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# An Estimation of Survival and Hazard Functions of Weighted Rayleigh Distribution

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**Abstract**
In this paper, we introduce a new class of Weighted Rayleigh Distribution based on two parameters, one is the scale parameter and the other is the shape parameter introduced in Rayleigh distribution. The main properties of this class are derived and investigated. The Moment method (ME) and the Maximum Likelihood method (MLE) are used to obtain estimators of parameters of this distribution. The probability density function, survival function, cumulative distribution and hazard function are derived and found. Real data sets are collected to investigate the two methods applied in this study. A comparison was made between the two methods of estimation and it is clear that MLE is better than ME according to the results of the mean squares error.

**Keywords:** weighted Rayleigh distribution, Maximum likelihood method, Moment method.

# تقدير دوال البقاء والمخاطرة لتوزيع رالي الموزون

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**الخلاصه**
في هذا البحث تقدمنا بتوزيع رالي الجديد بالاعتماد على معلمتين احدهما معلمة القياس والاخرى هي معلمة الشكل التي ادخلت في توزيع رالي الخواص الرياضيه الرئيسيه لتوزيع رالي المزوزن قد اشتقت وفحصت وتم تقديرالمعلمتين من خلال استخدام طريق العزوم وطريقة الامكان الاعظم للحصول على تقديرات للمعالم وتم اشتقاق دالة الكثافة الاحتمالية ، ودالة التوزيع التجميعي ، ودالة البقاء ودالة المخاطرة واوجدت من خلال تطبيقها على بيانات حقيقيه واستخدمنا الطريقتين في هذه الدراسه وقورنت النتائج باستخدام الطريقتين من خلال متوسط مربعات الخطأ وقد تبين ان طريقة الامكان الاعظم هي افضل من طريقه العزوم

## 1- Introduction

The Rayleigh distribution is one of the important continuous distributions; it has gained much attention in the literature as compared to any other distribution in lifetime sample modeling and data analysis. Many researchers developed various generalizations of Rayleigh probability density function to increase the flexibility in lifetime sample modeling. Azzalini, in 1985, introduced a new class of density function depending on the shape parameter in the normal distribution, which is known as weighted normal distribution or skew-normal distribution[1]. Gupta and Kundu, in 2009, used the idea

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of Azzalini to find the shape parameter to an exponential distribution, which is known as weighted exponential distribution, as well as to put the general mathematical formula to treat the weighted statistical distributions which are as follows: [2]

$$f_w(X) = \frac{1}{p[x_2 < \alpha x_1]} f(x_1)F(\alpha x_1) \qquad (1)$$

where:

$f_w(X)$ Weighted probability density function.

$f(x_1)$ Standard probability density function for $r.v(x_1)$

$F(\alpha x_1)$ Cumulative distribution functions with respect to weighted parameter $\alpha$ for standard distribution.

$P_r(x_2 < \alpha x_1)$ Probability for $r.v(x_2)$ with respect to the $r.v(x_1)$ and weighted function $(\alpha)$.

Shahbaz *et al.*, in 2010, studied weighed weibull distribution by using the idea of Azzalini and introduced the basic properties for this model [3]. Mervat, in 2011, studied the skewness parameter of a gamma distribution by using the idea of Azzalini, which resulted in a new class of weighted gamma distribution [4]. Amin and Hussian, in 2014, proposed the extension of the weighted Weibull distribution and the main properties of this class were investigated and derived [5]. Farahani and Khorram, in 2014, estimated Linley's approximation method for weighted exponential distribution by using a Monte Carlo simulation study [6]. Badmus *et al.*,in 2015, introduced two shape parameters to the existing weighted exponential distribution to develop the weighted Beta exponential distribution using the log of beta function[7]. Oguntunde, in 2015, proposed a model named the exponentiated weighted exponential distribution and some of the basic statistical properties of the proposed model were studied and provided [8]. Zahrani (2016) studied the nonparametric methods, such as the empirical method, kernel method, and the modified shrinkage method, which are provided in the weighted Weibull distribution [9]. AL-Noor and Hussein, in 2017, focused on Bayes estimation of weighted exponential distribution with fuzzy data[10]. Qguntunde *et al.* (2018) derived two parameters, where the inverted weighted exponential distribution and its various statistical properties were established [11]. Abbas *et al.* (2019) presented a new generalization weighted Weibull distribution using topple one family of distribution [12]. Iden *et al.*, in 2019, proposed a new class of weighted Rayleigh distribution by using the idea of Azzalini and introduced the main characteristics of this distribution [13].

The aims of this paper are to introduce a new weighted Rayleigh distribution with its properties which are discussed in an earlier work [13]. We applied this new distribution on real data to estimate the parameters by using two methods. We calculated the death density function, survival function, and hazard function for these two methods.

The remaining parts of this article are as follows; In section two we describe the new weighted Rayleigh distribution and its properties. In section we describe the three estimation methods. In section four we show the real data application. In section five we present the conclusions.

**2- Weighted Rayleigh Distribution**

In this section, we will provide the probability density function of the new weighted Rayleigh distribution, which is as follows:

$$f(x;\alpha,\theta) = \frac{\alpha^2+1}{\alpha^2}\theta x\, e^{-\frac{\theta}{2}x^2}\left[1 - e^{-\frac{\theta}{2}x^2\alpha^2}\right] \;,\; x > 0[13] \qquad (2)$$

The cumulative distribution of the new distribution is as follows:

$$F(x;\,\alpha,\theta) = 1 - \left[\frac{(\alpha^2+1)e^{-\frac{\theta}{2}x^2} - e^{-\frac{\theta}{2}x^2(\alpha^2+1)}}{\alpha^2}\right] \qquad (3)$$

The survival function and the hazard function are as follows:

$$S(x) = \frac{(\alpha^2+1)e^{-\frac{\theta}{2}x^2} - e^{-\frac{\theta}{2}x^2(\alpha^2+1)}}{\alpha^2} \qquad (4)$$

$$h(x) = \frac{\theta x(\alpha^2+1)\left[1 - e^{-\frac{\theta}{2}x^2 d^2}\right]}{\left[(\alpha^2+1) - e^{-\frac{\theta}{2}x^2\alpha^2}\right]} \qquad (5)$$

