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On e^*SS -Supplement Submodules

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Abstract

One aims this work is insert comprehensible of e^*SS -supplement submodules, weakly e^*SS -supplement submodules, e^*SS -supplemented modules, weakly e^*SS -supplemented modules, Cofinitely e^*SS -supplemented modules as generalization of supplement submodules, weakly supplement submodules, supplemented modules, weakly supplement submodules, weakly supplemented and Cofinitely supplemented modules consecutively. We will display a few characteristics of these concepts.

Keywords: e^*SS -supplement submodules, weak e^*SS -supplement submodules, e^*SS -supplemented modules, weakly e^*SS -supplemented modules.

حول المقاسات الجزئية المكملية من النمط e^*SS

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الخلاصة

الهدف من هذا البحث هو تقديم المفاهيم الرئيسية للمقاسات الجزئية التكميلية من الصنف e^*SS ، والمقاسات الجزئية التكميلية الضعيفة من الصنف e^*SS ، والمقاسات التكميلية من الصنف e^*SS ، والمقاسات التكميلية المحددة من الصنف e^*SS كتعميم على مفاهيم المقاسات الجزئية التكميلية، والمقاسات الجزئية التكميلية الضعيفة، والمقاسات التكميلية، والمقاسات التكميلية الضعيفة، والمقاسات التكميلية المحددة على التوالي. وسوف نبرهن بعض خصائص هذه المفاهيم.

1. Introduction

In this article M will be a unitary left R -module, and R be any ring with identity. Remember that a submodule S of M is named small ($S \ll M$) if any submodule J of M , $S + J = M$, then $J = M$, [1-3]. Many popularizations of small submodules were introduced see, [4-5]. A non-zero submodule E of M is considered to be essential if each $V \subseteq M, V \neq \{0\}$ where $E \cap V = \{0\}$, symbolizes that $E \subseteq_e M$ moreover M is known as the essential extension of E , see [2], [4], [6] and [7]. Baanoon in [8] introduced a latest type of

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submodules as a popularization of essential submodule named e^* -essential as follow:

For any non-zero cosingular submodules B of M , if $X \cap B \neq 0$, already X is named e^* -essential submodule in M , symbolized by $X \leq_{e^*} M$. The singular submodule is known as $Z(M) = \{t \in M : ann(t) \leq_e R\}$. If $Z(M) = M$, then M is called a singular module and a non-singular module if $Z(M) = 0$, this can be found in [9], and [10]. In [11] Özcan defined cosingular submodule as follows: $Z^*(M) = \{w \in M : wR \ll E(M)\}$, where $E(M)$ is the injective hull of the module M . Recall that, the injective hull of the module M is a minimal injective extension of M , [9]. A module M is named cosingular whenever $Z^*(M) = M$. A generalization of singular module had been given by Kamil and Davvaz in [12]. As well as, in [13], Kabban utilized e^* -essential submodule to insert a popularization to singular module as following: $Z_{e^*}(M) = \{v \in M : ann(v) \leq_{e^*} R\}$ if $Z_{e^*}(M) = M$, ($Z_{e^*}(M) = 0$), from that M will called an e^* -singular (e^* -non-singular), the concept of e^* -singular had been by Kabban in [13] to introduce an e^* S-essential submodule, when a submodule K of a module M is named e^* S-essential if for any non-zero e^* -singular submodule L , $K \cap L \neq 0$. A submodule T of M is termed a supplement of H in M if, $M = T + H$ with $T \cap H \ll T$, see [14-16]. In this work we have presented a new popularization of the concepts supplement submodules, weak supplement submodules supplemented modules, weakly supplemented modules, Cofinitely supplemented modules like in [17-18]. Many generalizations of supplemented, weakly supplemented modules were introduced in, [19-22]. In [23-25] one of popularization of supplemented modules define as follows, M is termed SS-supplemented whether any submodule P of M has a supplement Q in M then $P \cap Q$ is semi simple.

2. e^* SS-supplement submodules

This section is devoted to study a popularization of supplement submodules. Remember that, a submodule T of M is named e^* -singular essential small submodule of M (signified $T \ll_{e^*ss} M$), if whenever $M = T + H$, with H is e^* -singular essential submodule ($H \subseteq_{e^*s} M$), leads to, $M = H$.

Definition 2.1:

Suppose that Q and D are submodules of M . If $M = Q + D$ and $Q \cap D \ll_{e^*ss} P$, then Q is named e^* SS-supplement of D in M . If any submodule of M has e^* SS-supplement, thereafter M is named e^* SS-supplemented module.

Remark 2.2:

The submodule P is e^* SS-supplement of H in M if and only if P is minimal with references to $M = H + D$, $\forall D \subseteq_{e^*s} P$.

