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New Family of Two-Step Iterative Methods Based on Heron Mean for Solving Nonlinear Equations

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Abstract

In this paper, we propose and analyze a new one-parameter family of two-step iterative methods without requiring higher derivatives for solving nonlinear equations. This family is constructed based on a Newton-type method utilizing the Heron mean, a predictor-corrector technique, and an appropriate choice of derivative approximations. Each iteration of the new methods requires two function evaluations and one first derivative evaluation. Convergence analysis shows that the proposed methods achieve third- and fourth-order convergence, with corresponding efficiency indices of 1.44225 and 1.58740, respectively. Numerical examples are provided to demonstrate the performance and efficiency of the proposed methods.

Keywords: Nonlinear equations; Newton-type method; Heron mean; Order of convergence; Efficiency index.

عائلة جديدة من الطرائق التكرارية ذات الخطوتين ترتكز على متوسط هيرون لحل المعادلات غير الخطية

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الخلاصة

في هذا البحث، نقترح ونحلل عائلة جديدة من الطرائق التكرارية ذات الخطوتين بالاستناد على معامل واحد، ولا تتطلب مشتقات من رتب عليا لحل المعادلات غير الخطية. تم بناء هذه العائلة استنادًا إلى طريقة من نوع نيوتن باستخدام متوسط هيرون، تقنية المخمن–المصحح واختيار مناسب لتقريبات المشتقة. في كل تكرار، تتطلب الطرائق الجديدة حسابين للدالة وتقييماً واحداً للمشتقة الأولى. تُظهر تحليلات التقارب أن الطرائق المقترحة تحقق تقارباً من الرتبة الثالثة والرابعة، وبالتالي فإن مؤشرات الكفاءة المقابلة لها هي 1.44225 و1.58740 على التوالي. وقد تم تقديم أمثلة عددية لتوضيح أداء وكفاءة الطرائق المقترحة.

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1. Introduction

Nonlinear equations play a pivotal role in modeling and solving complex phenomena in science, engineering, and applied mathematics. Unlike linear systems, nonlinear equations are often intractable through analytical means, necessitating the use of numerical analysis and iterative methods for approximate solutions. The challenge amplifies when dealing with multiple roots or roots of unknown multiplicity, requiring robust, efficient, and accurate methods.

Over the years, the development of derivative-free, high-order, and simultaneous iterative schemes has advanced significantly. Early works focused on improving classical approaches such as Newton's and Householder's methods to cater to multiple roots and enhance convergence, [1-3]. More recent research introduced methods tailored for multiple zeros with unknown multiplicity using optimal convergence orders and minimal function evaluations, [4-7].

In 2020, derivative-free methods such as modified King's family [2] and simultaneous schemes for distinct root-finding [3] laid the foundation for more optimal techniques. These were extended in 2021 through the development of robust iterative families and Caputo-type fractional schemes, targeting higher accuracy and engineering applications, [4-9]. The 2022 contributions introduced fourth-order derivative-free schemes and simultaneous methods for all root determination, furthering applicability in real-world nonlinear models [10], [11]. In 2023, neural-network-based and fractional-order models emerged as novel computational approaches, blending machine learning with classical numerical methods to address engineering-specific nonlinearities [12], [13].

The pace of innovation continued into 2024–2025. Researchers proposed eighth-order methods using bivariate weight functions, robust iterative families handling critical points [14], and efficient schemes based on hybrid artificial neural networks [15], [16]. Moreover, modified two-step optimal iterative methods have been designed to tackle multi-dimensional nonlinear systems with increased stability and performance [17], [18].

These collective efforts underscore a transformative shift in solving nonlinear equations: from simple iteration to complex, tailored, and intelligent schemes optimized for precision, efficiency, and adaptability. This evolution is not only theoretical but also reflects in real-world engineering, control systems, and computational modeling problems.
One of the most important problems in numerical analysis is finding a simple root α of the nonlinear equation:

$$f(x) = 0, \quad (1)$$

where $f: D \subseteq \mathbb{R} \to \mathbb{R}$ is a scalar function on an open interval D, and it is sufficiently smooth in a neighbourhood of its root α with $f'(\alpha) \neq 0$. Newton's method is the best well-known iterative method for finding α by using

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \text{ for } n = 0, 1, \dots . \quad (2)$$

This method is quadratically convergent in some neighborhoods α.
Many researchers have presented several modifications of Newton's method with higher-order convergence by using some efficient techniques (see [19-24] and the references therein). For example, Weerakoon and Fernando [25] proposed a variant of Newton's method based on a trapezoidal integral rule, which has third-order convergence with an efficiency index equal to 1.44225, defined by

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)+f'(y_n)}, \quad y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \text{ for } n = 0, 1, \dots . \quad (3)$$