The $r^{th}$ moment of the new distribution is as follows:

$$E(x^r) = \frac{\alpha^2+1}{\alpha^2}\left(\frac{2}{\theta}\right)^{\frac{r}{2}} \Gamma\left(\frac{r}{2}+1\right)\left[1-\frac{1}{(\alpha^2+1)^{\frac{r}{2}+1}}\right] \qquad (6)$$

The mean of the new distribution is given by:

$$\text{M= E }(X)= \frac{\alpha^2+1}{\alpha^2}\left[\frac{\pi}{2\theta}\right]^{\frac{1}{2}}\left[1-\frac{1}{(\alpha^2+1)^{3/2}}\right] \qquad (7)$$

The variance is given by:

$$\sigma^2 = var(X) = \frac{4\alpha^4(\alpha^2+2)-\pi\left[(\alpha^2+1)^{\frac{3}{2}}-1\right]^2}{2\theta\alpha^4(\alpha^2+1)} \qquad (8)$$

The moment generated function of this distribution is as follows:

$$m.g.f = \frac{\alpha^2+1}{\alpha^2}\sum_{r=0}^{\infty}\frac{t^r}{r!}\left(\frac{2}{\theta}\right)^{\frac{r}{2}}\Gamma\left(\frac{r}{2}+1\right)\left[1-\frac{1}{(\alpha^2+1)^{r/2+1}}\right] \qquad (9)$$

The factorial moment generating function is:

$$M_x(t) = \frac{\alpha^2+1}{\alpha^2}\sum_{r=0}^{\infty}\frac{(\ell nt)^r}{r!}\left(\frac{2}{\theta}\right)^{\frac{r}{2}}\Gamma\left(\frac{r}{2}+1\right)\left[1-\frac{1}{(\alpha^2+1)^{\frac{r}{2}+1}}\right] \qquad (10)$$

The skewness and the kurtosis of this distribution is given by:

$$C.S = \frac{1}{\alpha^2}\Gamma\left(\frac{5}{2}\right)\left[\frac{(\alpha^2+1)^{\frac{5}{2}}-1}{(\alpha^2+1)^{\frac{3}{2}}}\right] \qquad (11)$$

$$C.k = \frac{2[(\alpha^2+1)^3-1]}{\alpha^2(\alpha^2+1)^2}-3 \qquad (12)$$

The Characteristic function of this distribution is as follows:

$$Qx\ (x) = \frac{\alpha^2+1}{\alpha^2}\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\left(\frac{2}{\theta}\right)^{r}\Gamma\left(\frac{r}{2}+1\right)\left[1-\frac{1}{(\alpha^2+1)^{\frac{r}{2}+1}}\right] \qquad (13)$$

**3- Estimation Methods**

In this section, we describe the estimation of the parameters of weighted Rayleigh distribution by employing two methods (maximum likelihood estimator method and moment estimator method).

**3-1: Maximum likelihood method**

MLE has been a famous classical method to find the estimators of parameters that maximize the likelihood function. The probability density function is:

$$f(\text{x}; \alpha, \theta) = \frac{\alpha^2+1}{\alpha^2}\,\theta \text{x}\, e^{-\frac{\theta}{2}x^2}\left[1-e^{-\frac{\theta}{2}x^2\alpha^2}\right] \qquad (14)$$

$$L\ (\alpha,\theta;x) = \frac{(\alpha^2+1)^n}{\alpha^{2n}}\theta^n\prod_{i=1}^{n}x_i\ e^{-\frac{\theta}{2}\sum x_i^2}\prod_{i=1}^{n}\left[1-e^{-\frac{\theta}{2}x^2\alpha^2}\right] \qquad (15)$$

$$LnL = nln\ (\alpha^2+1) - 2n\ ln(\alpha) + nln\theta + \sum_{i=1}^{n}Ln\ x_i - \frac{\Theta}{2}\sum_{i=1}^{n}x_i^2 + \sum_{i=1}^{n}Ln\left[1-e^{-\frac{\theta}{2}x_i^2\alpha^2}\right] \qquad (16)$$

$$\frac{dlnl}{d\theta} = \frac{n}{\theta} - \frac{1}{2}\sum_{i=1}^{n}x_i^2 + \sum_{i=1}^{n}\frac{\frac{1}{2}x_i^2\ \alpha^2 e^{-\frac{\theta}{2}x_i^2\alpha^2}}{\left[1-e^{-\frac{\theta}{2}x_i^2\alpha^2}\right]} = g_1(\theta) \qquad (17)$$

$$\frac{dlnl}{d\alpha} = \frac{2n\alpha}{(\alpha^2+1)} - \frac{2n}{\alpha} + \sum_{i=1}^{n}\frac{\theta\alpha\, x_i^2\ e^{-\frac{\theta}{2}x_i^2\alpha^2}}{\left[1-e^{-\frac{\theta}{2}x_i^2\alpha^2}\right]} = g_2\ (\alpha) \qquad (18)$$

$$\frac{dg_1(\theta)}{d\theta} = -\frac{n}{\theta^2} - \sum_{i=1}^{n} \frac{\frac{1}{4} x_i^4 \alpha^4 e^{-\theta x_i^2 \alpha^2}}{\left[1 - e^{-\frac{\theta}{2} x_i^2 \alpha^2}\right]^2} \tag{19}$$

$$\frac{dg_1(\theta)}{d\alpha} = \sum_{i=1}^{n} \frac{-\frac{1}{2}\theta\alpha^3 x_i^4 e^{-\frac{\theta}{2}x_i^2\alpha^2} + \alpha x_i^2 e^{-\frac{\theta}{2}x_i^2\alpha^2} - \alpha x_i^2 e^{-\theta x_i^2 \alpha^2}}{\left[1 - e^{-\frac{\theta}{2} x_i^2 \alpha^2}\right]^2} \tag{20}$$

$$\frac{dg_2(\alpha)}{d\theta} = \sum_{i=1}^{n} \frac{-\frac{1}{2}\theta\alpha^3 x_i^4 e^{-\frac{\theta}{2}x_i^2\alpha^2} + \alpha x_i^2 e^{-\frac{\theta}{2}x_i^2\alpha^2} - \alpha x_i^2 e^{-\theta x_i^2 \alpha^2}}{\left[1 - e^{-\frac{\theta}{2} x_i^2 \alpha^2}\right]^2} \tag{21}$$

$$\frac{dg_2(\alpha)}{d\alpha} = \frac{2n(1-\alpha^2)}{(\alpha^2+1)^2} + \frac{2n}{\alpha^2} + \sum_{i=1}^{n} \frac{\theta x_i^2 e^{-\frac{\theta}{2}x_i^2\alpha^2} - \theta x_i^2 e^{-\theta x_i^2\alpha^2} - \theta^2\alpha^2 x_i^4 e^{-\frac{\theta}{2}x_i^2\alpha^2}}{\left[1 - e^{-\frac{\theta}{2} x_i^2 \alpha^2}\right]^2} \tag{22}$$