Proof:

Suppose $M = H + D$, and $D \subseteq_{e^*s} P$. We have $M = H + P$ and $P \cap H \ll_{e^*ss} P$. We have to prove $D = P$. Since $P = M \cap P = (H + D) \cap P = H \cap P + D \cap P = D + H \cap P$, as $D \subseteq_{e^*s} P$. Hence, $P = D + H \cap P$, this is because $P \cap H \ll_{e^*ss} P$, therefore $P = D$.

Conversely, suppose P is minimal with regarding to property $M = H + P$, i.e., if there exists $X \subseteq_{e^*s} P$, satisfies

$M = H + X$, then $X = P$. To prove $P \cap H \ll_{e^*ss} P$, let $P \cap H + X = P$, $X \subseteq_{e^*s} P$, since $M = H + P$, so $M = P \cap H + X + H = X + H$, as $P \cap H \subseteq H$. Therefore, by minimality of P we get, $X = P$, and hence $P \cap H \ll_{e^*ss} P$.

Examples and remarks 2.3:

1. Every supplement submodule is e^* SS-supplement. The opposite is not true, for example, $\langle \bar{2} \rangle$ is e^* SS-supplement of Z_4 as Z -module since $\langle \bar{2} \rangle + Z_4 = Z_4$ and $\langle \bar{2} \rangle \cap Z_4 = \langle \bar{2} \rangle$

- $\ll_{e^*SS} \bar{2} \rangle$. But $\langle \bar{2} \rangle$ is not a supplement of Z_4 since $\langle \bar{2} \rangle$ is not small in $\langle \bar{2} \rangle$.
2. In any module M . The submodule $\{0\}$ is e^*SS -supplement of M in M and M is the e^*SS -supplement of $\{0\}$ in M .
 3. From (1) and (2), we see that an e^*SS -supplement is not unique since $\langle \bar{0} \rangle$ and $\langle \bar{2} \rangle$ are e^*SS -supplement of Z_4 in Z_4 .
 4. In generic the e^*SS -supplement submodule need not be existed. To explain that, in Z -module Z , $2Z$ has no e^*SS -supplement submodule. Since the only e^*S -essential small submodule of Z is $\{0\}$.
 5. In Z_{12} as Z -module, $\langle \bar{4} \rangle$ is not e^*SS -supplement of $\langle \bar{2} \rangle$, since $\langle \bar{4} \rangle + \langle \bar{2} \rangle = \langle \bar{2} \rangle \neq Z_{12}$.
 6. The e^*SS -supplement is non commute. To explain this the Z -module Z_{24} , $\langle \bar{2} \rangle$ has an e^*SS -supplement of $\langle \bar{3} \rangle$. But $\langle \bar{2} \rangle$ is not e^*SS -supplement of $\langle \bar{3} \rangle$, since $\langle \bar{2} \rangle \cap \langle \bar{3} \rangle = \langle \bar{6} \rangle$ and $\langle \bar{6} \rangle$ is not e^*SS -small in $\langle \bar{2} \rangle$. $\langle \bar{6} \rangle + \langle \bar{4} \rangle = \langle \bar{2} \rangle$, but $\langle \bar{4} \rangle$ not equal to $\langle \bar{2} \rangle$.
 7. Presume M is e^* -singular R -module, then supplemented and e^*SS -supplemented modules coincide (for e^* -singular modules $Z(M) = Z_{e^*}(M)$, [13]).
 8. The direct summand of any module M is an e^*SS -supplement. To see this let $T \subseteq_{\oplus} M$. By definition of direct summand there exists a submodule W such that $M = T + W$ and $T \cap W = \{0\}$. Since the submodule $\{0\}$ is always an e^*SS small. Hence, T is an e^*SS -supplement in M .

In following the main results of e^*SS -supplement submodules have been illustrated:

Proposition 2.4:

Suppose E and W are submodules of a module M achieve $W \subseteq E \subseteq M$. Let W be an e^*SS -supplement in M , then W is an e^*SS -supplement in E .

Proof:

Since W is an e^*SS -supplement in M , then there is a submodule D of M achieves $W + D = M$ and $W \cap D \ll_{e^*SS} W$. The submodule $E = E \cap M = E \cap (W + D)$ and by Modular law $E = W + (E \cap D)$. Hence, W is an e^*SS -supplement of $E \cap D$ in E since $W \cap (E \cap D) = W \cap D \ll_{e^*SS} W$. Hence, W is an e^*SS -supplement in E .

