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Finite Element Analysis of a Nonlinear Predator-Prey System with Robin Boundary Conditions

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Abstract.

A convex open boundary domain was used to define the predator-prey equations. This was then mathematically analysed using Robin boundary conditions. Using the Faedo-Galerkin methodology and compactness arguments, we demonstrate the existence of strong and weak solutions and their uniqueness. The solution findings exhibit a higher level of regularity. We construct a more regular model based on the initial data. Furthermore, we can demonstrate continuous dependence on initial conditions. To complement the theoretical results, numerical simulations were also conducted, confirming the validity of the model and illustrating the dynamic behaviour of the predator-prey system under Robin boundary conditions.

Keywords: Faedo-Galerkin, Predator-prey system, Robin boundary condition, Existence, uniqueness, Weak solution, Higher regularity.

Mathematics Subject Classification: 35A02 35A01 35K57

تحليل العناصر المنتهية لنظام المفترس – الفريسة غير الخطي مع شروط روبن الحدودية

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الخلاصة

تم استخدام مجال مفتوح ومحدب على نظام المفترس–الفريسة، ثم تم تحليلها رياضياً مع استخدام شروط روبن الحدودية. وبالاعتماد على منهجية فايدو–جاليركين وحجج التراص، نثبت وجود الحلول القوية والضعيفة ووحدانيتها. كما تُظهر نتائج الحل مستوى أعلى انتظام، مما يمكننا من بناء نموذج أكثر انتظامًا استنادًا إلى البيانات الأولية. بالإضافة إلى ذلك، نبرهن على الاعتماد المستمر للحلول على الشروط الأولية .ولتدعيم النتائج النظرية، أُجريت أيضًا محاكاة عددية أثبتت صحة النموذج ووضّحت السلوك الديناميكي لنظام المفترس والفريسة تحت شروط روبن الحدودية.

1. Introduction

Mathematics is a key tool in various scientific fields, aiding in analysis and application. It is used to retrieve time-wise coefficients in the heat equation [1], solve fuzzy stochastic ordinary differential equations with existence and uniqueness theorems [2], and find numerical solutions for singular ordinary differential equations using Wang-Ball polynomials [2,3]. Managing chemical reactions of systems under real conditions can be challenging.

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Additionally, while the reaction is in progress, the concentrations of several initial and final products are monitored in the process [4], this is one of the most important, as explained by Nicolis and Prigogine [5]. A cubic nonlinearity must be considered in the rate equations [6]. The predator-prey model [7] is the name given to this model. Using this model in chemical kinetics is useful for studying cooperative processes. Robin boundary conditions naturally arise in models where the boundary exhibits partial permeability or exchange proportional to the internal state variable. For example, in ecological systems, a Robin-type flux can represent a habitat with partial refuge where species interact with the environment at a rate depending on their density at the boundary. Similarly, in chemical engineering, these conditions model reactors with semi-permeable membranes, where substance exchange depends on concentration differences across boundaries. In contrast, Dirichlet conditions assume fixed concentrations or populations at the boundary, which may not reflect real scenarios with dynamic inflow and outflow. Neumann conditions enforce zero-flux or constant-flux constraints, potentially oversimplifying boundary interactions. Robin conditions offer a more flexible and realistic framework by interpolating between these extremes, making them particularly suitable for modeling coupled reaction–diffusion systems in biological and chemical contexts.

To produce ozone, atomic oxygen has to undergo a triple collision with ozone, followed by a trimolecular reaction. In addition to enzymatic reactions, plasma and laser physics have numerous couplings of specific modes.

The purpose of this paper is to study a predator-prey system having two coupled parts, reaction-diffusion equations with Robin boundary conditions [6]:

$$\frac{\partial p}{\partial t} = \Delta p + f_1(p,q), \qquad in\ \lambda_T\,, \tag{1}$$

$$\frac{\partial q}{\partial t} = \Delta q + f_2(p,q), \qquad in\ \lambda_T\,, \tag{2}$$

$$\frac{\partial p}{\partial v} + \gamma p = 0\,, \qquad \frac{\partial q}{\partial v} + \gamma q = 0\,, \qquad on\ \partial\lambda_T, \tag{3}$$

$$p(.,0) = p_0\,, \qquad q(.,0) = 0, \qquad in\ \lambda, \tag{4}$$

where $\rho^2 = p^2 + q^2$, and $\lambda_T = \lambda \times (0,T)$, λ is an open, bounded, convex domain in R^d (d=1, 2, 3), with $\partial\lambda$
of class C^2, $\partial\lambda_T = \partial\lambda \times (0,T)$, v denotes the exterior unit normal $f_1(p,q) = (1-\rho^2)p - (\omega_1 - \omega_2\rho^2)q$, $f_2(p,q) = (\omega_1 - \omega_2\rho^2)p + (1-\rho^2)q$ and ω_1, ω_2 and γ are positive constants. We assume two solutions for proving uniqueness. In addition to its relevance to chemical kinetics and thermodynamics, the coupled nonlinear system described above can be found in pattern formation; see [8]. Several recent studies have included analyzing this system theoretically or numerically. The system also contains several important applications [9,10].

The novelty of this work lies in the rigorous analytical treatment of a coupled nonlinear predator–prey system under Robin boundary conditions, which has not been addressed in this exact context in previous literature. While recent studies such as Al-Juaifri and Harfash [10], focused on reaction–diffusion systems with Robin boundaries. Our contribution advances the field by establishing higher regularity results, precise a priori energy bounds, and continuous dependence on initial data within a Faedo–Galerkin framework. Moreover, our approach integrates both theoretical analysis and practical numerical insight, extending the finite element methodology to nonlinear ecological models with complex boundary interactions.