The above equations are nonlinear and thus we must use the multivariate Newton-Raphson method, which is as follows:

$$\begin{bmatrix}\theta_{k+1} \\ \alpha_{k+1}\end{bmatrix} = \begin{bmatrix}\theta_k \\ \alpha_k\end{bmatrix} - j^{-1}\begin{bmatrix} g_1(\theta) \\ g_2(\alpha)\end{bmatrix} \tag{23}$$

where the Jacobean is as follows:

$$J = \begin{bmatrix} \frac{dg_1(\theta)}{d\theta} & \frac{dg_1(\theta)}{d\alpha} \\ \frac{dg_2(\alpha)}{d\theta} & \frac{dg_2(\alpha)}{d\alpha} \end{bmatrix} \tag{24}$$

where J refers to the Jacobean symmetric and square matrix.
The,n we find the stopping rule for obtaining convergence, which is as follows:

$$\left\| \begin{bmatrix}\theta_{k+1} \\ \alpha_{k+1}\end{bmatrix} - \begin{bmatrix}\theta_k \\ \alpha_k\end{bmatrix} \right\| \le \begin{vmatrix}\epsilon_\theta \\ \epsilon_\alpha\end{vmatrix} \tag{25}$$

**3-2: The moment estimation method**

This method assumes that sample moment will be equal to population moments, with solving the resulting equations to obtain estimators for parameters. By equating the first sample moment to the first population moment, we obtain:

$$M_1 = E(x) = \frac{\alpha^2+1}{\alpha^2}\left[\frac{\pi}{2\theta}\right]^{\frac{1}{2}}\left[1 - \frac{1}{(\alpha^2+1)^{3/2}}\right] \tag{26}$$

$$\grave{M}_1 = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{X}, \qquad M_1 = \grave{M}_1$$

$$\frac{\alpha^2+1}{\alpha^2}\left[\frac{\pi}{2\theta}\right]^{\frac{1}{2}}\left[1 - \frac{1}{(\alpha^2+1)^{3/2}}\right] = \bar{X} \tag{27}$$

By equating the second sample moment to the second population moment, we obtain:

$$M_2 = \grave{M}_2 \qquad , \frac{\sum_{i=1}^{n} x_i^2}{n} = \frac{2(\alpha^2+2)}{\theta(\alpha^2+1)} \tag{28}$$

$$\theta\frac{\sum xi^2}{n} = \frac{2(\alpha^2+2)}{(\alpha^2+1)} \qquad , \qquad \hat{\theta} = \frac{2n(\alpha^2+2)}{\sum x_i^2\,(\alpha^2+1)} \tag{29}$$

From equation (28), we get the following:

$$\frac{\alpha^2+1}{\alpha^2}\left[\frac{\pi}{2\theta}\right]^{\frac{1}{2}}\left[\frac{(\alpha^2+1)^{\frac{3}{2}} - 1}{(\alpha^2+1)(\alpha^2+1)^{\frac{1}{2}}}\right] = \bar{X} \tag{30}$$

$$\alpha^2 = \frac{1}{\bar{X}}\left[\frac{\pi}{2\hat{\theta}}\right]^{\frac{1}{2}}\frac{(\alpha^2+1)^{\frac{3}{2}} - 1}{(\alpha^2+1)^{\frac{1}{2}}} \tag{31}$$

$$\hat{\alpha} = \frac{1}{\sqrt{\bar{X}}}\left[\frac{\pi}{2\hat{\theta}}\right]^{\frac{1}{4}} \frac{\left[(\alpha^2+1)^{\frac{3}{2}}-1\right]^{\frac{1}{2}}}{(\alpha^2+1)^{\frac{1}{4}}} \quad (32)$$

$$f(\alpha) = \frac{1}{\sqrt{\bar{X}}}\left[\frac{\pi \sum x_i^2}{4n}\right]^{\frac{1}{4}} (\alpha^2+2)^{-\frac{1}{4}} \left[(\alpha^2+1)^{\frac{3}{2}}-1\right]^{\frac{1}{2}} \quad (33)$$

$$f`(\alpha) = \frac{\alpha}{2\sqrt{\bar{X}}}\left[\frac{\pi \sum x_i^2}{4n(\alpha^2+2)}\right]^{\frac{1}{4}} \frac{3(\alpha^2+2)(\alpha^2+1)^{\frac{1}{2}}-(\alpha^2+1)^{3/2}+1}{(\alpha^2+2)\left[(\alpha^2+1)^{3/2}-1\right]^{\frac{1}{2}}} \quad (34)$$

By using the Newton – Raphson method, we obtain:

$$\hat{\alpha}_{k+1} = \hat{\alpha}_k - \frac{f(\alpha)}{f`(\alpha)} \quad (35)$$

$$\hat{\alpha}_{k+1} = \hat{\alpha}_k - \frac{2}{\alpha}\frac{(\alpha^2+2)\left[(\alpha^2+1)^{3/2}-1\right]}{\left[3(\alpha^2+2)(\alpha^2+1)^{\frac{1}{2}}-(\alpha^2+1)^{3/2}+1\right]} \quad (36)$$

This can be solved iteratively to obtain an estimate of α , by using the error term, which is as follows:

$$|\hat{\alpha}_{k+1} - \hat{\alpha}_k| \leq |\epsilon_\alpha| \quad (37)$$

**4- Real data application**

In this section, real data for brain cancer disease is analyzed, since the disease is widespread and causes high mortality rate in Iraq The time between diagnosis to death was recorded for the period from 1/1/2018 to 31/12/2018 at the Medical City, Ministry of Health, Baghdad. The data of a sample size of 111 patients was considered as being a complete dataset.