Proposition 2.5: Let M be module,

- 1) Presume that T, K , and J are submodules of M , with $K \subseteq T$;
 - a) If $T \ll_{e^*SS} M$, then $K \ll_{e^*SS} M$ and $\frac{T}{K} \ll_{e^*SS} \frac{M}{K}$.
 - b) $T + J \ll_{e^*SS} M$ if and only if $T \ll_{e^*SS} M$ and $J \ll_{e^*SS} M$.
- 2) Suppose that $K \ll_{e^*SS} M$ and $\Psi: M \rightarrow N$, is a homomorphism, therefore, $\Psi(K) \ll_{e^*SS} N$. Special case, if $K \ll_{e^*SS} X \subseteq N$, subsequently $K \ll_{e^*SS} N$.
- 3) Assume that $C_1 \subseteq_{e^*S} M_1 \subseteq M$, $C_2 \subseteq_{e^*S} M_2 \subseteq M$ and $M = M_1 \oplus M_2$, then $C_1 \oplus C_2 \ll_{e^*SS} M_1 \oplus M_2$ if and only if $C_1 \ll_{e^*SS} M_1$ and $C_2 \ll_{e^*SS} M_2$.

Proof. 1) a) Let $L \subseteq_{e^*S} M$ and $L + K = M$, then $L + T = M$.

Thus, $L = M$ for $T \ll_{e^*SS} M$, so $K \ll_{e^*SS} M$. If $J \leq M$, with $\frac{J}{K} \subseteq_{e^*S} \frac{M}{K}$ and $\frac{J}{K} + \frac{T}{K} = \frac{M}{K}$, then $J + T = M$ and $J \subseteq_{e^*S} M$ by [14], hence $J = M$ and $\frac{J}{K} = \frac{M}{K}$.

b) $\Rightarrow$ Clear by (a).

$\Leftarrow$) Suppose $K \subseteq_{e^*S} M$, with $T + J + K = M$, then $J + K = M$, since $J + K \subseteq_{e^*S} M$ and $T \ll_{e^*SS} M$, thus $K = M$, since $J \ll_{e^*SS} M$.

2) Suppose $D \subseteq_{e^*s} N$, and $D + \Psi(K) = N$, it is enough to prove that $D = N$. By [14, Proposition 2.1.6] we obtain $\Psi^{-1}(D) \subseteq_{e^*s} M$. Let $t \in M$, hence $\Psi(t) \in N = D + \Psi(K)$, so there exists $k \in K$ and $d \in D$ achieves $\Psi(t) = d + \Psi(k)$, then $d = \Psi(t - k)$, this implies $t - k \in \Psi^{-1}(D)$ and $t = t - k + k \in \Psi^{-1}(D) + K$. Hence $\Psi^{-1}(D) + K = M$, since $K \ll_{e^*ss} M$, thus $\Psi^{-1}(D) = M$ and $\Psi(M) = \Psi(\Psi^{-1}(D)) = \Psi(D) \cap D$, so $\Psi(M) \subseteq D$, that means $\Psi(K) \subseteq D$. Therefore, $D = \Psi(K) + D = N$, and $\Psi(K) \ll_{e^*ss} N$.

3) Direct from (1) and (2).

Proposition 2.6:

Assume that T and L are submodules of M achieves $T \subseteq L \subseteq M$. Whenever L is an e^* SS-supplement in M , therefore $\frac{L}{T}$ is an e^* SS-supplement in $\frac{M}{T}$.

Proof:

Since L is an e^* SS-supplement in M , then there exists $Q \subseteq M$, such that $L + Q = M$, and $L \cap Q \ll_{e^*ss} L$, now, $\frac{M}{T} = \frac{L+Q}{T} = \frac{L}{T} + \frac{Q+T}{T}$ and $\frac{L}{T} \cap \frac{Q+T}{T} = \frac{L \cap (Q+T)}{T} = \frac{T + (L \cap Q)}{T}$, by Modular law. Since $L \cap Q \ll_{e^*ss} L$, then $\frac{T + (L \cap Q)}{T} \ll_{e^*ss} \frac{L}{T}$ [Proposition 2.5, (1)]. Therefore, $\frac{L}{T}$ is an e^* SS-supplement $\frac{Q+T}{T}$ in $\frac{M}{T}$.

Recall that, where M is a module such that $(M \neq 0)$ then is named an e^* -singular essential-hollow module (for brief e^* SS-hollow) if any submodule P of M such that $(P \neq M)$ is an e^* -singular essential small submodule of M .

Proposition 2.7:

Suppose that V be an e^* SS-hollow submodule of M . Then V is an e^* SS-supplement of each proper submodule P of M , such that $M = V + P$.