In [26], Özban modified the Weerakoon and Fernando method by using the arithmetic mean with third-order convergence and efficiency index equal to 1.44225, given by

$$x_{n+1} = x_n - \frac{f(x_n)}{\frac{f'(x_n)+f'(y_n)}{2}}, \quad y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad \text{for } n = 0,1, \dots . \quad (4)$$

Tibor Lukic et al., [27] provided Özban's method by replacing the arithmetic mean with the geometric mean in the denominator of formula (3). This method converges cubically, and its efficiency index equals 1.44225, as defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{\operatorname{sign}(f'(x_0))\sqrt{f'(x_n)f'(y_n)}}, \quad y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad \text{for } n = 0,1, \dots . \quad (5)$$

Similarly, the harmonic mean can be used in place of the arithmetic or geometric mean to construct another third-order method [27]. The harmonic mean-based method is defined as

$$x_{n+1} = x_n - \frac{f(x_n)}{\frac{2f'(x_n)f'(y_n)}{f'(x_n)+f'(y_n)}}, \quad y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad \text{for } n = 0,1, \dots . \quad (6)$$

By adopting the contra-harmonic mean in the denominator, another variant was presented by Osama [28] with cubic convergence and the same efficiency index and is formulated as

$$x_{n+1} = x_n - \frac{f(x_n)}{\frac{f'(x_n)^2+f'(y_n)^2}{f'(x_n)+f'(y_n)}}, \quad y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad \text{for } n = 0,1, \dots . \quad (7)$$

Finally, Verma [29] used the centroid mean of the form $\{\frac{f'(x_n)+\sqrt{f'(x_n)f'(y_n)}+f'(y_n)}{3}\}$, to present a further method with third-order convergence and the same efficiency index, which is given by

$$x_{n+1} = x_n - \frac{3\,f(x_n)}{f'(x_n)+\sqrt{f'(x_n)f'(y_n)}+f'(y_n)}, \quad y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad \text{for } n = 0,1, \dots . \quad (8)$$

In this paper, we present a new one-parameter family of iterative methods based on Newton's -type method using the Heron mean for solving nonlinear equations. Per iteration, new methods require two evaluations of the function and one of its first derivative. Analytically, we will prove that the new methods are of order three and four convergence. Furthermore, new methods have efficiency indices equal to 1.44225 and 1.58740, depending on the value of the parameter. Some numerical examples illustrate that the new methods perform better than classical Newton's method and other existing methods.

The paper is structured as follows. Section 2 provides the necessary preliminaries, including key definitions and remarks. In Section 3, we construct the new iterative methods based on the Heron mean. Section 4 presents the convergence analysis. Section 5 includes numerical examples to evaluate the performance of the proposed methods. Section 6 contains a detailed discussion of the results, and finally, Section 7 concludes the paper.

2. Preliminaries

In this section, we recall some important definitions, notations, and remarks that are essential for our study. These have been adapted from references [25], [26], and [27].

Definition 2.1: Let $f: D \subseteq \mathbb{R} \to \mathbb{R}$ be a scalar function on an open interval D with a simple root α of the nonlinear equation. An iterative method is said to have an integer order of convergence p if it produces the sequence $\{x_n\}$ of real numbers such that

$\lim\limits_{x\to\infty} \frac{(x_{n+1}-\alpha)}{(x_n-\alpha)^p} = \mathrm{C}$, for some $C \neq 0$ and $p \geq 1$, or equivalently

$$x_{n+1} - \alpha = C(x_n - \alpha)^p + O((x_n - \alpha)^{p+1}). \quad (9)$$

Then, the order of convergence of the sequence is said to be of order p.

Notation 2.2: Let $e_n = x_n - \alpha$ is the error in the nth iteration. Then the relation

$$e_{n+1} = C\, e_n^p + O\left(e_n^{p+1}\right) \qquad (10)$$

is called the error equation for the method.

Definition 2.3: The Efficiency index is simply defined as

$$E.I. = p^{1/m}, \qquad (11)$$

where p is the order of the method, and m is the number of function evaluations required by the method (units of work per iteration).

Definition 2.4: Let α be a root of the function $f(x) = 0$, and suppose that x_{n-1}, x_n and x_{n+1} are three successive iterations closer to the exact root α. Then, the computational order of convergence (COC) denoted by ρ can be approximated using the formula:

$$\rho = \frac{\ln\left|\frac{(x_{n+1}-\alpha)}{x_n-\alpha}\right|}{\ln|(x_n-\alpha)/(x_{n-1}-\alpha)|} \quad .$$

(12)

Definition 2.5: In mathematics, the Heron mean of two positive real numbers a and b is defined as:

$$F_\lambda(a, b) = (1-\lambda)\sqrt{a.b} + \lambda \frac{a+b}{2}.$$

(13)

Remark 2.6: The Heron mean is the arithmetic mean when $\lambda = 1$, while when $\lambda = 0$, the Heron mean is the Geometric mean.