X= { 23 10 14 4 14 11 20 15 20 9 7 15 10 16 12 7 11 15 16 13 14 15 12 9 14 14 21 28 16 16 10 5 5 13 17 9 9 9 9 22 10 13 18 22 8 19 20 10 6 12 10 20 5 12 10 26 8 9 21 12 16 12 14 14 19 17 28 6 10 5 20 6 8 11 14 17 9 18 24 9 10 9 10 14 14 8 16 8 8 7 13 11 5 14 24 7 11 15 2 18 10 11 15 20 28 14 19 9 15 7 9 }.

We then applied the chi square goodness of fit to test the hypothesis, as follows:

$H_0 = the\ data\ is\ distributed\ as\ weighted\ Rayleigh\ distribution.$

$H_1 = the\ data\ is\ not\ distributed\ as\ weighted\ Rayleigh\ distribution.$

The chi square goodness of fit statistic depends on the differences between the theoretical frequencies under the assumed distribution and the observed data, by applying the formula:

$$X^2 = \sum_{i=1}^{N} \frac{(O_i - E_i)^2}{E_i} \quad (38)$$

By applying the chi square goodness of fit to the data we get:

The calculated chi square test = 4.2963

The calculated chi square test = 35.17

Calculated chi- square < tabulated chi-square.

We accept the $H_0$, which indicates thatt the data is distributed in a weighted Rayleigh distribution.

We apply equations (23) and (25) in the maximum likelihood method, by using math lab program version (2014), to find the values of $\hat{\theta}$ and $\hat{\alpha}$.

Finally, we apply equations (29) and (37) in the moments method to find the values of $\hat{\theta}$ and $\hat{\alpha}$, then we get:

$\hat{\alpha}._{MLE}$ =1.9881, $\hat{\theta}_{MLE}$= 0.0118

$\hat{\alpha}._{ME}$ = 1.9679, $\hat{\theta}_{ME}$= 0.0119

**$A_1$- Applying the moment method**

The value of the estimated parameters $(\hat{\alpha}, \hat{\theta})$ was found to be 1.9679 and 0.0119, respectively. The values of the estimated probability density function, survival function, cumulative distribution and hazard function are illustrated in the Table-1.

**Table 1**-Estimated values of $\hat{f}$(t), $\hat{F}$(t), $\hat{S}$(t), $\hat{h}$(t) for the ME method.

| X | $\hat{S}$(t) | $\hat{f}$(t) | $\hat{F}$(t) | $\hat{h}$(t) |
|---|---|---|---|---|
| 2 | 0.998691 | 0.002557 | 0.001309 | 0.00256 |
| 4 | 0.981702 | 0.016688 | 0.018298 | 0.016999 |
| 5 | 0.959478 | 0.028092 | 0.040522 | 0.029279 |
| 5 | 0.959478 | 0.028092 | 0.040522 | 0.029279 |
| 5 | 0.959478 | 0.028092 | 0.040522 | 0.029279 |
| 5 | 0.959478 | 0.028092 | 0.040522 | 0.029279 |
| 5 | 0.959478 | 0.028092 | 0.040522 | 0.029279 |
| 6 | 0.925123 | 0.040673 | 0.074877 | 0.043965 |
| 6 | 0.925123 | 0.040673 | 0.074877 | 0.043965 |
| 6 | 0.925123 | 0.040673 | 0.074877 | 0.043965 |
| 7 | 0.878302 | 0.052751 | 0.121698 | 0.060061 |
| 7 | 0.878302 | 0.052751 | 0.121698 | 0.060061 |
| 7 | 0.878302 | 0.052751 | 0.121698 | 0.060061 |
| 7 | 0.878302 | 0.052751 | 0.121698 | 0.060061 |
| 7 | 0.878302 | 0.052751 | 0.121698 | 0.060061 |
| 8 | 0.820275 | 0.062882 | 0.179725 | 0.07666 |
| 8 | 0.820275 | 0.062882 | 0.179725 | 0.07666 |
| 8 | 0.820275 | 0.062882 | 0.179725 | 0.07666 |
| 8 | 0.820275 | 0.062882 | 0.179725 | 0.07666 |
| 8 | 0.820275 | 0.062882 | 0.179725 | 0.07666 |
| 8 | 0.820275 | 0.062882 | 0.179725 | 0.07666 |
| 9 | 0.753504 | 0.070135 | 0.246496 | 0.093078 |
| 9 | 0.753504 | 0.070135 | 0.246496 | 0.093078 |
| 9 | 0.753504 | 0.070135 | 0.246496 | 0.093078 |
| 9 | 0.753504 | 0.070135 | 0.246496 | 0.093078 |
| 9 | 0.753504 | 0.070135 | 0.246496 | 0.093078 |
| 9 | 0.753504 | 0.070135 | 0.246496 | 0.093078 |
| 9 | 0.753504 | 0.070135 | 0.246496 | 0.093078 |
| 9 | 0.753504 | 0.070135 | 0.246496 | 0.093078 |
| 9 | 0.753504 | 0.070135 | 0.246496 | 0.093078 |
| 9 | 0.753504 | 0.070135 | 0.246496 | 0.093078 |
| 9 | 0.753504 | 0.070135 | 0.246496 | 0.093078 |
| 9 | 0.753504 | 0.070135 | 0.246496 | 0.093078 |
| 10 | 0.681085 | 0.074165 | 0.318915 | 0.108892 |
| 10 | 0.681085 | 0.074165 | 0.318915 | 0.108892 |
| 10 | 0.681085 | 0.074165 | 0.318915 | 0.108892 |
| 10 | 0.681085 | 0.074165 | 0.318915 | 0.108892 |
| 10 | 0.681085 | 0.074165 | 0.318915 | 0.108892 |
| 10 | 0.681085 | 0.074165 | 0.318915 | 0.108892 |
| 10 | 0.681085 | 0.074165 | 0.318915 | 0.108892 |
| 10 | 0.681085 | 0.074165 | 0.318915 | 0.108892 |