Proof:

Presume that P is a proper submodule of M achieves $M = V + P$. So, $V \cap P$ is a proper submodule of V , if $V \cap P = V$. Then $V \subseteq P$ and $M = P$, which is a contrary the hypothesis. Since V is an e^* SS-hollow means $V \cap P \ll_{e^*ss} V$. Thus, V is an e^* SS-supplement of P .

Proposition 2.8:

Suppose that H and Q are submodules of M achieves H is an e^* SS-supplement of Q in M . If $M = L + H$, for some $L \subseteq Q$, then H is an e^* SS-supplement of L in M .

Proof:

Let $M = L + H$, for some $L \subseteq Q$ of Q and H is an e^* SS-supplement of Q in M , we have $M = H + Q$ and $H \cap Q \ll_{e^*ss} H$, since $L \subseteq Q$. So, $L \cap H \subseteq H \cap Q \ll_{e^*ss} H$ by [Proposition 2.5, (1)] and $M = L + H$. Therefore, H is an e^* SS-supplement to L in M .

Proposition 2.9:

Assume that P and G are submodules of M . If G is an e^* SS-supplement of P , then $\frac{G+K}{K}$ is an e^* SS-supplement of $\frac{P}{K}$ in $\frac{M}{K}$, for $K \subseteq P$.

Proof:

Presume that G is e^* SS-supplement of P in M . Then $M = P + G$ and $P \cap G \ll_{e^*ss} G$, for $K \subseteq P$, we have $P \cap (G + K) = (P \cap G) + K$, by Modular law, and $\frac{P}{K} \cap \left(\frac{G+K}{K}\right) = \frac{(P \cap G + K)}{K}$, since $(P \cap G) \ll_{e^*ss} G$, consequently, $\frac{(P \cap G + K)}{K} \ll_{e^*ss} \frac{G+K}{K}$, by [Proposition 2.5, (2)]. Now, $\frac{M}{K} = \frac{P+G}{K} = \frac{P}{K} + \frac{G+K}{K}$. Therefore, $\frac{G+K}{K}$ e^* SS-supplement of $\frac{P}{K}$ in $\frac{M}{K}$.

Proposition 2.10

Let T and H be submodules of M , the subsequent conditions are identical:

- 1) H is an e^* SS-supplement of T in M ;
- 2) $M = T + H$ and for every proper submodule X of H , achieve $X \subsetneq_{e^*s} H$, then $M \neq T + X$.

Proof:

(1) $\Rightarrow$ (2) Assume that H is an e^* SS-supplement of T in M and assume that $X \subsetneq_{e^*s} H$. Then by Modular law, $H = H \cap M = H \cap (T + X) = X + (T \cap H)$. Since H is an e^* SS-supplement of T in M and by (1) $(T \cap H) \ll_{e^*ss} H$, $X \subsetneq_{e^*s} H$ by Remark 2.2, we get $H = X$, which is a contradiction. Thus, $M \neq T + X$.

(2) $\Rightarrow$ (1) Assume that $M = H + T$. To show H is an e^* SS-supplement of T in M , it is enough to prove that $(H \cap T) \ll_{e^*ss} H$. Let C be submodule of H such that $H = (T \cap H) + C$, where $C \subseteq_{e^*s} H$. If C is a proper submodule of H , then by our assumption $M \neq T + X$. But $M = T + H = T + (T \cap H) + C = T + C$, that is a contradiction. Therefore, H is an e^* SS-supplement to T in M .

Proposition 2.11:

Assume that N, P and D are submodules of M . If N is an e^* SS-supplement of P in M , and P is an e^* SS-supplement of D in M , obtaining P is an e^* SS-supplement of N in M .

Proof:

Let $M = N + P = P + D$, $N \cap P \ll_{e^*ss} N$ and $P \cap D \ll_{e^*ss} P$. In order to show that $N \cap P \ll_{e^*ss} P$. Let Y be a submodule of P such that $Y \subseteq_{e^*s} P$ and $P = (N \cap Y) + Y$. Because, $M = P + D = (N \cap P) + Y + D$ and $Y \subseteq_{e^*s} P$, $D \subseteq_{e^*s} D$, then $Y + D \subseteq_{e^*s} P + D = M$, $N \cap P \ll_{e^*ss} M$, this implies $M = Y + D$.

Now, $P = P \cap M = P \cap (Y + D) = Y + (P \cap D)$, by Modular law, from that we have $P \cap D \ll_{e^*ss} P$, this gives $P = Y$. Thus, P is an e^* SS-supplement to N in M .

Remember that a fully invariant submodule N of M is defined as follows: for each $g \in \text{End}(M)$, $g(N) \subseteq N$ and M is named duo module if each submodule of M is a fully invariant, look [26] and [27].