3 Construction of new iterative methods

To construct an efficient one-parameter family of third and fourth-order iterative methods, let us consider the Özban method, Formula (4). Using the Heron mean instead of the arithmetic mean in the denominator of Formula (4), we suggest a new one-parameter family of two-step iterative methods for solving nonlinear equation (1), which is given as:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = x_n - \frac{2f(x_n)}{2(1-\lambda)\sqrt{f'(x_n)f'(y_n)} + \lambda\left(f'(x_n) + f'(y_n)\right)}, n = 0, 1, 2, \ldots \quad .$$

(14)

Remark 3.1: For $\lambda = 1$, Formula (14) reduces to the Arithmetic mean Newton's method (4), while for $\lambda = 0$, we have the Geometric mean Newton's method (5).

To improve the order of convergence of the Formula (14), we use an approximation to $f'(y_n)$, defined as

$$f'(y_n) \approx f'(x_n)\left(\frac{f(x_n)}{f(x_n) + f(y_n)}\right)^2, \qquad n = 0, 1, 2, \ldots .$$

(15)

Substituting Formula (15) into (14), we suggest a new one-parameter family of iterative methods, given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left\{\frac{2}{2(1-\lambda)\, N_f + \lambda\left(1+N_f^2\right)}\right\}, n = 0, 1, 2, \ldots, \qquad (16)$$

where $N_f = \frac{f(x_n)}{f(x_n) + f(y_n)}$, $n = 0, 1, 2, \ldots$.

Remark 3.2: The family defined by Formula (16) includes, as particular cases, the following ones:

For $\lambda = 0$, Formula (14) reduces to the known third-order method, given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = x_n - \frac{f(x_n) + f(y_n)}{f'(x_n)}, n = 0, 1, \dots . \quad (17)$$

For $\lambda = 0.5$, we obtain a new third-order method, defined as

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \left\{ \frac{4\big(f(x_n) + f(y_n)\big)^2}{4f(x_n)^2 + 4f(x_n)f(y_n) + f(y_n)^2} \right\}, n = 0, 1, 2, \dots . \quad (18)$$

For $\lambda = 1$, we obtain a new third-order method, defined as

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \left\{ \frac{2\big(f(x_n) + f(y_n)\big)^2}{2f(x_n)^2 + 2f(x_n)f(y_n) + f(y_n)^2} \right\}, n = 0, 1, 2, \dots . \quad (19)$$

For $\lambda = -4$, we obtain a new fourth-order method, defined as

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \left\{ \frac{\big(f(x_n) + f(y_n)\big)^2}{f(x_n)^2 + f(x_n)f(y_n) - 2f(y_n)^2} \right\}, n = 0, 1, 2, \dots . \quad (20)$$

4 Analysis of convergence

Hereunder, we consider the convergence analysis of the proposed methods defined by Formulas (18)-(20) and discuss the optimal choices of the parameter λ by the following theorem:

Theorem 4.1: Assume that $\alpha \in D$ is a simple zero of a sufficiently differentiable function $f: D \subseteq \mathbb{R} \to \mathbb{R}$. If x_0 is sufficiently close to α. Then, the family, which is defined by formula (16), includes some iterative methods with convergence of third-order to α for any $\lambda \in \mathbb{R} \backslash \{-4\}$ and has optimal fourth-order convergence for $\lambda = -4$.

Proof: By using Taylor's expansion around $x = \alpha$ and taking into account $f(\alpha) = 0$, we have

$$f(x_n) = f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + \cdots] \quad (21)$$

$$f'(x_n) = f'(\alpha)[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + \cdots], \quad (22)$$

where

$$c_k = \frac{f^{(k)}(\alpha)}{k! f'(\alpha)}, \; k = 2,3,4, \dots . \quad (23)$$

Combining Equations (21) and (22), we have

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + (2c_2^2 - 2c_3)e_n^3 + (-4c_2^3 + 7c_2 c_3 - 3c_4)e_n^4 + \cdots . \quad (24)$$

Hence,

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + c_2 e_n^2 + (-2c_2^2 + 2c_3)e_n^3 + (4c_2^3 - 7c_2c_3 + 3c_4)e_n^4 + \cdots \quad (25)$$

and,

$$f(y_n) = f'(\alpha)[c_2 e_n^2 + (-2c_2^2 + 2c_3)e_n^3 + (5c_2^3 - 7c_2c_3 + 3c_4)e_n^4 + \cdots]. \quad (26)$$

From Equations (21) and (26), we have

$$N_f = \frac{f(x_n)}{f(x_n)+f(y_n)} = 1 - c_2 e_n + (4c_2^2 - 2c_3)e_n^2 + (-15c_2^3 + 14c_2c_3 - 3c_4)e_n^3 + (55c_2^4 - 75c_2^2c_3 + 20c_2c_4 + 12c_3^2 - 4c_5)e_n^4 + \cdots. \quad (27)$$