| | | | | |
|---|---|---|---|---|
| 10 | 0.681085 | 0.074165 | 0.318915 | 0.108892 |
| 10 | 0.681085 | 0.074165 | 0.318915 | 0.108892 |
| 10 | 0.681085 | 0.074165 | 0.318915 | 0.108892 |
| 11 | 0.606204 | 0.075115 | 0.393796 | 0.12391 |
| 11 | 0.606204 | 0.075115 | 0.393796 | 0.12391 |
| 11 | 0.606204 | 0.075115 | 0.393796 | 0.12391 |
| 11 | 0.606204 | 0.075115 | 0.393796 | 0.12391 |
| 11 | 0.606204 | 0.075115 | 0.393796 | 0.12391 |
| 11 | 0.606204 | 0.075115 | 0.393796 | 0.12391 |
| 12 | 0.531731 | 0.073441 | 0.468269 | 0.138118 |
| 12 | 0.531731 | 0.073441 | 0.468269 | 0.138118 |
| 12 | 0.531731 | 0.073441 | 0.468269 | 0.138118 |
| 12 | 0.531731 | 0.073441 | 0.468269 | 0.138118 |
| 12 | 0.531731 | 0.073441 | 0.468269 | 0.138118 |
| 12 | 0.531731 | 0.073441 | 0.468269 | 0.138118 |
| 13 | 0.459997 | 0.069741 | 0.540003 | 0.151612 |
| 13 | 0.459997 | 0.069741 | 0.540003 | 0.151612 |
| 13 | 0.459997 | 0.069741 | 0.540003 | 0.151612 |
| 13 | 0.459997 | 0.069741 | 0.540003 | 0.151612 |
| 14 | 0.392722 | 0.064619 | 0.607278 | 0.164541 |
| 14 | 0.392722 | 0.064619 | 0.607278 | 0.164541 |
| 14 | 0.392722 | 0.064619 | 0.607278 | 0.164541 |
| 14 | 0.392722 | 0.064619 | 0.607278 | 0.164541 |
| 14 | 0.392722 | 0.064619 | 0.607278 | 0.164541 |
| 14 | 0.392722 | 0.064619 | 0.607278 | 0.164541 |
| 14 | 0.392722 | 0.064619 | 0.607278 | 0.164541 |
| 14 | 0.392722 | 0.064619 | 0.607278 | 0.164541 |
| 14 | 0.392722 | 0.064619 | 0.607278 | 0.164541 |
| 14 | 0.392722 | 0.064619 | 0.607278 | 0.164541 |
| 14 | 0.392722 | 0.064619 | 0.607278 | 0.164541 |
| 14 | 0.392722 | 0.064619 | 0.607278 | 0.164541 |
| 15 | 0.331052 | 0.058616 | 0.668948 | 0.17706 |
| 15 | 0.331052 | 0.058616 | 0.668948 | 0.17706 |
| 15 | 0.331052 | 0.058616 | 0.668948 | 0.17706 |
| 15 | 0.331052 | 0.058616 | 0.668948 | 0.17706 |
| 15 | 0.331052 | 0.058616 | 0.668948 | 0.17706 |
| 15 | 0.331052 | 0.058616 | 0.668948 | 0.17706 |
| 15 | 0.331052 | 0.058616 | 0.668948 | 0.17706 |
| 16 | 0.275634 | 0.052178 | 0.724366 | 0.189302 |
| 16 | 0.275634 | 0.052178 | 0.724366 | 0.189302 |
| 16 | 0.275634 | 0.052178 | 0.724366 | 0.189302 |
| 16 | 0.275634 | 0.052178 | 0.724366 | 0.189302 |
| 16 | 0.275634 | 0.052178 | 0.724366 | 0.189302 |
| 16 | 0.275634 | 0.052178 | 0.724366 | 0.189302 |

| | | | | |
|---|---|---|---|---|
| 17 | 0.226722 | 0.045656 | 0.773278 | 0.201372 |
| 17 | 0.226722 | 0.045656 | 0.773278 | 0.201372 |
| 17 | 0.226722 | 0.045656 | 0.773278 | 0.201372 |
| 18 | 0.184263 | 0.039311 | 0.815737 | 0.213341 |
| 18 | 0.184263 | 0.039311 | 0.815737 | 0.213341 |
| 18 | 0.184263 | 0.039311 | 0.815737 | 0.213341 |
| 19 | 0.147978 | 0.033333 | 0.852022 | 0.225253 |
| 19 | 0.147978 | 0.033333 | 0.852022 | 0.225253 |
| 19 | 0.147978 | 0.033333 | 0.852022 | 0.225253 |
| 20 | 0.117433 | 0.027848 | 0.882567 | 0.237137 |
| 20 | 0.117433 | 0.027848 | 0.882567 | 0.237137 |
| 20 | 0.117433 | 0.027848 | 0.882567 | 0.237137 |
| 20 | 0.117433 | 0.027848 | 0.882567 | 0.237137 |
| 20 | 0.117433 | 0.027848 | 0.882567 | 0.237137 |
| 20 | 0.117433 | 0.027848 | 0.882567 | 0.237137 |
| 21 | 0.092093 | 0.022932 | 0.907907 | 0.249006 |
| 21 | 0.092093 | 0.022932 | 0.907907 | 0.249006 |
| 22 | 0.071369 | 0.018618 | 0.928631 | 0.260869 |
| 22 | 0.071369 | 0.018618 | 0.928631 | 0.260869 |
| 23 | 0.054656 | 0.014906 | 0.945344 | 0.272728 |
| 24 | 0.041364 | 0.011772 | 0.958636 | 0.284587 |
| 24 | 0.041364 | 0.011772 | 0.958636 | 0.284587 |
| 26 | 0.022863 | 0.007049 | 0.977137 | 0.308303 |
| 28 | 0.012052 | 0.004001 | 0.987948 | 0.332019 |
| 28 | 0.012052 | 0.004001 | 0.987948 | 0.332019 |
| 28 | 0.012052 | 0.004001 | 0.987948 | 0.332019 |

From Table-1, we find that the death density function is increasing with the failure times until (0.075115) when t=11 then decreasing with failure times from (0.073441) when t=12 until the end of failure times .We found that the survival function is decreasing, whereas the hazard function is increasing, with increasing failure times.

**$A_2$-Maximum Likelihood Method** The value of the estimated parameters $(\hat{\alpha}, \hat{\theta})$ was found to be 1.9881 and 0.0118, respectively, when this method was applied. The estimated values of the probability density function, survival function, cumulative distribution and hazard functions are illustrated in the Table-2.

