}$ if $\widetilde{\langle .,. \rangle}$ achieves the following:

i. for all $\tilde{v}_{f_{G(e)}} \in \widetilde{H}$ yields $\langle \tilde{v}_{f_{G(e)}}, \tilde{v}_{f_{G(e)}} \rangle \cong \tilde{0}$;

ii. $\langle \tilde{v}_{f_{G(e)}}, \tilde{v}_{f_{G(e)}} \rangle \cong \tilde{0}$ if and only if $\tilde{v}_{f_{G(e)}} \cong \tilde{0}$;

iii. for all $\tilde{v}^1_{f_1 G(e_1)}$, $\tilde{v}^2_{f_2 G(e_2)} \in \widetilde{H}$ gains $\langle \tilde{v}^1_{f_1 G(e_1)}, \tilde{v}^2_{f_2 G(e_2)} \rangle \cong \overline{\langle \tilde{v}^2_{f_2 G(e_2)}, \tilde{v}^1_{f_1 G(e_1)} \rangle}$;

iv. for all $\tilde{v}^1_{f_1 G(e_1)}$, $\tilde{v}^2_{f_2 (e_2)} \in \widetilde{H}$ *and* $\tilde{r} \in \mathcal{C}(\mathcal{A})$, attains $\langle \tilde{r}\, \tilde{v}^1_{f_1 G(e_1)}, \tilde{v}^2_{f_2 G(e_2)} \rangle \cong \tilde{r} \langle \tilde{v}^1_{f_1 G(e_1)}, \tilde{v}^2_{f_2 G(e_2)} \rangle$;

v. for all $\tilde{v}^1_{f_1 G(e_1)}$, $\tilde{v}^2_{f_2 G(e_2)}$, $\tilde{v}^3_{f_3 G(e_3)}, \in \widetilde{H}$ we yield $\langle \tilde{v}^1_{f_1 G(e_1)} + \tilde{v}^2_{f_2 G(e_2)}, \tilde{v}^3_{f_3 G(e_3)} \rangle \cong \langle \tilde{v}^1_{f_1 G(e_1)}, \tilde{v}^3_{f_3 G(e_3)} \rangle + \langle \tilde{v}^2_{f_2 G(e_2)}, \tilde{v}^3_{f_3 G(e_3)} \rangle$.

The FS-V $\widetilde{H}$ with $\widetilde{\langle .,. \rangle}$ indicated by $(\widetilde{H}, \widetilde{\langle .,. \rangle})$namely fuzzy soft inner product space (FS-IP space).

Definition 2.8. [36] A fuzzy soft normed space $(\widetilde{H}, \widetilde{\|.\|})$ is called fuzzy soft complete if every fuzzy soft Cauchy sequence in it is fuzzy soft convergent sequence in it.

Theorem 2.9. [35] The fuzzy soft inner product space $(\widetilde{U}, \widetilde{\langle .,. \rangle}$ can be considered a fuzzy soft normed space with the fuzzy soft norm $\left\| \tilde{v}_{f_{G(e)}} \right\| \cong \sqrt{\langle \tilde{v}_{f_{G(e)}}, \tilde{v}_{f_{G(e)}} \rangle}$ for all $\tilde{v}_{f_{G(e)}} \in \mathrm{FSV}(\widetilde{U})$.

Definition 2.10. [34] Assume $(\widetilde{H}, \widetilde{\langle .,. \rangle})$ be a fuzzy soft inner product space, then is fuzzy soft Hilbert space when its fuzzy soft complete, denoted by FSH-space.

Definition 2.11. [25] Let $\widetilde{H}$ be FSH-space, the operator $\widetilde{\Lambda}: \widetilde{H} \to \widetilde{H}$, named fuzzy soft linear operator (FS-operator), if for all $\tilde{v}^1_{f_1 G(e_1)}$, $\tilde{v}^2_{f_2 G(e_2)} \in \widetilde{H}$ and $\tilde{\sigma}, \tilde{\rho} \in \mathcal{C}(\mathcal{A})$, then

$\widetilde{\Lambda}(\tilde{\sigma}\, \tilde{v}^1_{f_1 G(e_1)} +, \tilde{\rho}\, \tilde{v}^2_{f_2 G(e_2)}) \cong \tilde{\sigma}\widetilde{\Lambda}\left(\tilde{v}^1_{f_1 G(e_1)} \right) + \tilde{\rho}\widetilde{\Lambda}\left(\tilde{v}^2_{f_2 G(e_2)} \right)$.