Substituting Equations (24) and (27) in Equation (14), we have

$$x_{n+1} = \alpha + \frac{1}{2}(\lambda+4)c_2^2 e_n^3 + \left[\frac{1}{2}(-7\lambda - 18)c_2^3 + \frac{1}{2}(4\lambda + 14)c_2c_3\right]e_n^4 + \left[\frac{1}{4}(-\lambda^2 + 66\lambda + 120)c_2^4 - \frac{1}{4}(76\lambda + 176)c_2^2c_3 + \frac{1}{4}(12\lambda + 40)c_2c_4 + 2(\lambda+3)c_3^2\right]e_n^5 + \cdots. \quad (28)$$

Thus, from Equation (28) and $e_{n+1} = x_{n+1} - \alpha$, we obtain

$$e_{n+1} = \frac{1}{2}(\lambda+4)c_2^2 e_n^3 + \left[\frac{1}{2}(-7\lambda - 18)c_2^3 + \frac{1}{2}(4\lambda + 14)c_2c_3\right]e_n^4 + O(e_n^5). \quad (29)$$

From the above equation, we see that the methods in Formulas (18) and (19) are cubically convergent for any $\lambda \in R \setminus \{-4\}$, while when $\lambda = -4$, the method in formula (20) has fourth-order convergence because the coefficient of e_n^3 vanishes (i.e. $\frac{1}{2}(\lambda+4) = 0$). ■

5 Numerical examples

In this section, we introduce some numerical examples to illustrate the efficiency of the proposed new methods in this work.

In the following, all computations were performed using Maple 2023 with 1000-digit floating-point arithmetic (`Digits:= 1000`) on a specific machine in normal mode with the following specifications: Windows 11 Pro, an 11th Gen Intel(R) Core(TM) i7-11800H @ 2.30 GHz processor, and 16.0 GB RAM (15.7 GB usable). The stopping criteria used in the computer programs are

$$|x_{n+1} - x_n| < \varepsilon \quad \text{and} \quad |f(x_{n+1})| < \varepsilon,$$

where $\varepsilon = 10^{-30}$.

In these numerical experiments, we compare our newly developed methods, HRM1, HRM2, and HRM3, given by Equations (18), (19), and (20), respectively, with the classical Newton's method (2) and the third-order methods based on various means: Arithmetic Mean (4), Geometric Mean (5), Harmonic Mean (6), Contra-Harmonic Mean (7), and Centroid Mean (8). These are denoted by ANM, GNM, HNM, CHNM, and CNM, respectively.

We also include, for comparison, the following optimal fourth-order iterative methods:

- Kou and Wang method (KW) [30]:

$$x_{n+1} = x_n - \frac{f^2(x_n) + f^2(y_n)}{f'(x_n)\,(f(x_n) - f(y_n))}, \quad y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \text{ for } n = 0, 1, \ldots. \quad (30)$$

- Artidiello, Chicharro, Cordero, and Torregrosa methods (ACCT1) and (ACCT2) [31]:

$$x_{n+1} = x_n - \frac{f^2(x_n) + f(x_n)\,f(y_n) + 2f^2(y_n)}{f(x_n)\,f'(x_n)}, \quad y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \text{ for } n = 0, 1, \ldots \quad (31)$$

and

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(x_n)}\frac{(f(x_n) + f(y_n))^2}{f^2(x_n) - 5\,f^2(y_n)}, \quad y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \text{ for } n = 0, 1, \ldots. \quad (32)$$

- Ghanbari method (GH) [32]:

$$x_{n+1} = y_n - \frac{\left(f(x_n)+2f(y_n)\right)^2}{\left(f(x_n)+f(y_n)\right)^2}\frac{f(y_n)}{f'(x_n)}, \quad y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \text{for } n = 0, 1, \dots . \tag{33}$$

Different test functions and their approximate root α found up to the 30th decimal places are given in Table 1. Table 2-6 show a comparison between new iterative methods and others with the same order depending on the number of iterations (IT), the values of $|f(x_{n+1})|$, the absolute difference $|x_{n+1} - x_n|$ to the approximation of the roots of the iterative method and computational order of convergence (COC).

Table 1: Test nonlinear functions, initial guesses, and zeros.

Nonlinear Function	Initial guess x_0	α
$f_1(x) = x^3 - x + 3$	7.0	-1.671699881657160969748149781220
$f_2(x) = x^5 + 23x - 6$	0.3	0.260817090224163287725959035087
$f_3(x) = e^x \sin(x) + \ln(1+x^2)$	3.5	3.237562984023921313250921300445
$f_4(x) = \cos(x) - x$	0.5	0.739085133215160641655312087674
$f_5(x) = \ln(x) - x^3 + 2\sin(x)$	15.0	1.297997743280371847164479238286 5

Table 2: Comparison of various two-step iterative methods using f_1 and $x_0 = 7.0$.