**Table 2-**Estimated values of $\hat{f}$(t), $\hat{F}$(t), $\hat{S}$(t), $\hat{h}$(t) for MLE method

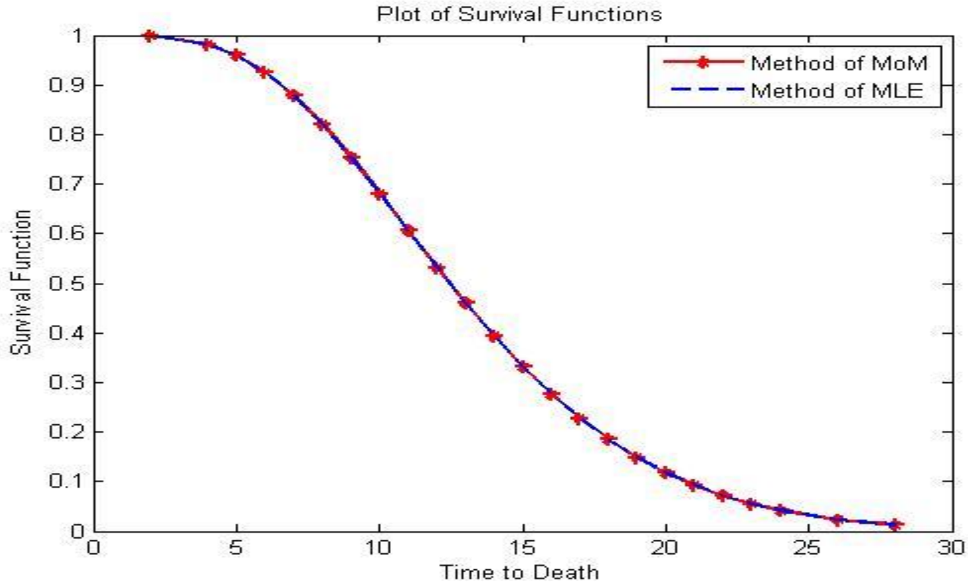
| X | $\hat{S}$(t) | $\hat{f}$(t) | $\hat{F}$(t) | $\hat{h}$(t) |
|---|---|---|---|---|
| 2 | 0.998678 | 0.002583 | 0.001322 | 0.002586 |
| 4 | 0.981538 | 0.016821 | 0.018462 | 0.017137 |
| 5 | 0.959154 | 0.028277 | 0.040846 | 0.029481 |
| 5 | 0.959154 | 0.028277 | 0.040846 | 0.029481 |
| 5 | 0.959154 | 0.028277 | 0.040846 | 0.029481 |
| 5 | 0.959154 | 0.028277 | 0.040846 | 0.029481 |
| 5 | 0.959154 | 0.028277 | 0.040846 | 0.029481 |
| 6 | 0.924601 | 0.040877 | 0.075399 | 0.04421 |

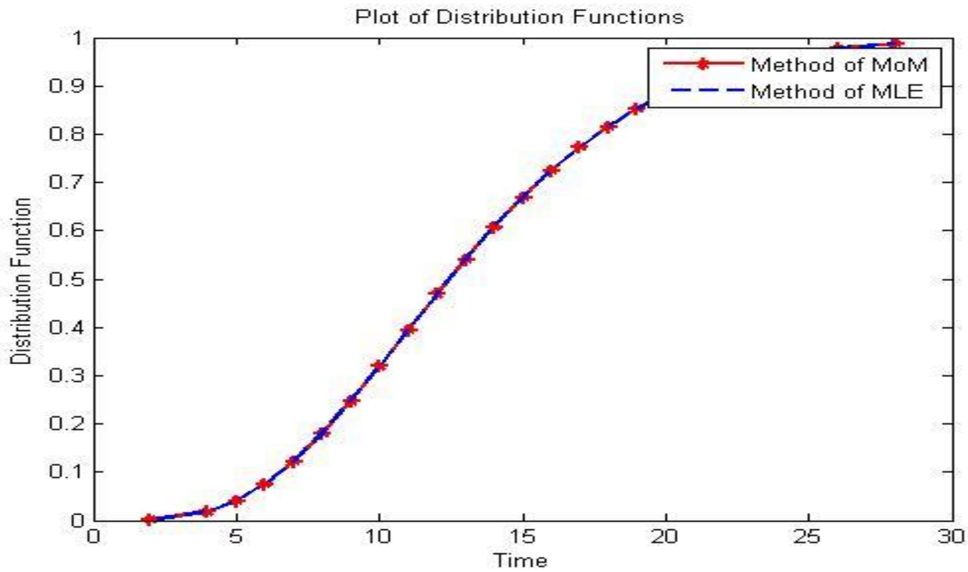
| | | | | |
|---|---|---|---|---|
| 6 | 0.924601 | 0.040877 | 0.075399 | 0.04421 |
| 6 | 0.924601 | 0.040877 | 0.075399 | 0.04421 |
| 7 | 0.877585 | 0.052931 | 0.122415 | 0.060314 |
| 7 | 0.877585 | 0.052931 | 0.122415 | 0.060314 |
| 7 | 0.877585 | 0.052931 | 0.122415 | 0.060314 |
| 7 | 0.877585 | 0.052931 | 0.122415 | 0.060314 |
| 7 | 0.877585 | 0.052931 | 0.122415 | 0.060314 |
| 8 | 0.819409 | 0.062997 | 0.180591 | 0.076881 |
| 8 | 0.819409 | 0.062997 | 0.180591 | 0.076881 |
| 8 | 0.819409 | 0.062997 | 0.180591 | 0.076881 |
| 8 | 0.819409 | 0.062997 | 0.180591 | 0.076881 |
| 8 | 0.819409 | 0.062997 | 0.180591 | 0.076881 |
| 8 | 0.819409 | 0.062997 | 0.180591 | 0.076881 |
| 9 | 0.752565 | 0.070164 | 0.247435 | 0.093233 |
| 9 | 0.752565 | 0.070164 | 0.247435 | 0.093233 |
| 9 | 0.752565 | 0.070164 | 0.247435 | 0.093233 |
| 9 | 0.752565 | 0.070164 | 0.247435 | 0.093233 |
| 9 | 0.752565 | 0.070164 | 0.247435 | 0.093233 |
| 9 | 0.752565 | 0.070164 | 0.247435 | 0.093233 |
| 9 | 0.752565 | 0.070164 | 0.247435 | 0.093233 |
| 9 | 0.752565 | 0.070164 | 0.247435 | 0.093233 |
| 9 | 0.752565 | 0.070164 | 0.247435 | 0.093233 |
| 9 | 0.752565 | 0.070164 | 0.247435 | 0.093233 |
| 9 | 0.752565 | 0.070164 | 0.247435 | 0.093233 |
| 9 | 0.752565 | 0.070164 | 0.247435 | 0.093233 |
| 10 | 0.680162 | 0.074106 | 0.319838 | 0.108953 |
| 10 | 0.680162 | 0.074106 | 0.319838 | 0.108953 |
| 10 | 0.680162 | 0.074106 | 0.319838 | 0.108953 |
| 10 | 0.680162 | 0.074106 | 0.319838 | 0.108953 |
| 10 | 0.680162 | 0.074106 | 0.319838 | 0.108953 |
| 10 | 0.680162 | 0.074106 | 0.319838 | 0.108953 |
| 10 | 0.680162 | 0.074106 | 0.319838 | 0.108953 |
| 10 | 0.680162 | 0.074106 | 0.319838 | 0.108953 |
| 10 | 0.680162 | 0.074106 | 0.319838 | 0.108953 |
| 10 | 0.680162 | 0.074106 | 0.319838 | 0.108953 |
| 10 | 0.680162 | 0.074106 | 0.319838 | 0.108953 |
| 11 | 0.605378 | 0.074984 | 0.394622 | 0.123864 |
| 11 | 0.605378 | 0.074984 | 0.394622 | 0.123864 |
| 11 | 0.605378 | 0.074984 | 0.394622 | 0.123864 |
| 11 | 0.605378 | 0.074984 | 0.394622 | 0.123864 |
| 11 | 0.605378 | 0.074984 | 0.394622 | 0.123864 |
| 11 | 0.605378 | 0.074984 | 0.394622 | 0.123864 |
| 12 | 0.53106 | 0.073266 | 0.46894 | 0.137963 |
| 12 | 0.53106 | 0.073266 | 0.46894 | 0.137963 |
| 12 | 0.53106 | 0.073266 | 0.46894 | 0.137963 |
| 12 | 0.53106 | 0.073266 | 0.46894 | 0.137963 |