This study discusses boundary value problems of type Robin using spectral theory. We begin by using the Faedo-Galerkin approach [11] as well as the Alaoglu compactness argument [12]. The weak solution for the problem (H) is shown to exist and to be unique. In this work, truncated eigenfunction expansions are used so that infinite-dimensional dynamical systems can be converted to finite-dimensional ones. It is then demonstrated that the solutions in the finite weak form of the predator-prey system are local and unique in finite time based on Picard's existence argument. In $L^2(\lambda)$, we also infer existence, uniqueness, and continuous dependence on the initial conditions. The results of these studies can be used to obtain an estimation of energy based on the Alaoglu compactness theorem. Additionally, we use Grisvard's methods to find regularity for the Robin boundary value problem [13]. In $H^1(\lambda)$ we establish the existence, uniqueness, and continuous dependence of higher regularity on the initial conditions. It is possible to produce these results by estimating the regularity of the initial conditions with a lower degree of regularity.

We begin with basic terminology and concepts in Section 2. An investigation of the weak form of the predator-prey system is presented in Section 3. It presents a description of the important findings of this section, which reveals that the predator-prey reaction-diffusion system has a unique weak solution. For weak formulations, weak solutions are established locally and globally in Sections 3.1, 3.2, and 3.3. Galerkin approximations are utilized to pass to the limit. Uniqueness and continuation of a solution are established in Section 3.4. We infer more regularity findings for the weak from additional estimates in Section 4, which leads to results for strong solutions. We provide a regularity argument for the elliptic boundary value problem with Robin boundary conditions in Section 4.1. Finally, numerical experiments in one and two space dimensions are undertaken in Section 5.

2. Auxiliary Results

In this paper, we adopt the standard notation for Sobolev spaces, denoting the norm of $W^{l,\delta}(\lambda), l \in N, \delta \in [1,\infty]$ by $\|.\|_{l,\delta}$ and semi-norm by$|.|_{l,\delta}$. For $\delta = 2, W^{l,2}(\lambda)$ will be denoted by $H^l(\lambda)$ with norm $\|.\|_l$ and semi-norm $|.|_l$ and if $l = 0$, the $L^2(\lambda)$ inner product over λ with norm $\|.\|_0 = |.|_0$ is denoted by $(.,.)$. As well as, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $(H^1(\lambda))\acute{}$ and $H^1(\lambda)$ where $(H^1(\lambda))\acute{}$ is the dual space of $H^1(\lambda)$. A norm on $(H^1(\lambda))\acute{}$ is given by:

$$\|\varphi(t)\|_{(H^1(\lambda))\acute{}} = \sup_{\beth\neq 0} \frac{|\langle\varphi,\beth\rangle|}{\|\beth\|=1} = \sup_{\|\beth\|_1=1} \|\langle\varphi,\beth\rangle\|. \tag{5}$$

Let Y be a Banach space and $(1 \leq \delta \leq \infty)$. Denote $L^\delta(0,T;X)$ to be the Banach space of all measurable functions $\varphi(t) : [0,T] \to Y$ such that $t \to \|\varphi(t)\|_Y$ is in $L^\delta(0,T)$, with norm

$$\| \varphi(t) \|_{L\varphi(0,T;Y)} = \left(\int_0^T |\varphi(t)|_Y^\delta \, dt\right)^{\frac{1}{\delta}}, \tag{6}$$

$$\| \varphi(t)_{(L^\infty)(0,T;Y)} = ess \sup_{t\in[0,T]} \| \varphi(t) \|_Y. \tag{7}$$

We also define $L^\delta(\lambda_T) = L^\delta\left(0,T;L^\delta(\lambda)\right)$. Further, we define $C([0,T];Y)$, the space of continuous functions from $[0,T]$ into Y which consists $\varphi(t) : [0,T] \to Y$ such that as $\varphi(t) \to \varphi(t_0)$ in Y as $t \to t_0$. Previously we had that $C([0,T]):Y)$ is a Banach space with the corresponding norm (see [14] p.43).

$$\|\varphi(t)\|_{C[0,T];Y} = \sup_{t\in[0,T]} \|\varphi(t)\|_Y. \tag{8}$$

Some of the Sobolev results one knows are:

$$H^1(\lambda) \hookrightarrow L^{\delta}(\lambda) \hookrightarrow (H^1(\lambda))' \text{ holds for } \delta \in \begin{cases} [1,\infty] & if\ d=1, \\ [1,\infty) & if\ d=2, \\ [1,6] & if\ d=3, \end{cases} \quad (9)$$

where $\hookrightarrow$ denotes the continuous embedding. Further, we have from the Rellich-Kondrachov theorem. ([15] p. 114 and [16] p. 8), that the embedding in (7) is compact with $\delta \in [1,6]$ replaced by $\delta \in [1,6)$ in the case d=3. The compact embedding will be denoted by the symbol $\overset{c}{\hookrightarrow}$.

There is also a need for Hölders inequality: for $1 \leq a_1 a_2 \leq \infty$ such that $\frac{1}{a_1}+\frac{1}{a_2}=1$ if $\phi \in L^{a_1}(\lambda)$ and $\chi \in L^{a_2}(\lambda)$ then $\varphi\chi \in L^1(\lambda)$ and

$$\left\|\varphi\, \chi\right\|_{0.1} \leq \left\|\varphi\right\|_{0,a_1} \|\, \chi\|_{0,a_2}. \quad (10)$$

The following inequality results from applying the above inequality twice

$$\|\varphi v o\|_{0,1} \leq \|\varphi\|_{0,a_1}\|v\|_{0,a_2}\|o\|_{0,a_3} \quad (11)$$

for $1 \leq a_1, a_2, a_3 \leq \infty$, such that $\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}=1$.