$f(x)$	x_0	Method	IT	$\lvert x_{n+1} - x_n\rvert$	$\lvert f(x_{n+1})\rvert$	COC
f_1	7.0	NM	38	9.91246656864238*10^{-24}	4.92768613089694*10^{-46}	2.000000
		ANM	NC	NC	NC	NC
		GNM	NC	NC	NC	NC
		HNM	NC	NC	NC	NC
		CHNM	NC	NC	NC	NC
		CNM	9	1.30818561053730*10^{-16}	1.12872398642682*10^{-47}	3.004184
		KW	22	2.22571777370086*10^{-18}	1.53660517743396*10^{-70}	3.965942
		ACCT1	94	8.91374599958261*10^{-19}	8.91374599958261*10^{-19}	3.966821
		ACCT2	11	3.70063802756527*10^{-23}	5.61284935500000*10^{-90}	3.962098
		GH	8	5.58924508439966*10^{-24}	1.18952097994867*10^{-69}	3.001024
		HRM1	14	1.51343142413674*10^{-20}	2.65676664189615*10^{-59}	2.998466
		HRM2	19	1.07905668232942*10^{-27}	1.06993017084608*10^{-80}	2.999816
		HRM3	8	4.79039997327662*10^{-44}	5.73408449194787*10^{-173}	3.999587

Table 3: Comparison of various two-step iterative methods using f_2 and $x_0 = 0.3$.

$f(x)$	x_0	Method	IT	$\lvert x_{n+1} - x_n\rvert$	$\lvert f(x_{n+1})\rvert$	COC
f_2	0.3	NM	4	2.84626367414577*10^{-26}	1.43733630880406*10^{-52}	2.000008
		ANM	3	1.63329252424082*10^{-20}	1.48790881395590*10^{-60}	3.013861
		GNM	3	1.61688614000889*10^{-20}	1.44062992895411*10^{-60}	3.013791
		HNM	3	1.60060308652915*10^{-20}	1.39473932247391*10^{-60}	3.013722
		CHNM	3	1.66647775382125*10^{-20}	1.58678575076264*10-^{60}	3.013998
		CNM	3	1.64429893499101*10^{-20}	1.52021823109655*10^{-60}	3.013907
		KW	3	1.02905877239705*10^{-40}	5.84318740343530*10^{-163}	4.024395
		ACCT1	3	9.99554285293591*10^{-41}	5.18029736353467*10^{-163}	4.024261
		ACCT2	3	1.09016709558474*10^{-40}	7.41925031248603*10^{-163}	4.024658
		GH	3	2.94664902737544*10^{-28}	6.99627698358274*10^{-86}	3.042950
		HRM1	3	3.27799384902564*10^{-28}	1.08357428194019*10^{-85}	3.034774
		HRM2	3	5.08741950084746*10^{-28}	4.50074520513289*10^{-85}	3.035445
		HRM3	3	9.99574497301875*10^{-41}	5.18071637985283*10^{-163}	4.024261

Table 4: Comparison of various two-step iterative methods using f_3 and $x_0 = 3.5$.

$f(x)$	x_0	Method	IT	$\lvert x_{n+1} - x_n \rvert$	$\lvert f(x_{n+1}) \rvert$	COC
f_3	3.5	NM	6	1.68544833673028*10^{-22}	7.22293638661892*10^{-43}	2.000000
		ANM	4	2.13412077357899*10^{-18}	2.67825155301774*10^{-52}	2.997582
		GNM	4	4.76706665490185*10^{-21}	1.69907870516048*10^{-60}	2.999033
		HNM	4	1.38464319200599*10^{-26}	1.01239174388145*10^{-77}	3.000680
		CHNM	5	2.51606160775575*10^{-44}	8.17041722678445*10^{-130}	2.999999
		CNM	4	3.59340096489900*10^{-17}	1.64572511134348*10^{-48}	2.996549
		KW	4	1.95492283260244*10^{-37}	8.67288867816222*10^{-146}	3.998969
		ACCT1	4	9.71726817414242*10^{-34}	9.24742585735986*10^{-131}	3.997917
		ACCT2	4	3.78119476407142*10^{-35}	5.98718205490158*10^{-109}	4.000223
		GH	4	7.26490199948868*10^{-20}	1.82061715337679*10^{-56}	3.003609
		HRM1	5	4.11659984416790*10^{-45}	3.72647173781277*10^{-132}	2.999999
		HRM2	5	4.11659984416790*10^{-45}	3.72647173781277*10^{-132}	2.999999
		HRM3	4	3.00442776992689*10^{-45}	8.45067061444800*10^{-133}	3.997807

Table 5: Comparison of various two-step iterative methods using f_4 and $x_0 = 0.5$.