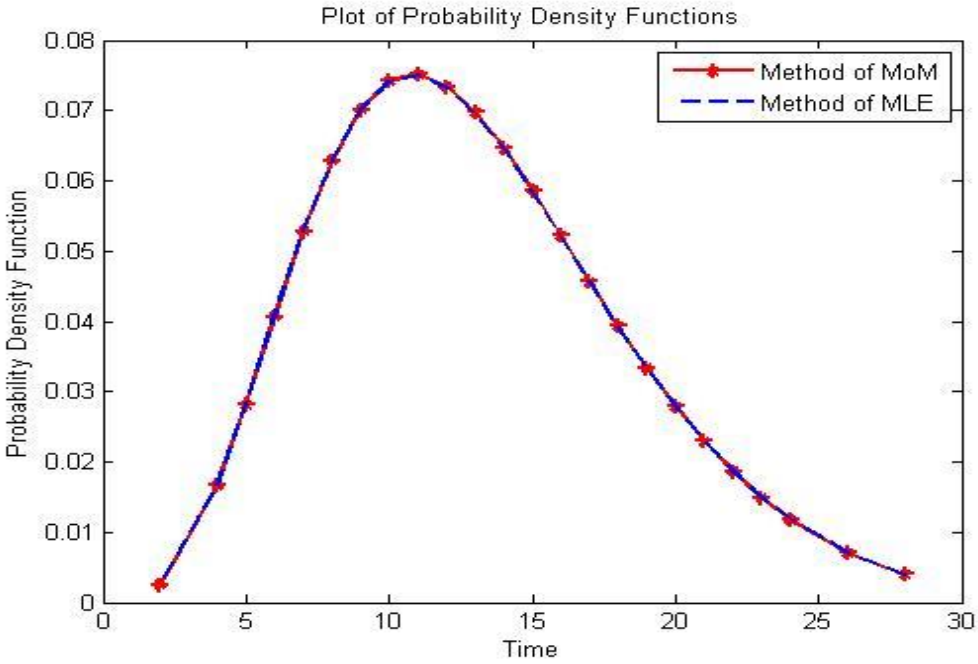
| | | | | |
|---|---|---|---|---|
| 12 | 0.53106 | 0.073266 | 0.46894 | 0.137963 |
| 12 | 0.53106 | 0.073266 | 0.46894 | 0.137963 |
| 13 | 0.459511 | 0.06955 | 0.540489 | 0.151356 |
| 13 | 0.459511 | 0.06955 | 0.540489 | 0.151356 |
| 13 | 0.459511 | 0.06955 | 0.540489 | 0.151356 |
| 13 | 0.459511 | 0.06955 | 0.540489 | 0.151356 |
| 14 | 0.392426 | 0.064435 | 0.607574 | 0.164197 |
| 14 | 0.392426 | 0.064435 | 0.607574 | 0.164197 |
| 14 | 0.392426 | 0.064435 | 0.607574 | 0.164197 |
| 14 | 0.392426 | 0.064435 | 0.607574 | 0.164197 |
| 14 | 0.392426 | 0.064435 | 0.607574 | 0.164197 |
| 14 | 0.392426 | 0.064435 | 0.607574 | 0.164197 |
| 14 | 0.392426 | 0.064435 | 0.607574 | 0.164197 |
| 14 | 0.392426 | 0.064435 | 0.607574 | 0.164197 |
| 14 | 0.392426 | 0.064435 | 0.607574 | 0.164197 |
| 14 | 0.392426 | 0.064435 | 0.607574 | 0.164197 |
| 14 | 0.392426 | 0.064435 | 0.607574 | 0.164197 |
| 14 | 0.392426 | 0.064435 | 0.607574 | 0.164197 |
| 15 | 0.330929 | 0.058456 | 0.669071 | 0.176641 |
| 15 | 0.330929 | 0.058456 | 0.669071 | 0.176641 |
| 15 | 0.330929 | 0.058456 | 0.669071 | 0.176641 |
| 15 | 0.330929 | 0.058456 | 0.669071 | 0.176641 |
| 15 | 0.330929 | 0.058456 | 0.669071 | 0.176641 |
| 15 | 0.330929 | 0.058456 | 0.669071 | 0.176641 |
| 15 | 0.330929 | 0.058456 | 0.669071 | 0.176641 |
| 16 | 0.275656 | 0.05205 | 0.724344 | 0.188823 |
| 16 | 0.275656 | 0.05205 | 0.724344 | 0.188823 |
| 16 | 0.275656 | 0.05205 | 0.724344 | 0.188823 |
| 16 | 0.275656 | 0.05205 | 0.724344 | 0.188823 |
| 16 | 0.275656 | 0.05205 | 0.724344 | 0.188823 |
| 16 | 0.275656 | 0.05205 | 0.724344 | 0.188823 |
| 17 | 0.226855 | 0.045562 | 0.773145 | 0.200842 |
| 17 | 0.226855 | 0.045562 | 0.773145 | 0.200842 |
| 17 | 0.226855 | 0.045562 | 0.773145 | 0.200842 |
| 18 | 0.184472 | 0.03925 | 0.815528 | 0.212767 |
| 18 | 0.184472 | 0.03925 | 0.815528 | 0.212767 |
| 18 | 0.184472 | 0.03925 | 0.815528 | 0.212767 |
| 19 | 0.148234 | 0.033299 | 0.851766 | 0.224641 |
| 19 | 0.148234 | 0.033299 | 0.851766 | 0.224641 |
| 19 | 0.148234 | 0.033299 | 0.851766 | 0.224641 |
| 20 | 0.11771 | 0.027837 | 0.88229 | 0.236489 |
| 20 | 0.11771 | 0.027837 | 0.88229 | 0.236489 |
| 20 | 0.11771 | 0.027837 | 0.88229 | 0.236489 |
| 20 | 0.11771 | 0.027837 | 0.88229 | 0.236489 |
| 20 | 0.11771 | 0.027837 | 0.88229 | 0.236489 |
| 20 | 0.11771 | 0.027837 | 0.88229 | 0.236489 |