Proposition 2.12:

Assume that M be an e^* SS-supplemented module, and L be a fully invariant submodule of M , therefore $\frac{M}{L}$ is an e^* SS-supplemented.

Proof:

Let $\frac{P}{L} \subseteq \frac{M}{L}$, we have to prove $\frac{P}{L}$ has an e^* SS-supplement in $\frac{M}{L}$, where $P \subseteq M$. Because M is an e^* SS-supplemented, then there is $F \subseteq M$, achieves, $M = P + F$, and $P \cap F \ll_{e^*ss} F$. Now, $\frac{M}{L} = \frac{P+F}{L} = \frac{P}{L} + \frac{F+L}{L}$, to prove $\frac{P}{L} \cap \frac{F+L}{L} \ll_{e^*ss} \frac{F+L}{L}$, let $\left(\frac{P}{L} \cap \frac{F+L}{L}\right) + \frac{V}{L} = \frac{F+L}{L}$, with $\frac{V}{L} \subseteq_{e^*s} \frac{M}{L}$, so $\left(\frac{P \cap (F+L)}{L}\right) = \frac{L+(P \cap F)}{L}$, then $\frac{L+(P \cap F)}{L} + \frac{V}{L} = \frac{F+L}{L}$ and $L + (P \cap F) + V = F + L$, since $L \subseteq V$, then $(P \cap F) + V = F + L$, but $P \cap F \ll_{e^*ss} F \subseteq F + L$, then $P \cap F \ll_{e^*ss} F + L$ by [Proposition 2.5, (1)], therefore $V = F + L$ and $\frac{V}{L} = \frac{F+L}{L}$.

Corollary 2.13:

The homomorphic image of an e^* SS-supplemented module is e^* SS-supplemented.

Proof: As every homomorphic image is isomorphic a quotient module, hence the proof has been completed.

Remark 2.14:

In general, the opposite of Proposition 2.12 is not right. To explain that, we have $\frac{Z}{6Z} \cong Z_6$ as Z -module is an e^* SS-supplemented module. However, the Z -module Z is not an e^* SS-supplemented module.

3. Weak e^* SS-supplemented submodules

The generalization of weak supplement submodules has been introduced in this section, and some of facts of this concept have been proved as well. Keep in mind that a submodule D of M is named a weak supplement of a submodule H in M , if $M = D + H$ and $D \cap H << M$.

Definition 3.1:

Let A and T be submodules of a module M . If $M = A + T$ and $A \cap T <<_{e^*SS} M$, then A is called a weak e^* SS-supplement of T in M . A module M is named a weakly e^* SS-supplemented if each submodule of M has a weak e^* SS-supplement in M .

Examples and remarks 3.2:

1. Presume that M is an e^* -singular module, then weakly supplemented and weakly e^* SS-supplemented modules are coincide.
2. In the Z -module Z_{12} , $\langle \bar{4} \rangle$ is not a weakly e^* SS-supplement of $\langle \bar{2} \rangle$, as $\langle \bar{4} \rangle + \langle \bar{2} \rangle = \langle \bar{2} \rangle \neq Z_{12}$.
3. Every e^* SS-supplement submodule is a weak e^* SS-supplement. The opposite is not satisfied, to explain that, the Z -module Z_{24} , $\langle \bar{2} \rangle$ is weak e^* SS-supplement of $\langle \bar{3} \rangle$, but $\langle \bar{2} \rangle$ is not an e^* SS-supplement of $\langle \bar{3} \rangle$ in Z_{24} , since $\langle \bar{2} \rangle \cap \langle \bar{3} \rangle = \langle \bar{6} \rangle$ and $\langle \bar{6} \rangle + \langle \bar{4} \rangle = \langle \bar{2} \rangle$, $\langle \bar{4} \rangle \subseteq_{e^*SS} \langle \bar{2} \rangle$, but $\langle \bar{4} \rangle \neq \langle \bar{2} \rangle$.
4. The Z as Z -module is not a weakly e^* SS-supplemented, since $2Z$ has no e^* SS-supplement (weak e^* SS-supplement) submodule.

Proposition 3.3:

Assume that E, G be two submodules of M , and assume that E be a weakly e^* SS-supplemented module. If $E + G$ has a weak e^* SS-supplement in M then does G .