We shall frequently need the following simple version of Young's inequality

$$\chi_1\chi_2 \leq \alpha^{\frac{\beta_1}{\beta_2}} \frac{\chi_1^{\beta_1}}{\beta_1} + \alpha^{-1}\frac{\chi_2^{\beta_2}}{\beta_2}, \qquad \frac{1}{\beta_1}+\frac{1}{\beta_2}=1, \quad (12)$$

valid for any $\chi_1, \chi_2 \geq 0, \alpha > 0$ and $\beta_1, \beta_2 > 1$. Additionally, Young's inequality has the following are useful implications:

$$\chi_1\chi_2 \leq -\alpha\frac{\chi_1^2}{2} + \alpha^{-1}\frac{\chi_2^2}{2}, \qquad \forall\chi_1,\chi_2 \in \mathbb{R}, \forall\alpha > 0. \quad (13)$$

We also require the following lemma:

Lemma 2.1. (Grönwall lemma): let $\Omega(t) \in W^{1.1}(0,T)$ and $\theta(t),\ \Upsilon(t),\ \Lambda(t) \in L^1(0,T)$ which are nonnegative functions. It follows from

$$\frac{d\Omega(t)}{dt} + \Upsilon(t) \leq \Lambda(t)\Omega(t) + \theta(t) \ \ a.e.\ t \in [0,T] \quad (14)$$

that

$$\Omega(T) + \int_0^T \Upsilon(T)dt \leq e^{\Lambda(s)}\Omega(0) + e^{\Lambda(s)} \int_0^T \theta(\iota)d\iota, \quad (15)$$

where $\Lambda(s) = \int_0^T \Lambda(\iota)d\iota$.

3. Weak solutions

We obtain the weak formulation of the system (1)-(4).

Find p(.,t), q(.,t) $\in H^1(\lambda)$ such that p(.,0) = p_0(.), q(.,0) = q_0 (.), ∀$ $\in H^1(\lambda)$ and that is mean t ∈ (0,T)

$$\left(\frac{\partial p}{\partial t}, \varpi\right) + (\nabla p, \nabla\varpi) + \gamma\int_{\partial\lambda} p^k\varpi\, dA = \left((1-\rho^2)p, \varpi\right) - \left((w_1 - w_2\rho^2)q, \varpi\right), \quad (16)$$

$$\left(\frac{\partial q}{\partial t}, \varpi\right) + (\nabla q, \nabla\varpi) + \gamma\int_{\partial\lambda} q\varpi\, dA = \left((w_1 - w_2\rho^2)p, \varpi\right) + \left((1-\rho^2)q, \varpi\right). \quad (17)$$

Theorem 3.1. Assume $\lambda \subset \mathbb{R}^d$ and $p_0(.), q_0(.) \in L^2(\lambda)$, then the system (1)-(4) has a unique solution {p,q} satisfying

$$p(x,t), q(x,t) \in L^2\big(0,T;H^1(\lambda)\big) \cap L^\infty\big(0,T;L^2(\lambda)\big) \cap L^4(\lambda_T) \cap C\big([0,\mathrm{T}];\mathrm{L}^2(\lambda)\big), \quad (18)$$

and the system (1)-(4) holds as equalities in $L^1(0, T;(H^1(\lambda))')$ and $L^2(0, T;(H^1(\lambda))')$, respectively.

Proof: Four parts of the proof are presented in the following order:

3.1 Local existence of the approximations

This section will utilize the Faedo-Galerkin method [11,17]. Suppose that $\{yi\}_{i=1}^{\infty}$ be an orthogonal basis for $H^1(\lambda)$ and an orthonormal basis for $L^2(\lambda)$. Suppose that the Robin eigenvalue problem:

$$-\Delta y_i + y_i = \eta_i y_i \,, \text{ in } \lambda, \qquad \frac{\partial y_i}{\partial \eta} + \gamma y_i = 0 \text{ on } \partial\lambda, \quad (19)$$

where

$1 \leq \eta_1 \leq \eta_2 \leq \eta_3 \leq \cdots \leq \eta_k \leq \cdots$with

$$lim_{i\to\infty}\eta_i = \infty\,. \quad (20)$$

Corresponding eigenvalues are infinite. Such that $(y_i, y_j)_{H^1(\lambda)} = \eta_I \delta_{IJ}$ and $(y_i, y_j)_{L^2(\lambda)} = \delta_{ij}$. Denote by Ψ^K the finite-dimensional span in $H^1(\lambda)$ of $\{y_i\}_{i=0}^{k}$. Setting the L^2 projection onto Ψ^k, $P^k : L^2(\lambda) \longmapsto \Psi^K$, by $P^k\, v = \sum_{i=1}^{k}(v, y_j)y_j$, we also notice that $(P^k\, v, \varpi^k) = (v, \varpi^k)$ for all $\varpi^k \in \psi^k$. A definition such as this makes sense for elements such as $H^1(\lambda) \subset L^2(\lambda)$.

The problem (16)-(17) can be rephrased as follows:

Find $P^K(.,t), q^k(.,0) \in \Psi^k$ in this way $p^k(.,0) = p_0^k, q^k(.,0) = q_0^k$ and for almost every $t \in (0,T)$ for all $\varpi^k \in \Psi^k$

$$\left(\frac{\partial p^k}{\partial t}, \varpi^k\right) + (\nabla p^k, \nabla\varpi^k) + \gamma \int_{\partial\lambda} p^k \varpi^k\, d = \left((1-\rho^2)p^k, \varpi^k\right) - \left((w_1 - w_2\rho^2)q^k, \varpi^k\right), \quad (21)$$

$$\left(\frac{\partial q^k}{\partial t}, \varpi^k\right) + (\nabla q^k, \nabla\varpi^k) + \gamma \int_{\partial\lambda} q^k \varpi^k\, dA = \left((w_1 - w_2\rho^2)p^k, \varpi^k\right) + \left((1-\rho^2)q^k, \varpi^k\right) \quad (22)$$

As a result, we write p^k, q^k as a Galerkin form

$$p^k(.,t) = \sum_{i=1}^{k} \alpha_{ik}(t) y_i(.),\; q^k(.,t) = \sum_{i=1}^{k} \beta_{ik}(t) y_i(.), \quad (23)$$

where the α_{ik}(t) and β_{ik}(t) are to be determined, and set $\varpi^k = y_j$ for j = 1, ..., k.