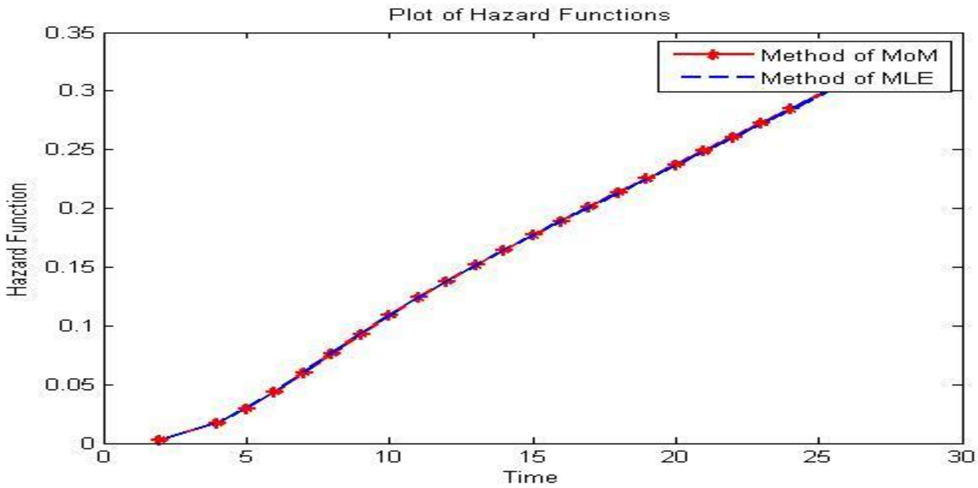
| | | | | |
|---|---|---|---|---|
| 21 | 0.092372 | 0.022938 | 0.907628 | 0.248324 |
| 21 | 0.092372 | 0.022938 | 0.907628 | 0.248324 |
| 22 | 0.071635 | 0.018636 | 0.928365 | 0.260153 |
| 22 | 0.071635 | 0.018636 | 0.928365 | 0.260153 |
| 23 | 0.0549 | 0.014932 | 0.9451 | 0.27198 |
| 24 | 0.04158 | 0.011801 | 0.95842 | 0.283806 |
| 24 | 0.04158 | 0.011801 | 0.95842 | 0.283806 |
| 26 | 0.02302 | 0.007078 | 0.97698 | 0.307456 |
| 28 | 0.012156 | 0.004025 | 0.987844 | 0.331107 |
| 28 | 0.012156 | 0.004025 | 0.987844 | 0.331107 |
| 28 | 0.012156 | 0.004025 | 0.987844 | 0.331107 |

From Table-2, we find that the death density function is increasing with the failure times until (0.074984) when t=11, then decreasing with failure times from (0.073266) when t=12 until the end of failure times .We found that the survival function is decreasing, whereas the hazard function is increasing, with increasing failure times .We find that






**5 : Conclusions**

1. The two estimation methods used in this study showed an inverse relationship between the probability survival function and the failure times was observed.

A very clear direct relationship between the times of failure and the probability hazard function was also observed.

The values of the estimated probability density function, using both the two methods, were increasing until reaching a value of times of failure of x = 11), but they were decreasing with increasing the failure times. This shows that the relationship between the probability density function and the failure times is not constant.

As expected, we clearly observed that the values of the cumulative distribution function are increasing with increasing values of failure times.

Observing that the estimation values of the parameters which is shape parameter and the scale parameter in the weighted Rayleigh distribution. It is clear that, for the $\hat{\alpha}$ values are one close to the other in the MLE, ME method but on the other hand the value of $\hat{\theta}$ is close to one unit of the other.

The estimated values of the two parameters for the weighted Rayleigh distribution are shown in Table -3.

| The methods | $\hat{\alpha}$ | $\hat{\theta}$ |
|---|---|---|
| **MLE** | 1.9881 | 0.0118 |
| **ME** | 1.9679 | 0.0119 |

It is known that the values of the probability hazard function depend on the value of the shape parameter. Thus, the probability hazard function was increased , reaching a value of $\alpha > 1$, for the two estimation methods used.

The mean squares error values for the MLE, ME methods By using the equation (40) finding the MSE for MLE and ME are very close to one another and eventually the MLE method is better than ME as below:

| **MSE** | |
|---|---|
| **OLS** | **MLE** |
| | **0.0006** |

We note that the MLE method is better than the ME method.

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