Proof:

From the assumption there is U achieve $U \subseteq M$, therefore, $U + (E + G) = M$, and $U \cap (E + G) <<_{e^*SS} M$. Since E is weakly e^* SS-supplemented module there exists $T \subseteq E$, such that $(U + G) \cap E + T = E$ and $(U + G) \cap T <<_{e^*SS} E$. Thus, $G + U + T = M$ and $(U + G) \cap T <<_{e^*SS} E$ and by [Proposition 2.5 (1)] $(U + G) \cap T <<_{e^*SS} M$, that is T is a weak e^* SS-supplement of $U + G$ in M . We have to prove that $U + T$ is a weak e^* SS-supplement of G in M . It is clear that $(U + T) + G = M$, it is enough to show $(U + T) \cap G <<_{e^*SS} M$. Notice that, $(U + T) \cap G = U \cap T + T \cap G$, and $U \cap (E + G) = U \cap E + U \cap G$, and $(U + G) \cap T = U \cap T + G \cap T$. This leads to $(U + T) \cap G \subseteq U \cap (E + G) + (U + G) \cap T <<_{e^*SS} M$, thus, $(U + T) \cap G <<_{e^*SS} M$. Therefore, $U + T$ is a weak e^* SS-supplement of G in M .

Corollary 3.4:

Suppose that M be a module and $M = A_1 + A_2$. If A_1 and A_2 are a weakly e^* SS-supplemented modules, then M is a weakly e^* SS-supplemented.

Proof:

Suppose $E \subseteq M$. Since $A_1 + (A_2 + E) = M$, then by Proposition 3.3, we obtain $(A_2 + E)$ has a weak e^* SS-supplement in M , and again by Proposition 3.3, E has a weak e^* SS-supplement in M . Therefore, M is a weakly e^* SS-supplemented.

Proposition 3.5:

Assume that M is weakly e^* SS-supplemented module and $Y \subseteq D \subseteq M$, if $Y <<_{e^*SS} M$ implies that $Y <<_{e^*SS} D$, then D is an e^* SS-supplement submodule of M .

Proof:

Suppose that M is a weakly e^* SS-supplemented. So, $M = D + N$, $N \subseteq M$, and $D \cap N <<_{e^*SS} M$. By our assumption we get $D \cap N <<_{e^*SS} D$. Hence, D is an e^* SS-supplement of N in M .

Proposition 3.6:

Assume that M be a weakly e^* SS-supplemented module then for $C, F \subseteq M$, with $C + F = M$, there is a weak e^* SS-supplement K of C in M with $K \subseteq F$.

Proof:

Suppose $C, F \subseteq M$, with $C + F = M$. Because M is a weakly e^* SS-supplemented, $C \cap F$ has a weak e^* SS-supplement T in M , thus $M = (C \cap F) + T$ and $(C \cap F) \cap T <<_{e^*SS} M$. As $M = C + F$ and $F = (C \cap F) + (F \cap T)$, then, $M = C + (C \cap F) + (F \cap T)$, since $(C \cap F) \subseteq C$, hence $M = C + (F \cap T)$. Let $L = F \cap T$. Then, $M = C + L$ and $C \cap L = C \cap F \cap T <<_{e^*SS} M$. Hence, L is a weak e^* SS-supplement of C in M with $L \subseteq F$.

Proposition 3.7:

Assume that P is a weak e^* SS-supplement of W in a module M . Consequently for $G \subseteq W$, $\frac{P+G}{G}$ is a weak e^* SS-supplement of $\frac{W}{G}$ in $\frac{M}{G}$.

Proof:

By assumption P is a weak e^* SS-supplement of W in M . So $M = P + W$ and $P \cap W <<_{e^*SS} M$, for $G \subseteq W$. Now, $\frac{M}{G} = \frac{W+G}{G} = \frac{W}{G} + \frac{P+G}{G}$ and $\frac{W}{G} \cap \left(\frac{P+G}{G}\right) = \frac{(W \cap P) + G}{G}$, by Modular law, and Since $W \cap P <<_{e^*SS} M$, therefore $\frac{(W \cap P) + G}{G} <<_{e^*SS} \frac{M}{G}$ by using Proposition 2.5, (1). Therefore, $\frac{P+G}{G}$ is a weak e^* SS-supplement of $\frac{W}{G}$ in $\frac{M}{G}$.

Proposition 3.8

The epimorphic image of a weakly e^* SS-supplemented module is weakly e^* SS-supplemented.