Lemma 3.1.1 For any p ∈ $H^1(\lambda)$ we have

$$(\nabla(P^k p), \nabla\varpi^k) = (\nabla p, \nabla\varpi^k) \quad \forall\varpi^k \in \psi^k. \quad (24)$$

By straightforwardly calculating this projection operator, the following properties can be demonstrated:

$$\left\|\nabla P^k p\right\|_0 \leq \|\nabla p\|_0 \qquad \forall p \in H^1(\lambda)\,. \quad (25)$$

The following are the initial approximations:

$$p^K(.,0) := P^k\, p_0^k \qquad q^K(.,0) := P^k\, q_0^k\,, \quad (26)$$

when the following property is met:

$$\{p_0^k, q_0^k\} \longmapsto \{p_0, q_0\} \text{ in } L^2(\lambda) \text{ as } k \longmapsto \infty. \quad (27)$$

Note that (21) and (22) can be formulated as an ODE system in α_{ik} (.) and β_{ik} (.). For this system of ODEs the equivalent composite form is then given by

$$\frac{dp^k}{dt} = \Delta p^k + P^k f_1(p^k, q^k), \qquad p^K(.,0) := \mathrm{P}^k p_0^k, \tag{28}$$
$$\frac{dq^k}{dt} = \Delta q^k + P^k f_2(p^k, q^k), \qquad q^K(.,0) := \mathrm{P}^k q_0^k. \tag{29}$$

We must demonstrate that local Lipschitz nonlinearity is present in the system of ODEs. Those are the functions we handle f_1 and f_2 as follows

$$|f_1(p_1, q_1) - f_1(p_2, q_2)| + |f_2(p_1, q_1) - f_2(p_2, q_2)|$$
$$\leq (1 + w_1)|p_1 - p_2| + (1 + w_2)|p_1^3 - p_2^3| + (1 + w_1)|q_1 - q_2|$$
$$+ (1 + w_2)|q_1^3 - q_2^3| + |p_2 q_2^2 - p_1 q_1^2 - p_1^2 q_1 - p_2^2 q_2|$$
$$\leq \varphi_1|p_1 - p_2| + \varphi_2|q_1 - q_2| + \varphi_3(|p_1 - p_2| + |q_1 - q_2|), \tag{30}$$

where $\varphi_1 = 2 + w_1 + w_2 + p_1^2 + p_1 p_2 + p_2^2$, $\varphi_2 = 2 + w_1 + w_2 + q_1^2 + q_1 q_2 + q_2^2$ and $\varphi_3 = -p_1 q_1^2 + p_2 q_2^2 - p_1^2 q_1 - p_2^2 q_2$

Thus, based on Lipschitz and Picard's theorem, f_1 and f_2 are locally Lipschitz ([18], p. 9), therefore, ODEs provide a unique solution $\{p^k, q^k\}$ on $(0, t_k)$.

3.2 Global existence of the approximations

To prove the global existence of Galerkin approximations, we need independent k bounds on p^k and q^k, in some spaces. Then, to show that the solution can exist by the Galerkin method, we do as follow:

Estimate I: Setting $\varpi^k = p^k$ in (21) and $\varpi^k = q^k$ in (22), consequently, the equations resulting from this process are then summed, as a result

$$\frac{1}{2}\frac{d}{dt}\|p^k\|_0^2 + \frac{1}{2}\frac{d}{dt}\|q^k\|_0^2 + |p^k|_1^2 + |q^k|_1^2 + \gamma\|p^k\|_{L^2(\partial\lambda)}^2 + \gamma\|q^k\|_{L^2(\partial\lambda)}^2$$
$$= \int_\lambda (1 - \rho^2)(|p^k|^2 + |p^k|^2)\, dx\ . \tag{31}$$

By substituting $\rho^2 = p^2 + q^2$ on the right, we have that

$$\frac{1}{2}\frac{d}{dt}\|p^k\|_0^2 + \frac{1}{2}\frac{d}{dt}\|q^k\|_0^2 + |p^k|_1^2 + |q^k|_1^2 + \|\rho^k\|_{0,4}^4 \leq \|p^k\|_0^2 + \|q^k\|_0^2. \tag{32}$$

Application Lemma 2.1, hence we obtain

$$\sup_{0\leq t\leq T}\|p^k(T)\|_0^2 + \sup_{0\leq t\leq T}\|q^k(T)\|_0^2 + \|p^k\|_{L^2(0,T;H^1(\lambda))}^2 + \|q^k\|_{L^2(0,T;H^1(\lambda))}^2 + \|\rho^k\|_{L^4(\lambda_T)}^4$$
$$\leq e^{2T}[\|p^k(0)\|_0^2 + \|q^k(0)\|_0^2]. \tag{33}$$

Since $p_0^k, q_0^k \in L^2(\lambda)$ we have $p^k, q^k \in L^\infty(0,T;\, L^2(\lambda)) \cap L^2(0,T;\, H^1(\lambda)) \cap L^4(\lambda_T)$.

Estimate II: From the Equation (31), and noting $\rho^2 = p^2 + q^2$, we obtain

$$\frac{d}{dt}\|\rho^k\|_0^2 + 2\|\nabla\rho^k\|_0^2 + 2\|\rho^k\|_{0,4}^4 \leq 2\|\rho^k\|_0^2. \tag{34}$$

Application Lemma 2.1, we obtain

$$\sup_{0\leq t\leq T}\|p^k(T)\|_0^2 + 2\|p^k\|_{L^2(0,T;H^1(\lambda))}^2 + 2\|\rho^k(T)\|_{L^4(\lambda_T)}^4 \leq e^{2CT}[\|\rho^k(0)\|_0^2. \tag{35}$$

Thus, we have $\rho^k \in L^\infty(0,T;L^2(\lambda)) \cap L^2(0,T;H^1(\lambda)) \cap L^4(\lambda_T)$.