Definition 2.12. [35] Assume $\widetilde{H}$ be FSH-space and $\widetilde{\Lambda}$: $\widetilde{H} \to \widetilde{H}$ be FS-operator named fuzzy soft bounded operator (FSB-operator), if $\exists\ \tilde{\mathfrak{L}} \tilde{>} \tilde{0} \in \tilde{R}_A$ such that $\left\| \widetilde{\Lambda}\left(\tilde{v}_{f_{G(e)}} \right) \right\| \tilde{\leq} \tilde{\mathfrak{L}} \left\| \tilde{v}_{f_{G(e)}} \right\|$ for all $\tilde{v}_{f_{G(e)}} \in \widetilde{H}$.

Definition 2.13. [36] Assume $\widetilde{H}$ be FSH-space and $\widetilde{\Lambda}: \widetilde{H} \to \widetilde{H}$ be fuzzy soft bounded operator, then the fuzzy soft adjoint operator $\widetilde{\Lambda}^*$is defined by

$\langle \widetilde{\Lambda}\, \tilde{v}^1_{f_1 G(e_1)}, v^2_{f_2 G(e_2)} \rangle \cong \langle \tilde{v}^1_{f_1 G(e_1)}, \widetilde{\Lambda^*}\, \tilde{v}^2_{f_2 G(e_2)} \rangle$, $\forall\ \tilde{v}^1_{f_1 G(e_1)}, \tilde{v}^2_{f_2 G(e_2)} \in \widetilde{H}$.

3. Main outcomes

In this section we present a new class of fuzzy soft hyponormal operators by utilizing the Drazin inverse is called the fuzzy soft power Drazin inverse hyponormal operator.

At first, the definition that showed in [9] it generalized in the following:

Definition 3.1. Assume FS operator $\widetilde{\Lambda} \in \mathrm{FSB}(\widetilde{H})$ be FS Drazin invertible if there exists a unique FS operator $\widetilde{\Lambda}^D \in \mathrm{FSB}(\widetilde{H})$. $\widetilde{\Lambda}^D$is the FS Drazin invertible of $\widetilde{\Lambda}$ verifting:

1- $\widetilde{\Lambda}\widetilde{\Lambda}^D \cong \widetilde{\Lambda}^D\widetilde{\Lambda}$;

2- $\widetilde{\Lambda}^D\widetilde{\Lambda}\widetilde{\Lambda}^D \cong \widetilde{\Lambda}^D$;

3- $\widetilde{\Lambda}^{n+1} \cong \widetilde{\Lambda}^n$ for some n∈N .

Definition 3.2. An operator $\widetilde{\Lambda} \cong \tilde{B}(\widetilde{H})$, named fuzzy soft (D,n)-normal operator if $\widetilde{\Lambda}^{*n}\,\widetilde{\Lambda}^{D}$- $\widetilde{\Lambda}^{D}\widetilde{\Lambda}^{*n} \cong \widetilde{0}$, for all n ∈ ℕ.

Definition 3.3. Assume that $\widetilde{\Lambda}\in\widetilde{B}(\widetilde{H})^{D}$ be named fuzzy soft (D,n)-hyponormal operator if $\widetilde{\Lambda}^{*n}\,\widetilde{\Lambda}^{D} \cong \widetilde{\Lambda}^{D}\widetilde{\Lambda}^{*n}$, for all n ∈ ℕ.