Proof:

Suppose that $\Phi: M \rightarrow M'$ be an R -epimorphism, and also M is a weakly e^* SS-supplemented. Assume that $B \subseteq M'$, then $\Phi^{-1}(B) \subseteq M$, because M is e^* SS-supplemented. Therefore, there is $P \subseteq M$ achieves $M = \Phi^{-1}(B) + P$ and $\Phi^{-1}(B) \cap P <<_{e^*SS} M$. Now, $M' = B + \Phi(P)$, to show $B \cap \Phi(P) <<_{e^*SS} M'$ it is enough to show that, $B \cap \Phi(P) \subseteq \Phi(\Phi^{-1}(B) \cap P)$. Let $w \in B \cap \Phi(P)$, then $w = a$ and $w = \Phi(b)$, $b \in P$, so $b \in \Phi^{-1}(w) \in \Phi^{-1}(B)$, hence $b \in \Phi^{-1}(B)$, $b \in P$, so $w = \Phi(b) \in \Phi(\Phi^{-1}(B) \cap P)$, but $\Phi^{-1}(B) \cap P <<_{e^*SS} M$, hence $\Phi^{-1}(B) \cap P <<_{e^*SS} \Phi(M) = M'$, and therefore $B \cap \Phi(P) <<_{e^*SS} M$.

Corollary 3.9:

Let M is a weakly e^* SS-supplemented and $T \subseteq M$, subsequently $\frac{M}{T}$ is a weakly e^* SS-supplemented.

Remember that a submodule K of M is named coclosed of M (symbolize by $K \subseteq_{CC} M$), if whenever $\frac{K}{L} << \frac{M}{L}$ implies that $K = L$, see [28]. We can develop this definition as follows, let $T \subseteq M$, we say that T is an e^* SS-coclosed submodule of M (denoted by $T \subseteq_{e^*SS-CC} M$) if $\frac{T}{L} <<_{e^*SS} \frac{M}{L}$, implies that $T = L$, where $L \subseteq T$, $L \subseteq_{e^*SS} M$.

Now, we can show the below:

Proposition 3.10

Suppose that B is a submodule of M . Observe the following:

- 1) A submodule B is an e^* SS-supplement of M ;
- 2) A submodule B is an e^* SS-coclosed in M ;
- 3) For every submodule Y of B , if $Y \ll_{e^*SS} M$, then $Y \ll_{e^*SS} B$.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) if M is an e^* -Singular. If M is a weakly e^* SS-supplemented, then (3) $\Rightarrow$ (1).

Proof:

(1) $\Rightarrow$ (2).

Assume B be e^* SS-supplement of H in M , then $M = B + H$ and $B \cap H \ll_{e^*SS} B$.

To show B is an e^* SS-coclosed, assume that $\frac{B}{Y} \ll_{e^*SS} \frac{M}{Y}$ for $Y \subseteq_{e^*S} M, Y \subseteq B$. Since $M = B + H$, then $\frac{M}{Y} = \frac{B}{Y} + \frac{H+Y}{L}$, and we have $\frac{H+Y}{H}$ is e^* -Singular, by [14]. But, $\frac{B}{Y} \ll_{e^*SS} \frac{M}{Y}$, then $\frac{H+Y}{H} = \frac{M}{Y}$ implies that $M = H + Y$. Not that $B = B \cap M = B \cap (H + Y) = Y + (B \cap H)$. But $B \cap H \ll_{e^*SS} B$ and $Y \subseteq_{e^*S} M$. Therefore, $B = H$. This gives $B = Y$. So, B is an e^* SS-coclosed.

(2) $\Rightarrow$ (3) Suppose B is an e^* SS-coclosed in M and $T \ll_{e^*SS} M$, let $B = T + D$ and $D \subseteq_{e^*S} M$. Since B is e^* SS-coclosed in M , it is sufficient to show $\frac{B}{D} \ll_{e^*SS} \frac{M}{D}$. Let $\frac{M}{D} = \frac{B}{D} + \frac{H}{D}$ for $\frac{H}{D} \subseteq_{e^*SS} \frac{M}{D}$, then $M = B + H = T + D + H = T + H$ since M is an e^* -Singular thus $H \subseteq_{e^*S} M$. But $T \ll_{e^*SS} M$, thus, $M = H$.

(3) $\Rightarrow$ (1)

Because M is a weakly e^* SS-supplemented, there is a submodule D of M such that $M = B + D$ and $B \cap D \ll_{e^*SS} M$. By assumption, $B \cap D \ll_{e^*SS} B$. Thus, B is an e^* SS-supplement of M .

4. Cofinitely e^* SS-supplemented module

Recall that a submodule L of M is called cofinite if $\frac{M}{L}$ is finitely generated (f.g), see [29], and [30]. This section is devoted to define cofinitely e^* SS-supplemented module with main properties.

Definition 4.1:

A module M is named cofinitely e^* SS-supplemented (in short Cof e^* SS-supplemented) if each cofinite submodule of M has an e^* SS-supplement in M .