3.3 Passage to the limit

We will now apply Alaoglu's compactness theorem. $[19-22]$, on the sequences of the functions $\{p^k\}_{k=1}^\infty$ and $\{q^k\}_{k=1}^\infty$ that are bounded uniformly, we determine convergent subsequences, put $\{p^k\}, \{q^k\}$, such as

$$\{p^k, q^k\} \rightharpoonup \{p, q\} \text{ in } L^2(0,T;L^2(\lambda)) \cap L^4(\lambda_T) \text{ as } k \to \infty, \tag{36}$$

and

$$\{p^k, q^k\} \rightharpoonup^* \{p, q\} \text{ in } L^\infty(0,T;L^2(\lambda)) \quad \text{as} \quad k \to \infty, \tag{37}$$

$' \rightharpoonup '$ and $' \rightharpoonup ' *$ refer to weak and weak^* Convergence, as a consequence.
A Composite Galerkin approximation is performed in the first step; we demonstrate passage to the limit of the terms (28). We will have similar arguments to those in (29). Let $P^k f_1(p^k, q^k)$, this can be easily demonstrated

$$|f_1(p^k, q^k)| \leq C(|p^k|^4 + |q^k|^4 + |p^k|^3 + |q^k|^3 + |p^k|^2 + |q^k|^2 + |p^k| + |q^k|) \quad (38)$$

Thus, we find

$$\int_0^T \int_\lambda |f(p^k, q^k)| dxdt \leq$$
$$C \int_0^T \int_\lambda (|p^k|^4 + |q^k|^4 + |p^k|^3 + |q^k|^3 + |p^k|^2 + |q^k|^2 + |p^k| + |q^k|) dxdt. \quad (39)$$

From the bounds in (33), and $L^4(\lambda_T) \hookrightarrow L^3(\lambda_T), L^4(\lambda_T) \hookrightarrow L^2(\lambda_T)$ and $L^4(\lambda_T) \hookrightarrow L^1(\lambda_T)$ we have that $f_1(p^k, q^k) \in L^1(\lambda_T)$, from weak compactness arguments, there exists some $\tau \in L^1(\lambda_T)$ then

$$f_1(p^k, q^k) \rightharpoonup \tau \text{ in } L^1(\lambda_T) \quad \text{as, } k \to \infty. \quad (40)$$

We also prove that $P^k f_1(p^k, q^k)$ converges weakly to τ in $L^1(\lambda_T)$. Let us define $\mathcal{N}^k := I - P^k$, which represents the projection orthogonal to P^k. Let's recall this $(P^k v, \tau^k)_\psi = (v, \rho^k)_\psi \ \forall \tau^k \in \psi^k, v \in H^1(\lambda)$, holds $\|P^k v - v\|_1 \leq \|v - \tau^k\|_1 \ \forall \tau^k \in \psi^k, v \in H^1(\lambda)$.
Such as ψ^k is dense in $H^1(\lambda)$, there are $P^k p \longrightarrow p$ in $H^1(\lambda)$, $\forall\, p \in H^1(\lambda)$, $i.e.$ $\mathcal{N}^k p \longrightarrow 0$ in $H^1(\lambda)$ as
$k \longrightarrow \infty$. In addition, there are $H^1(\lambda) \hookrightarrow L^\infty(\lambda)$ therefore, $\mathcal{N}^k p \longrightarrow 0$ in $L^\infty(\lambda)$ $\forall\, p \in H^1(\lambda)$. The consideration of any function $\theta \in L^\infty(\lambda_T)$, then utilizing (9) and the orthogonality of $\mathcal{N}^k$, thus

$$\left|\int_0^T (P^k f_1(p^k, q^k) - \tau, \theta) dt\right| = \left|\int_0^T [(f_1(p^k, q^k) - \tau, \theta) - (f_1(p^k, q^k), \mathcal{N}^k \theta) dt]\right|$$
$$\leq \left|\int_0^T (f_1(p^k, q^k) - \tau, \theta) dt\right| + \int_0^T \|f_1(p^k, q^k)\|_{0,1} \|\mathcal{N}^k \theta\|_{0,\infty} dt \longrightarrow 0 \quad \text{as } k \to \infty.$$

By noting the strong convergence of $\mathcal{N}^k \theta$ to 0 in $L^\infty(\lambda)$ and (40), we arrive at:

$$P^k f_1(p^k, q^k) \rightharpoonup \tau \text{ in } L^1(\lambda_T) \text{ as } k \to \infty. \quad (41)$$

Similarly, we have

$$|f_2(p^k, q^k)| \leq C(|p^k|^4 + |q^k|^4 + |p^k|^3 + |q^k|^3 + |p^k|^2 + |q^k|^2 + |p^k| + |q^k|), \quad (42)$$

Then, it follows that.

$$\int_0^T \int_\lambda |f_2(p^k, q^k)| dxdt$$
$$\leq C \int_0^T \int_\lambda (|p^k|^4 + |q^k|^4 + |p^k|^3 + |q^k|^3 + |p^k|^2 + |q^k|^2 + |p^k| + |q^k|) dxdt. \quad (43)$$

Noting the bounds (33), and the injection $L^4(\lambda_T) \hookrightarrow L^3(\lambda_T)$, $L^4(\lambda_T) \hookrightarrow L^2(\lambda_T)$ And $L^4(\lambda_T) \hookrightarrow L^1(\lambda_T)$ this led to $f_2(p^k, q^k)$ is uniformly bounded in $L^1(\lambda_T)$and therefore, from weak compactness arguments, there exists $\vartheta \in L^1(\lambda_T)$ such that

$$f_2(p^k, q^k) \rightharpoonup \vartheta \in L^1(\lambda_T) \quad \text{as } k \to \infty. \quad (44)$$

In our study, we also demonstrate. $P^k f_2(p^k, q^k)$ converges weakly to ϑ in $L^1(\lambda_T)$. We have $P^k q \longrightarrow 0 \in H^1(\lambda)$ as $k \to \infty$. Furthermore, it is a consequence of injection. $H^1(\lambda) \hookrightarrow$