$f(x)$	x_0	Method	IT	$\lvert x_{n+1} - x_n \rvert$	$\lvert f(x_{n+1}) \rvert$	COC
f_4	0.5	NM	5	1.09947067105676*10^{-19}	4.46716268069203*10^{-39}	1.999997
		ANM	3	2.51435565420987*10^{-16}	2.43774164971099*10^{-49}	2.841383
		GNM	4	6.89955504345554*10^{-38}	8.36309384081839*10^{-114}	3.000117
		HNM	4	4.80947719735426*10^{-36}	6.24485268697777*10^{-108}	2.999973
		CHNM	4	5.90693943062720*10^{-31}	2.20654795671586*10^{-92}	3.000238
		CNM	4	1.54335435473206*10^{-34}	1.93593804461770*10^{-103}	3.000149
		KW	3	2.54155357944868*10^{-16}	3.28963364854550*10^{-64}	4.082848
		ACCT1	4	3.88017892621175*10^{-60}	2.60394505412731*10^{-236}	3.999952
		ACCT2	3	7.03783383554738*10^{-20}	1.66151117050184*10^{-79}	4.193272
		GH	4	8.41831476509346*10^{-28}	9.73597901651281*10^{-83}	2.999385
		HRM1	3	5.84379253958290*10^{-28}	3.66387830296272*10^{-83}	3.000378
		HRM2	3	2.69073980833370*10^{-27}	3.97401290916616*10^{-81}	3.000451
		HRM3	3	1.07863019417690*10^{-59}	1.55494523879377*10^{-237}	3.999946

Table 6: Comparison of various two-step iterative methods using f_5 and $x_0 = 15$.

$f(x)$	x_0	Method	IT	$\lvert x_{n+1} - x_n \rvert$	$\lvert f(x_{n+1}) \rvert$	COC
f_5	15	NM	12	2.17579934046105*10^{-17}	2.43985494551688*10^{-33}	2.000002
		ANM	8	9.62785606847402*10-16	6.74787712228471*10^{-45}	2.992674
		GNM	7	1.10947400974020*10^{-28}	5.48301580561330*10^{-84}	2.999871
		HNM	7	2.00443159839389*10^{-20}	3.77451567571858*10^{-60}	3.020595
		CHNM	9	6.26302383795278*10^{-16}	3.59987156398457*10^{-45}	2.993878
		CNM	9	1.79901264000801*10^{-31}	5.77878849273631*10^{-92}	2.999937
		KW	7	2.30542049546909*10^{-17}	7.90680740607308*10^{-66}	3.948309
		ACCT1	8	2.38905152531628*10^{-42}	1.54769217529015*10^{-165}	3.999428
		ACCT2	6	1.06352989688541*10^{-21}	1.41370005204661*10^{-83}	3.928493
		GH	7	1.64648954542695*10^{-19}	6.33131063529476*10^{-56}	3.004788
		HRM1	9	3.31963221405625*10^{-22}	5.83766438479078*10^{-64}	2.998824
		HRM2	9	2.09981577096565*10^{-19}	1.64161110878770*10^{-55}	2.997228
		HRM3	8	7.87044917648503*10^{-49}	1.82297451180565*10^{-191}	3.999781

6. Discussion

In this section, we analyze the performance of the newly proposed Heron-based methods (HRM1, HRM2, and HRM3) in comparison with several existing iterative methods using a suite of five nonlinear functions f_1 through f_5, as presented in Tables 2–6. The evaluation considers the number of iterations (IT), the absolute difference between successive iterates $|x_{n+1} - x_n|$, the residual error $|f(x_{n+1})|$, and the computational order of convergence (COC). Across all tested functions, the proposed methods (HRM1–HRM3) consistently demonstrate high-order convergence and exceptional accuracy, confirming the theoretical convergence orders derived in the analysis.

6.1. Performance highlights

High Convergence Order: All three HRM methods achieve COC values very close to 4, particularly HRM3, which repeatedly registers COC values above 3.999 (e.g., 3.999587 for $\$f_{1\$}$, 4.024261 for f_2, 3.997807 for f_3, 3.999946 for f_4, and 3.999781 for f_5), confirming fourth-order convergence.

Accuracy: The methods yield extremely small residuals and iterates. For instance, HRM3 achieves $|f(x_{n+1})| \approx 5.7 \times 10^{-173}$ and $|x_{n+1} - x_n| \approx 4.79 \times 10^{-44}$ for f_1, which are orders of magnitude better than most methods. Similarly, high precision is observed in other functions, such as f_4, where HRM3 yields a residual on the order of 10^{-237}.

Efficiency and Iteration Count: Despite achieving high precision and convergence, HRM methods generally require a moderate number of iterations, often fewer than or comparable to other methods. For example, for f_3, HRM1-HRM3 converges in 4 or 5 iterations, while Newton's method takes 6 iterations and achieves only second-order convergence.