Example 3.4. The $\widetilde{\Lambda} \cong \begin{bmatrix}(0.5,0) & (0.2,0)\\ (0.3,0) & (0.1,0)\end{bmatrix}$ is fuzzy soft (D,n)-hyponormal operator, since $\widetilde{\Lambda}^{*} \cong \begin{bmatrix}(0.5-,0) & (0.2,0)\\ (0.3,0) & (0.1-,0)\end{bmatrix}$ for $n = 2 \in \mathbb{N}$, then $\widetilde{\Lambda}^{*2} \cong \begin{bmatrix}(0.25-,0) & (0.14,0)\\ (0.09,0) & (0.05-,0)\end{bmatrix}$ and $\widetilde{\Lambda}^{D} \cong \begin{bmatrix}(0.1-,0) & (0.6,0)\\ (0.4,0) & (0.2-,0)\end{bmatrix}$, then $\widetilde{\Lambda}^{*2}\,\widetilde{\Lambda}^{D}$- $\widetilde{\Lambda}^{D}\widetilde{\Lambda}^{*2} \cong \widetilde{0}$.

Theorem 3.5. Assume $\widetilde{\Lambda}$: $\widetilde{H} \to \widetilde{H}$ be a FS-(D,n)-hyponormal operator on FSH-space $\widetilde{H}$, Then $(\widetilde{\Lambda}^{*n}\,\widetilde{\Lambda}^{D})^{m} \cong (\widetilde{\Lambda}^{D}\widetilde{\Lambda}^{*n})^{m}$ be FS-(D,n)-hyponormal operator.

Proof: By utilizing mathematical induction, since $\widetilde{\Lambda}$ is an FS-(D,n)-hyponormal operator, then $(\widetilde{\Lambda}^{*n}\,\widetilde{\Lambda}^{D})^{m} \cong (\widetilde{\Lambda}^{D}\widetilde{\Lambda}^{*n})^{m}$ is FS-(D,n)-hyponormal operator.

For m=1, then $(\widetilde{\Lambda}^{*n}\,\widetilde{\Lambda}^{D})^{1} \cong (\widetilde{\Lambda}^{D}\widetilde{\Lambda}^{*n})^{1}$. … (i)

Given this outcome is attained for m=k, consequently

$(\widetilde{\Lambda}^{*n}\,\widetilde{\Lambda}^{D})^{k} \cong (\widetilde{\Lambda}^{D}\widetilde{\Lambda}^{*n})^{k}$. … (ii)

Then, to demonstrate the validity of the outcome for m=k+1,

$(\widetilde{\Lambda}^{*n}\,\widetilde{\Lambda}^{D})^{k+1} \cong (\widetilde{\Lambda}^{D}\widetilde{\Lambda}^{*n})^{k}(\widetilde{\Lambda}^{D}\widetilde{\Lambda}^{*n})^{1}$,

from Equality (i) and Equality (ii), we yield

$(\widetilde{\Lambda}^{*n}\,\widetilde{\Lambda}^{D})^{k+1} \cong (\widetilde{\Lambda}^{D}\widetilde{\Lambda}^{*n})^{k+1}$.

Therefore, $(\widetilde{\Lambda}^{*n}\,\widetilde{\Lambda}^{D})^{m} \cong (\widetilde{\Lambda}^{D}\widetilde{\Lambda}^{*n})^{m}$ is an FS-(D,n)-hyponormal operator.

Theorem 3.6. Assume $\widetilde{\Lambda}$ and $\widetilde{W}$ be FS-(D,n)-hyponormal operator on FSH-space $\widetilde{H}$. If $\widetilde{W}\widetilde{\Lambda} \cong \widetilde{\Lambda}\widetilde{W}$ and $\widetilde{\Lambda}^{*n}\widetilde{W} \cong \widetilde{W}\widetilde{\Lambda}^{*n}$. Then $(\widetilde{\Lambda} + \widetilde{W})$ is FS-(D,n)-hyponormal operator.

Proof: Since $\widetilde{\Lambda}$ and $\widetilde{W}$ are FS-(D,n)- hyponormal then $\widetilde{W}\widetilde{\Lambda} \cong \widetilde{\Lambda}\widetilde{W}$ and $\widetilde{\Lambda}^{*n}\widetilde{W} \cong \widetilde{W}\widetilde{\Lambda}^{*n}$.