$L^1(\lambda_T)$ that $P^k q \longrightarrow 0 \in L^\infty(\lambda)\,.\forall_q \in L^\infty(\lambda)$. Let $\theta \in L^\infty(\lambda_T)$, then by utilizing (9) and $\mathcal{N}^k$Its orthogonality, we get

$$\left|\int_0^T (P^k f_2(p^k,q^k) - \vartheta, \theta)dt\right| \le \left|\int_0^T (f_2(p^k,q^k)) - \vartheta, \theta)dt\right| + \left|\int_0^T (f_2(p^k,q^k),\theta)dt\right|$$
$$\le \left|\int_0^T (f_2(p^k,q^k) - \vartheta,\theta)dt\right| + \int_0^T \|f_2(p^k,q^k)\|_{0,1}\|\mathcal{N}^k\theta\|_{0,\infty}\,dt \longrightarrow 0 \text{ as } k \to \infty.$$

Thus, it follows

$$P^k f_2(p^k,q^k) \rightharpoonup \vartheta \text{ in } L^1(\lambda_T) \quad \text{as } k \to \infty. \tag{45}$$

It is worth noting $\Delta p^k \in L^2\left(0,T;\left(H^1(\lambda)\right)'\right)$ and $P^k f_1(p^k,q^k) \in L^1(\lambda_T)$ it follows from (28) that
$\frac{dp^k}{dt} \in L^2\left(0,T;\left(H^1(\lambda)\right)'\right) \cap L^1(\lambda_T)$ and from weak compactness arguments
$\frac{dp^k}{dt}$ tends weakly to some $\dot{\theta}$ in $L^2\left(0,T;\left(H^1(\lambda)\right)'\right) + L^1(\lambda_T)$. As a result of the weak convergence′its uniqueness, we can calculate $\dot{\theta} = \frac{dp}{dt}$, that means,

$$\frac{dp^k}{dt} \rightharpoonup \frac{dp}{dt} \text{ in } L^2\left(0,T;\left(H^1(\lambda)\right)'\right) + L^1(\lambda_T) \quad \text{as } k \to \infty. \tag{46}$$

Now, refer to(36) that $\{p^k\} \rightharpoonup \{p\}$ in $L^2\left(0,T;H^1(\lambda)\right)$ with dual space $L^2\left(0,T;\left(H^1(\lambda)\right)'\right)$. Now set a random choice. $\theta(t) \in C_0^\infty(0,T;(H^1(\lambda))$. Integrating by parts and by $p^k \rightharpoonup p$ in $L^2\left(0,T;\left(H^1(\lambda)\right)'\right)$ yields

$$\int_0^T \left(\frac{dp^k}{dt},\theta\right)dt = -\int_0^T \left(p^k,\frac{d\theta}{dt}\right)dt + \gamma\int_{\partial\lambda}\frac{\partial p^k}{dt}|\theta^k|^2 dA \to$$
$$-\int_0^T \left(p,\frac{d\theta}{dt}\right)dt + \gamma\int_{\partial\lambda}|\theta^k|^2 dA = \int_0^T \left(\frac{dp}{dt},\theta\right)dt,$$

after noting $\frac{d\theta}{dt} \in C_0^\infty(0,T;(H^1(\lambda))$. From the weak convergence of $\frac{dp^k}{dt}$ to $\dot{\varpi}$ in $L^2\left(0,T;\left(H^1(\lambda)\right)'\right)$,
we have

$$\int_0^T \left(\frac{dp^k}{dt},\theta\right)dt \longrightarrow \int_0^T (\dot{\varpi},\theta)dt,$$

We have therefore, weak limits are unique. $\frac{dp}{dt} = \dot{\varpi}$, that is mean, $\frac{dp^k}{dt} \rightharpoonup \frac{dp}{dt} \in L^2\left(0,T;\left(H^1(\lambda)\right)'\right)$ as $k \to \infty$.
Now, as $p^k \rightharpoonup p \in L^2(0,T;(H^1(\lambda))$ (see[20], p. 204) and $\Delta p^k \rightharpoonup \Delta p \in L^2\left(0,T;\left(H^1(\lambda)\right)'\right)$.

Therefore, the passage to the limit on every term in $L^2\left(0,T;\left(H^1(\lambda)\right)'\right)$. We derive $\tau = f_1(p,q)$
in (40) with the help of a few classical theorems. By the Lions-Aubin theorem [11] we have

$$W = \{\varpi : \varpi \in L^2(0,T;(H^1(\lambda)); \frac{d\eta}{dt} \in L^2\left(0,T\left(H^1(\lambda)\right)'\right)\} \overset{C}{\hookrightarrow} L^2(\lambda_T).$$

We have $p^k \in W$ Subsequences can be extracted, denoted still p^k, such that $p^k \to p$ in $L^2(\lambda_T)$, thus $p^k \to p$ pointwise in λ_T. As f_1 is locally Lipschitz in λ_T by continuity, we can conclude that

$f_1(p^k,q^k) \to f_1(p,q)$ (pointwise) almost everywhere in λ_T. We now apply Lemma 1.3 of Lions [11], and then we get

$$f_1(p^k,q^k) \to f_1(p,q) \in L^1(\lambda_T), \tag{47}$$

because weak limits are unique, we can deduce $\tau = f_1(p,q)$, as required. Similarly, to show $\frac{dq^k}{dt} \rightharpoonup \frac{dq^k}{dt}$ in $L^2\left(0,T;\left(H^1(\lambda)\right)'\right) + L^1(\lambda_T)$, $f_2(p^k,q^k) \rightharpoonup f_2(p,q) \in L^1(\lambda_T)$and $\Delta q^k \rightharpoonup \Delta q \in L^2\left(0,T;\left(H^1(\lambda)\right)'\right)$. So, we have reached the' limits of (29) in $L^2\left(0,T;\left(H^1(\lambda)\right)'\right)$.
Lastly, it remains to prove $p,q \in C([0,T];L^2(\lambda))$ [18]. We have shown $p,q \in L^2(0,T;(H^1(\lambda)) \cap L^4(\lambda_T)$ and $\frac{dp}{dt},\frac{dq}{dt} \in L^2\left(0,T;\left(H^1(\lambda)\right)'\right) + L^1(\lambda_T)$. Since $L^2\left(0,T;\left(H^1(\lambda)\right)'\right)$
and $L^2\left(0,T;\left(H^1(\lambda)\right)'\right) + L^1(\lambda_T)$ are the dual spaces of $L^2(0,T;(H^1(\lambda))$ and $L^2(0,T;(H^1(\lambda)) \cap L^\infty(\lambda_T)$, respectively, then we deduce that $p,q \in C([0,T];L^2(\lambda))$.