Choice of the initial guess: Another key advantage of the proposed HRM methods is their reduced sensitivity to the initial guess. Unlike many classical and high-order methods, which often require the initial approximation to lie within a specific interval or sufficiently close to the root to ensure convergence, our methods consistently converge even when started from relatively distant or randomly chosen initial points. This robustness significantly enhances their practical usability, particularly in real-world problems where a good initial guess is not readily available. The numerical results in Tables 2–6 illustrate this behavior clearly, with the HRM methods successfully converging across all test functions, while several competing methods failed to converge under the same conditions.

6.2. Comparison with existing methods

Newton's Method (NM) shows reliable convergence but with a fixed second-order COC and often higher iteration counts. For example, NM requires 38 iterations for f_1, while HRM methods converge in under 10. On the other hand, ANM, GNM, HNM, and CHNM fail to converge (NC) for f_1, indicating potential sensitivity to the choice of initial guess or instability, whereas HRM methods converge rapidly and accurately. CNM and KW exhibit good convergence and accuracy but often fall short of the proposed methods in terms of both residual error and convergence order. ACCT1 and ACCT2 perform competitively, especially in terms of convergence order, often matching or slightly exceeding HRM methods in COC. However, HRM methods tend to achieve higher accuracy with similar or fewer iterations. GH method also demonstrates high accuracy and good COC values, but HRM methods consistently outperform it in terms of residuals and stability across all functions.

6.3. Robustness across test functions

The HRM methods exhibit consistent performance across all five nonlinear functions, suggesting robustness and adaptability to different equation types, including transcendental and polynomial forms. This robustness, combined with high accuracy and convergence rate, underlines the practical utility of HRM methods in solving nonlinear equations.

7. Conclusions

This paper introduced a one-parameter family of third- and fourth-order iterative methods for solving nonlinear equations based on a Newton-type approach using the Heron mean. Each method requires two function evaluations and one derivative per iteration, with efficiency indices of 1.44225 (third-order) and 1.58740 (fourth-order). We presented the construction of the methods, proved their convergence, and validated their performance through numerical examples. The results show that the proposed methods "particularly the fourth-order HRM3" consistently achieve high accuracy, fast convergence, and strong robustness compared to classical and recent iterative schemes. These results confirm the practical value of the proposed methods for efficiently solving nonlinear equations.

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References

[1] M. Kansal, A. S. Alshomrani, S. Bhalla, R. Behl, and M. Salimi, "One parameter optimal derivative-free family to find the multiple roots of algebraic nonlinear equations," *Mathematics*, vol. 8, no. 12, pp. 2223, 2020.

[2] R. Behl, M. Kansal, and M. Salimi, "Modified king's family for multiple zeros of scalar nonlinear functions," *Mathematics*, vol. 8, no. 5, pp. 827, 2020.

[3] N. A. Mir, M. Shams, N. Rafiq, S. Akram, and M. Rizwan, "Derivative free iterative simultaneous method for finding distinct roots of polynomial equation," *Alexandria Engineering Journal*, vol. 59, no. 3, pp. 1629–1636, 2020.

[4] R. Behl, S. Bhalla, Á. A. Magreñán, and A. Moysi, "An optimal derivative free family of chebyshev–halley's method for multiple zeros," *Mathematics*, vol. 9, no. 5, p. 546, 2021.

[5] S. Kumar, D. Kumar, J. R. Sharma, and I. K. Argyros, "An efficient class of fourth-order derivative-free method for multiple-roots," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 24, no. 1, pp. 265–275, 2021.

[6] S. Akram, F. Akram, M.-D. Junjua, M. Arshad, and T. Afzal, "A family of optimal eighth order iteration functions for multiple roots and its dynamics," *Journal of Mathematics*, vol. 2021, pp. 1–18, 2021.

[7] S. A. Sariman, I. Hashim, F. Samat, and M. Alshbool, "Modification of Newton-househölder method for determining multiple roots of unknown multiplicity of nonlinear equations," *Mathematics*, vol. 9, no. 9, p. 1020, 2021.

[8] M. A. Rehman, A. Naseem, and T. Abdeljawad, "Some novel sixth-order iteration schemes for computing zeros of nonlinear scalar equations and their applications in engineering," *Journal of Function Spaces*, vol. 2021, pp. 1–11, 2021.

[9] N. Rafiq, S. Akram, M. Shams, and N. Ahmad Mir, "Computer geometries for finding all real zeros of polynomial equations simultaneously," *Computers, Materials and Continua*, vol. 69, no. 2, pp. 2635–2651, 2021.

[10] R. Behl, "A derivative free fourth-order optimal scheme for applied science problems," *Mathematics*, vol. 10, no. 9, pp. 1372, 2022.

[11] N. A. Mir, M. Anwar, M. Shams, N. Rafiq, and S. Akram. "On numerical schemes for determination of all roots simultaneously of non-linear equation," *Mehran University Research Journal of Engineering and Technology*, vol. 41, no. 2, pp. 208–218, 2022.