$$\begin{aligned}(\widetilde{\Lambda} + \widetilde{W})^{*n}\,(\widetilde{\Lambda} + \widetilde{W})^{D} &\cong (\widetilde{\Lambda}^{*n} + \widetilde{W}^{*n})(\widetilde{\Lambda}^{D} + \widetilde{W}^{D})\\ &\cong \widetilde{\Lambda}^{*n}\widetilde{\Lambda}^{D}+\widetilde{\Lambda}^{*n}\widetilde{W}^{D} + \widetilde{W}^{*n}\widetilde{\Lambda}^{D}+\widetilde{W}^{*n}\widetilde{W}^{D}\\ &\cong \widetilde{\Lambda}^{D}\widetilde{\Lambda}^{*n}+\widetilde{W}^{D}\widetilde{\Lambda}^{*n} + \widetilde{\Lambda}^{D}\widetilde{W}^{*n}+\widetilde{W}^{D}W^{*n}\\ &\cong (\widetilde{\Lambda}^{D} + \widetilde{W}^{D})\,(\widetilde{\Lambda}^{*n} + \widetilde{W}^{*n})\\ &\cong (\widetilde{\Lambda} + \widetilde{W})^{D}(\widetilde{\Lambda} + \widetilde{W})^{*n}.\end{aligned}$$

Hence, $(\widetilde{\Lambda} + \widetilde{W})$ is an FS-(D,n)-hyponormal operator.

Theorem 3.7. Assume $\widetilde{\Lambda}$ and $\widetilde{W}$ be FS-(D,n)-hyponormal operator on FSH-space $\widetilde{H}$. If $\widetilde{W}\widetilde{\Lambda} \cong \widetilde{\Lambda}\widetilde{W}$ and $\widetilde{\Lambda}^{*n}\widetilde{W} \cong \widetilde{W}\widetilde{\Lambda}^{*n}$ Then, $\widetilde{\Lambda}\widetilde{W}$ is an FS-(D,n)-hyponormal operator.

Proof: Since $\widetilde{\Lambda}$ and $\widetilde{W}$ are FS-(D,n)-hyponormal operator then $\widetilde{W}\widetilde{\Lambda} \cong \widetilde{\Lambda}\widetilde{W}$ and $\widetilde{\Lambda}^{*n}\widetilde{W} \cong \widetilde{W}\widetilde{\Lambda}^{*n}$.

$$\begin{aligned}(\widetilde{\Lambda}\widetilde{W})^{*n}\,(\widetilde{\Lambda}\widetilde{W})^{D} \cong\ \widetilde{\Lambda}^{*n}\widetilde{W}^{*n}\,\widetilde{\Lambda}^{D}\widetilde{W}^{D} &\cong \widetilde{\Lambda}^{*n}\,\widetilde{\Lambda}^{D}\,\widetilde{W}^{*n}\widetilde{W}^{D}\\ &\cong \widetilde{\Lambda}^{D}\widetilde{\Lambda}^{*n}\,\widetilde{W}^{D}\,\widetilde{W}^{*n}\\ &\cong \widetilde{\Lambda}^{D}\widetilde{W}^{D}\widetilde{\Lambda}^{*n}\,\widetilde{W}^{*n}\\ &\cong (\widetilde{\Lambda}\widetilde{W})^{D}(\widetilde{\Lambda}\widetilde{W})^{*n}.\end{aligned}$$

Hence, $\widetilde{\Lambda}\widetilde{W}$ is an FS-(D,n)-hyponormal operator.

Theorem 3.8. Assume $\widetilde{\Lambda}_1, \widetilde{\Lambda}_2, \widetilde{\Lambda}_3, \dots\dots, \widetilde{\Lambda}_m$ be FS-(D,n)-hyponormal operator in $\tilde{B}(\widetilde{H})^D$ $\forall\, k = 1,2,3, \dots\dots, m$. Then, $(\widetilde{\Lambda}_1 \otimes \widetilde{\Lambda}_2 \otimes \dots\dots \otimes \widetilde{\Lambda}_m)$ is an FS-(D,n)-hyponormal operator.