Estimate III: Let $\chi \in L^4(0,T;H^1(\lambda))$ in (16), and noting the bounds (36) and (37), $L^4(0,T;(H^1(\lambda)) \hookrightarrow L^2(0,T;H^1(\lambda))$, $L^4(0,T;H^1(\lambda)) \hookrightarrow L^4(\lambda_T) \hookrightarrow L^2(\lambda_T)$,
and trace embedding theorems $H^1(\lambda) \hookrightarrow L^2(\partial\lambda)$ (see[21], p. 257 − 261), we have that

$$\begin{aligned}\left|\int_0^T \int_\lambda \frac{dp^k}{dt}\,\chi\,\mathrm{dxdt}\right| \le C\Big[&\|p^k\|_{L^2(0,T;H^1(\lambda))}\|\chi\|_{L^2(0,T;H^1(\lambda))}\\ &+\|p^k\|_{L^2(0,T;L^4(\lambda))}\|p^k\|_{L^4(\lambda_T)}\|\chi\|_{L^4(\lambda_T)}\|p^k\|_{L^2(0,T;L^4(\lambda))}\|q^k\|_{L^4(\lambda_T)}\|\chi\|_{L^4(\lambda_T)}\\ &+\|p^k\|_{L^2(\lambda_T)}\|\chi\|_{L^2(\lambda_T)}\Big] \le C\|\chi\|_{L^4(0,T;H^1(\lambda))},\end{aligned} \tag{48}$$

and thus, we have $\frac{dp^k}{dt} \in L^2\left(0,T;\left(H^1(\lambda)\right)'\right) \subset L^1\left(0,T;\left(H^1(\lambda)\right)'\right)$. Since we have

$$W_z = \left\{z : z \in L^2(0,T;H^1(\lambda));\ \frac{dz^k}{dt} \in L^1\left(0,T;\left(H^1(\lambda)\right)'\right)\right\} \overset{C}{\hookrightarrow} L^2(0,T;L^\wr(\lambda)),$$

and this implies to $p^k \in W_z$ therefore, we can find a subsequence $\{p^k\}$ such that $\{p^k\} \to \{p\}$ in
$L^2(0,T;L^\varsigma(\lambda))$, satisfies for

$$\wr \in \begin{cases} [2,\infty] & if\ d=1,\\ [2,\infty) & if\ d=2,\\ [2,6] & if\ d=3.\end{cases}$$

Similar to this idea, let $\chi \in L^4(0,T;H^1(\lambda))$ in (17), after noting the bounds (36) and (37), $L^2(0,T;H^1(\lambda)) \hookrightarrow L^2(\lambda_T)$, we can show that $\frac{dq^k}{dt} \in L^2\left(0,T;\left(H^1(\lambda)\right)'\right) \subset L^1\left(0,T;\left(H^1(\lambda)\right)'\right)$ and since we have $q^k \in W_z$ it follows that $\{q^k\} \to \{q\}$ in $L^2(0,T;L^\wr(\lambda))$. This leads to $p,q \in L^2(0,T;L^\wr(\lambda))$, $\frac{dp}{dt},\frac{dq}{dt} \in L^1\left(0,T;\left(H^1(\lambda)\right)'\right)$.

3.4 Uniqueness

In order to prove uniqueness (as done [23 − 26]), Suppose that there are two solutions to the weak from (16) and (17) which are p_1, p_2 and q_1, q_2 respectively, with initial conditions $p_1(.,0) = p_{1,0}(.)\ p_2(.,0) = p_{2,0}(.)$, and $q_1(.,0) = q_{1,0}(.)\ q_2(.,0) = q_{2,0}(.)$, respectively.

Setting $\psi_1 = p_1 - p_2$ and $\psi_2 = q_1 - q_2$, and setting $\varpi = \psi_1, \varpi = \psi_2$ in (16) and (17), and by subtracting the results, we get:

$$\frac{1}{2}\frac{d}{dt}\|\psi_1\|_0^2 + \frac{1}{2}\frac{d}{dt}\|\psi_2\|_0^2 + |\psi_1|_1^2 + |\psi_2|_1^2 + \gamma\|\psi_1\|_{L^2(\partial\lambda)}^2 + \gamma\|\psi_2\|_{L^2(\partial\lambda)}^2$$
$$= (p_1 - p_1\rho_1^2 - p_2 + p_2\rho_2^2, \psi_1) - (w_1 q_1 - w_2\rho_1^2 q_1 + w_1 q_2 - w_2\rho_2^2 q_2, \psi_1)$$
$$+(w_1 p_1 - w_2\rho_1^2 p_1 - w_1 p_2 + w_2\rho_2^2 p_2, \psi_2) + (q_1 - q_1\rho_1^2 - q_2 + q_2\rho_2^2, \psi_2). \quad (49)$$

Applying (9) and (12) on first to last terms on the right-side yields that

$$(p_1 - p_1\rho_1^2 - p_2 + p_2\rho_2^2, \psi_1) \le \|\psi_1\|_0^2 - \left(\|p_1\|_{0,\infty}\|\rho_1\|_{0,4}\|\psi_1\|_0\right)$$
$$\le \|\psi_1\|_0^2 + C\|\psi_1\|_0^2\left[\|p_1\|_{0,\infty}^2 + \|\rho_1\|_{0,4}^2\right]. \quad (50)$$