Examples and remarks 4.2:

- 1) The Z -module Z_6 is Cof e^* SS-supplemented module.
- 2) The Z -module Z is not Cof e^* SS-supplemented. Because $\langle 2 \rangle$ is a cofinite submodule of Z which has no-supplement.
- 3) Obviously all e^* SS-supplemented module is Cof e^* SS-supplemented. In general, the opposite is not valid, to explain that, the Z -module Q is Cof e^* SS-supplemented, because the only cofinite submodule of Q is Q itself which has an e^* SS-supplement, then Q is not an e^* SS-supplemented.

The following proposition explains the sufficient condition which makes an e^* SS-supplemented module and a Cof e^* SS-supplemented are equivalent.

Proposition 4.3:

Assume that M is a finitely generated module (f.g). Consequently M is an e^* SS-supplemented

if and only if M is a Cof e^* SS-supplemented.

Proof: $\Leftarrow$

Presume that M is Cof e^* SS-supplemented to manifest M is an e^* SS-supplemented. Suppose $T \subseteq M$, because M is (f.g), then $\frac{M}{T}$ is (f.g), thus T is Cofinite submodule of M . Because M is Cof e^* SS-supplemented, we get T has e^* SS-supplement in M . Thus, M is an e^* SS-supplemented.

$\Rightarrow$) Clearly by Examples and remarks 4.2, (3).

Proposition 4.4:

Assume that M is a Cof e^* SS-supplemented, and Q be a submodule of M , therefore $\frac{M}{Q}$ is a Cof e^* SS-supplemented for any fully invariant submodule Q of M .

Proof:

Presume that $\frac{V}{Q}$ be a Cofinite submodule of $\frac{M}{Q} \cong \frac{\frac{M}{Q}}{\frac{V}{Q}}$ which is (f.g), hence $\frac{M}{V}$ is (f.g) by Third Isomorphism Theorem. So, V is a Cofinite of M . Thus, there is $Y \subseteq M$, moreover $M = Y + V$ and $Y \cap V <<_{e^*SS} M$. Now, $\frac{M}{Q} = \frac{Y+V}{Q} = \frac{Y}{Q} + \frac{Y+Q}{L}$, $\left(\frac{V}{Q} \cap \frac{Y+Q}{Q}\right) = \frac{V \cap (Y+Q)}{Q} = \frac{(Y \cap V) + Q}{Q} <<_{e^*SS} \frac{M}{Q}$, by Modular law and Proposition 2.5.

In general, the opposite of Proposition 4.4, is not correct. To explain that, in the Z -module Z , $\frac{Z}{6Z} \cong Z_6$ is a Cof e^* SS-supplemented, however Z is not Cof e^* SS-supplemented.

Corollary 4.5:

Let M be a Cof e^* SS-supplemented duo module, then any direct summand of M is Cof e^* SS-supplemented.

Proposition 4.6:

Let C, E be submodules of M and C is a Cof e^* SS-supplemented, E is a Cofinite in M and $C + E$ has e^* SS-supplement D in M . Therefore $C \cap (E + D)$ has e^* SS-supplement W in C . In addition, $W + D$ is e^* SS-supplement of E in M .

Proof: Presume that D be an e^* SS-supplement of $C + E$ in M . Thus, $M = D + (C + E)$ and $D \cap (C + E) <<_{e^*SS} D$. Now, by The Second Isomorphism Theorem $\frac{C}{C \cap (E+D)} \cong \frac{C+(E+D)}{E+D} = \frac{\frac{M}{E+D}}{\frac{E}{E+D}}$ which is f.g, therefore $C \cap (E + D)$ is Cofinite in C . However, C is Cof e^* SS-supplemented. As a submodule W of C exists, achieves W is an e^* SS-supplement of $C \cap (E + D)$ in C . Thus, $C = W + [C \cap (E + D)]$ and $W \cap [C \cap (E + D)] = W \cap (E + D) <<_{e^*SS} W$. To prove $D + W$ is an e^* SS-supplement of E in M , notice because $M = D + C + E = D + W + [C \cap (E + D)] + E = D + W + E$, since $E \cap (D + W) \subseteq [D \cap (E + W)] + [W \cap (E + D)] <<_{e^*SS} W + D$, therefore, $W + D$ is e^* SS-supplement of E in M .

5. Conclusions

We certain the below outcomes

- 1) Every supplemented module is an e^* SS-supplemented.
- 2) Every supplemented module is SS-supplement. The opposite is not valid, for instant Q as Z -module.
- 3) The direct summand of any module is an e^* SS-supplemented.
- 4) Every e^* SS-hollow module is an e^* SS-supplemented.
- 5) The image homomorphism of the e^* SS-supplemented module is an e^* SS-supplemented.

6) If M is an e^* -singular module, then supplemented and e^* SS-supplemented modules coincide.

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