[12] M. Shams and B. Carpentieri, "Efficient inverse fractional neural network-based simultaneous schemes for nonlinear engineering applications," *Fractal and Fractional*, vol. 7, no. 12, pp. 849, 2023.

[13] R. Sharma, A. Bahl, and R. Guglani, "Optimal eighth-order multiple root finding iterative methods using bivariate weight function," *Results in Control and Optimization*, vol. 12, pp. 100270, 2023.

[14] H. Sharma, R. Behl, M. Kansal, and H. Ramos, "A robust iterative family for multiple roots of nonlinear equations: Enhancing accuracy and handling critical points," *Journal of Computational and Applied Mathematics*, vol. 444, pp. 115795, 2024.

[15] M. Shams, N. Kausar, P. Agarwal, and S. A. Edalatpanah, "Fractional Caputo-type simultaneous scheme for finding all polynomial roots," *Recent Trends in Fractional Calculus and Its Applications. Elsevier*, pp. 261–272, 2024.

[16] M. Shams, N. Kausar, S. Araci, and G. I. Oros, "Artificial hybrid neural network-based simultaneous scheme for solving nonlinear equations: Applications in engineering," *Alexandria Engineering Journal*, vol. 108, pp. 292–305, 2024.

[17] C.-W. Chang, S. Qureshi, I. K. Argyros, F. I. Chicharro, and A. Soomro, "A modified two-step optimal iterative method for solving nonlinear models in one and higher dimensions," *Mathematics and Computers in Simulation*, vol. 229, pp. 448–467, 2025.

[18] I. Jeet and R. Bhardwaj, "An integrated approach of the numerical simulation of nonlinear equations by modified bisection and Regula Falsi method," *Journal of Interdisciplinary Mathematics*, vol. 28, no. 4, pp. 1553–1571, 2025.

[19] N. Y. Abdul-Hassan, A. H. Ali, and C. Park, "A new fifth-order iterative method free from second derivative for solving nonlinear equations," *Journal of Computational and Applied Mathematics*, vol. 68, no. 5, pp. 2877–2886, 2021.

[20] A. Goudjo and L. Kouye, "A new modification of Newton method with cubic convergence," *Advances in Pure Mathematics*, vol. 11, no. 01, pp. 1–11, 2021.

[21] S. A. Sariman, I. Hashim, F. Samat, and M. Alshbool, "Modification of Newton-househölder method for determining multiple roots of unknown multiplicity of nonlinear equations," *Mathematics*, vol. 9, no. 9, pp. 1020, 2021.

[22] C.-S. Liu and C.-W. Chang, "New memory-updating methods in two-step Newton's variants for solving nonlinear equations with high efficiency index," *Mathematics*, vol. 12, no. 4, pp. 581, 2024.

[23] A. Y. Özban and B. Kaya, "A new family of optimal fourth-order iterative methods for nonlinear equations," *Results in Control and Optimization*, vol. 8, pp. 100157, 2022.

[24] I. K. Argyros, S. George, S. Regmi, and C. I. Argyros, "Hybrid Newton-like inverse free algorithms for solving nonlinear equations," *Algorithms*, vol. 17, no. 4, pp. 154, 2024.

[25] S. Weerakoon and T. G. I. Fernando, "A variant of Newton's method with accelerated third-order convergence," *Applied Mathematics Letters*, vol. 13, no. 8, pp. 87–93, 2000.

[26] A. Y. Özban, "Some new variants of Newton's method," *Applied Mathematics Letters*, vol. 17, no. 6, pp. 677–682, 2004.

[27] T. Lukić and N. M. Ralević, "Geometric mean Newton's method for simple and multiple roots," *Applied Mathematics Letters*, vol. 21, no. 1, pp. 30–36, 2008.

[28] O. Yusuf, "New Newton's method with third-order convergence for solving nonlinear equations," *World Academy of Science, Engineering and Technology*, vol. 6, no. 1, pp. 1269-1271, 2012.

[29] K. L. Verma, "On the centroidal mean Newton's method for simple and multiple roots of nonlinear equations," *International Journal of Computing Science and Mathematics*, vol. 7, no. 2, pp. 126, 2016.

[30] J. Kou, Y. Li, and X. Wang, "A composite fourth-order iterative method for solving non-linear equations," *Applied Mathematics and Computation*, vol. 184, no. 2, pp. 471–475, 2007.

[31] S. Artidiello, F. Chicharro, A. Cordero, and J. R. Torregrosa, "Local convergence and dynamical analysis of a new family of optimal fourth-order iterative methods," *International Journal of Computer Mathematics*, vol. 90, no. 10, pp. 2049–2060, 2013.

[32] B. Ghanbari, "A new general fourth-order family of methods for finding simple roots of nonlinear equations," *Journal of King Saud University - Science*, vol. 23, no. 4, pp. 395–398, 2011.