Proof:

$(\widetilde{\Lambda}_1 \otimes \widetilde{\Lambda}_2 \otimes \dots.\otimes \widetilde{\Lambda}_m)^{*n}(\widetilde{\Lambda}_1 \otimes \widetilde{\Lambda}_2 \otimes \dots.\otimes \widetilde{\Lambda}_m)^D \cong (\widetilde{\Lambda}_1^{*n} \otimes \widetilde{\Lambda}_2^{*n} \otimes \dots.\otimes \widetilde{\Lambda}_m^{*n})(\widetilde{\Lambda}_1^D \otimes \widetilde{\Lambda}_2^D \otimes \dots.\otimes \widetilde{\Lambda}_m^D)$

$$\cong (\widetilde{\Lambda}_1^{*n}\widetilde{\Lambda}_1^D \otimes \widetilde{\Lambda}_2^{*n}\widetilde{\Lambda}_2^D \otimes \dots.\otimes \widetilde{\Lambda}_m^{*n}\widetilde{\Lambda}_m^D)$$
$$\gtrsim (\widetilde{\Lambda}_1^D\widetilde{\Lambda}_1^{*n} \otimes \widetilde{\Lambda}_2^D\widetilde{\Lambda}_2^{*n} \otimes \dots.\otimes \widetilde{\Lambda}_m^D\widetilde{\Lambda}_m^{*n})$$
$$\cong (\widetilde{\Lambda}_1^D \otimes \widetilde{\Lambda}_2^D \otimes \dots.\otimes \widetilde{\Lambda}_m^D)(\widetilde{\Lambda}_1^{*n} \otimes \widetilde{\Lambda}_2^{*n} \otimes \dots.\otimes \widetilde{\Lambda}_m^{*n}).$$

Hence, $(\widetilde{\Lambda}_1 \otimes \widetilde{\Lambda}_2 \otimes \dots\dots \otimes \widetilde{\Lambda}_m)$ is an FS-(D,n)-hyponormal operator.

Theorem 3.9. Assume $\widetilde{\Lambda}_1, \widetilde{\Lambda}_2, \widetilde{\Lambda}_3, \dots\dots, \widetilde{\Lambda}_m$ be FS-(D,n)-hyponormal operator in $\tilde{B}(\widetilde{H})^D$ for all $k = 1,2,3, \dots, m$, then $(\widetilde{\Lambda}_1 \oplus \widetilde{\Lambda}_2 \oplus \dots\dots \oplus \widetilde{\Lambda}_m)$ is an FS-(D,n)-hyponormal operator .

Proof:

$(\widetilde{\Lambda}_1 \oplus \widetilde{\Lambda}_2 \oplus \dots.\oplus \widetilde{\Lambda}_m)^{*n}(\widetilde{\Lambda}_1 \oplus \widetilde{\Lambda}_2 \oplus \dots.\oplus \widetilde{\Lambda}_m)^D \cong (\widetilde{\Lambda}_1^{*n} \oplus \widetilde{\Lambda}_2^{*n} \oplus \dots.\oplus \widetilde{\Lambda}_m^{*n})(\widetilde{\Lambda}_1^D \oplus \widetilde{\Lambda}_2^D \oplus \dots.\oplus \widetilde{\Lambda}_m^D)$

$$\cong (\widetilde{\Lambda}_1^{*n}\widetilde{\Lambda}_1^D \oplus \widetilde{\Lambda}_2^{*n}\widetilde{\Lambda}_2^D \oplus \dots.\oplus \widetilde{\Lambda}_m^{*n}\widetilde{\Lambda}_m^D)$$
$$\gtrsim (\widetilde{\Lambda}_1^D\widetilde{\Lambda}_1^{*n} \oplus \widetilde{\Lambda}_2^D\widetilde{\Lambda}_2^{*n} \oplus \dots.\oplus \widetilde{\Lambda}_m^D\widetilde{\Lambda}_m^{*n})$$
$$\cong (\widetilde{\Lambda}_1^D \oplus \widetilde{\Lambda}_2^D \oplus \dots.\oplus \widetilde{\Lambda}_m^D)(\widetilde{\Lambda}_1^{*n} \oplus \widetilde{\Lambda}_2^{*n} \oplus \dots.\oplus \widetilde{\Lambda}_m^{*n}).$$

Hence, $(\widetilde{\Lambda}_1 \oplus \widetilde{\Lambda}_2 \oplus \dots\dots \oplus \widetilde{\Lambda}_m)$ is an FS-(D,n)-hyponormal operator.

Theorem 3.10. If $\widetilde{\Lambda}$, $\widetilde{W} \in$ FS-(D,n)-hyponormal operator, then $(\widetilde{\Lambda} \oplus \widetilde{W}) \in$ FS-(D,n)-hyponormal operator .

Proof: Assume $\widetilde{\Lambda}, \widetilde{W} \in$ FS-(D,n)-hyponormal operator, then:

$(\widetilde{\Lambda} \oplus \widetilde{W})^{*n}\,(\widetilde{\Lambda} \oplus \widetilde{W})^D \cong (\widetilde{\Lambda}^{*n} \oplus \widetilde{W}^{*n})(\widetilde{\Lambda}^D \oplus \widetilde{W}^D)$

$$\cong \widetilde{\Lambda}^{*n}\widetilde{\Lambda}^D \oplus \widetilde{W}^{*n}\widetilde{W}^D$$
$$\gtrsim \widetilde{\Lambda}^D\widetilde{\Lambda}^{*n} \oplus \widetilde{W}^D\widetilde{W}^{*n}$$
$$\cong (\widetilde{\Lambda}^D \oplus \widetilde{W}^D)\,(\widetilde{\Lambda}^{*n} \oplus \widetilde{W}^{*n})$$
$$\cong\ (\widetilde{\Lambda} \oplus \widetilde{W})^D(\widetilde{\Lambda} \oplus \widetilde{W})^{*n}.$$

Hence, $(\widetilde{\Lambda} \oplus \widetilde{W})$ is of class FS-(D,n)- hyponormal operator.

Theorem 3.11. Assume $\widetilde{\Lambda}$, $\widetilde{W} \in \widetilde{B}(\widetilde{H})^D$, if $\widetilde{W}$ is FS-(D,n)- hyponormal operator, and $\widetilde{\Lambda}$ is an FS-unitary equivalent to $\widetilde{W}$. Then, $\widetilde{\Lambda}$ FS-(D,n)- hyponormal operator.

Proof:

Assume $\widetilde{W}$ be an FS-(D,n)- hyponormal operator and $\widetilde{\Lambda} \in \tilde{B}(\widetilde{H})^D$ is FS-unitary equivalent to $\widetilde{W}$.

there exists a FS-unitary equivalent $\tilde{V} \in \tilde{B}(\widetilde{H})$,fulfilling $\widetilde{\Lambda} \cong \tilde{V}^*\widetilde{W}\tilde{V}$. So, $\widetilde{\Lambda}^{*n} \cong \tilde{V}^*\widetilde{W}^{*n}\tilde{V}$ and $\widetilde{\Lambda}^D \cong \tilde{V}^*\widetilde{W}^D\tilde{V}$.

We obtain

$$\widetilde{\Lambda}^{*n}\,\widetilde{\Lambda}^D \cong \tilde{V}^*\widetilde{W}^{*n}\tilde{V}\tilde{V}^*\widetilde{W}^D\tilde{V}$$
$$\cong\ \tilde{V}^*\widetilde{W}^{*n}\widetilde{W}^D\tilde{V}$$
$$\gtrsim \tilde{V}^*\widetilde{W}^D\widetilde{W}^{*n}\tilde{V}$$
$$\gtrsim \tilde{V}^*\widetilde{W}^D\tilde{V}\tilde{V}^*\widetilde{W}^{*n}\tilde{V}$$
$$\cong\ \widetilde{\Lambda}^D\widetilde{\Lambda}^{*n}.$$

Hence , $\widetilde{\Lambda}^{*n}\,\widetilde{\Lambda}^D \gtrsim \widetilde{\Lambda}^D\widetilde{\Lambda}^{*n}$.

4. Conclusions

In this paper, a modest attempt is made to present and study a certain class of fuzzy soft operators defined on FS-H. A novel class of fuzzy soft power Drazin inverse hyponormal operators is identified and presented. They proved the identical condition for this considered fuzzy operator. Moreover, the analytic and algebraic merits of this new operator are examined. The n-times direct sum and tensor product for this new fuzzy operator are also discussed.

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