In the same manner on the second terms, we have that

$$-(w_1 q_1 - w_2\rho_1^2 q_1 + w_1 q_2 - w_2\rho_2^2 q_2, \psi_1)$$
$$\le -w_1\|\psi_1\|_0\|\psi_2\|_0 + \|\psi_1\|_0^2\left[\|q_1\|_{0,\infty}^2 + \|\rho_1\|_{0,4}^2\right]. \quad (51)$$

Similarly,

$$(w_1 p_1 - w_2\rho_1^2 p_1 - w_1 p_2 + w_2\rho_2^2 p_2, \psi_2) \le w_1\|\psi_1\|_0\|\psi_2\|_0$$
$$+w_2\|\psi_2\|_0^2\left[\|p_1\|_{0,\infty}^2 + \|\rho_1\|_{0,4}^2 + \|p_2\|_{0,\infty}^2 + \|\rho_2\|_{0,4}^2\right]. \quad (52)$$

Furthermore, it follows from (9) and (12) that

$$(q_1 - q_1\rho_1^2 - q_2 + q_2\rho_2^2, \psi_2) \le \|\psi_2\|_0^2 + \|\rho_1\|_{0,4}\|q_1\|_{0,\infty}\|\psi_2\|_0 + \|\rho_2\|_{0,4}\|q_2\|_{0,\infty}\|\psi_2\|_0$$
$$\le \|\psi_2\|_0^2 + C\|\psi_2\|_0^2\left[\|\rho_1\|_{0,4}^2 + \|\rho_2\|_{0,4}^2 + \|q_1\|_{0,\infty}^2 + \|q_2\|_{0,\infty}^2\right]. \quad (53)$$

By substituting the results (50)-(53) in (49), multiplying the results by 2, after ignoring the left-hand side's last four terms, we arrive at

$$\frac{d}{dt}\|\psi_1\|_0^2 + \frac{d}{dt}\|\psi_2\|_0^2 \le C[1 + \|\rho_1\|_{0,\infty}^2 + \|\rho_2\|_{0,\infty}^2 + \|p_1\|_{0,\infty}^2$$
$$+\|q_1\|_{0,\infty}^2 + \|p_2\|_{0,\infty}^2 + \|q_2\|_{0,\infty}^2][\|\psi_1\|_0^2 + \|\psi_2\|_0^2]. \quad (54)$$

Applied Lemma 2.1, gives

$$\|\psi_1\|_0^2 + \|\psi_2\|_0^2 \le \exp[C\int_0^T (1 + \|\rho_1\|_{0,\infty}^2 + \|\rho_2\|_{0,\infty}^2 + \|p_1\|_{0,\infty}^2$$
$$+\|q_1\|_{0,\infty}^2 + \|p_2\|_{0,\infty}^2 + \|q_2\|_{0,\infty}^2)dt](\|\psi_1(0)\|_0^2 + \|\psi_2(0)\|_0^2). \quad (55)$$

From **Estimate I, Estimate II** and $L^\infty(0,T;(H^1(\lambda)) \hookrightarrow L^2(0,T;(L^\infty(\lambda))$. Therefore,

$$\|\psi_1\|_0^2 + \|\psi_2\|_0^2 \le C(\|\psi_1(0)\|_0^2 + \|\psi_2(0)\|_0^2). \quad (56)$$

Thus if $(p_1(0), q_1(0)) = (p_2(0), q_2(0))$ hence we deduce uniqueness $(p_1(t) = p_2(t)$ and $q_1(t) = q_2(t))$ for all t. However, if $(p_1(0), q_1(0)) \ne (p_2(0), q_2(0))$, as a result, continuous dependence in $L^2(\lambda)$.

4. Higher regularity

Theorem 4.1. Assume that $p_0, q_0 \in H^1(\lambda)$, then the system (H) possesses a unique solution $\{p, q\}$ satisfying

$$p(x,t), q(x,t) \in L^2\left(0,T; H^2(\lambda)\right) \cap L^\infty\left(0,T; L^4(\lambda)\right)$$
$$\cap L^\infty\left(0,T; H^1(\lambda)\right) \cap L^6(\lambda_T) \cap C\left([0,T]; H^1(\lambda)\right), \quad (57)$$
$$\frac{\partial p(x,t)}{\partial t}, \frac{\partial q(x,t)}{\partial t} \in L^2(\lambda_T) \quad (58)$$

and the system (H) assume equality $L^2(\lambda_T)$. Moreover, the $\left(p_0(.), q_0(.)\right), \mapsto$ $(p(.,t; p_0, q_0), q(.,t; p_0, q_0))$
is continuous in $H^2(\lambda)$

Proof: We can achieve further regularity results by using more a priori estimates to obtain the existence and uniqueness of strong solutions.

4.1 Existence

Estimate IV: Set $\varpi^k = (p^k)^3, \varpi^k = (q^k)^3$ in the weak from $(21)-(22)$ the results, when combined, yield

$$\frac{1}{4}\frac{d}{dt}\|p^k\|_{0,4}^4 + \frac{1}{4}\frac{d}{dt}\|q^k\|_{0,4}^4 + 3\|p^k\nabla q^k\|_0^2 + 3\|q^k\nabla p^k\|_0^2 + \gamma\|p^k\|_{L^4(\partial\lambda)}^4 + \gamma\|p^k\|_{L^4(\partial\lambda)}^4 + \|p^k\|_{0,6}^6 + \|q^k\|_{0,6}^6 = \|p^k\|_{0,4}^4 + \|q^k\|_{0,4}^4 + (\mathcal{w}_1 - \mathcal{w}_2\rho^2), p^k(q^k)^3) - ((\mathcal{w}_1 - \mathcal{w}_2\rho^2), q^k(p^k)^3). \quad (59)$$

Using Young's inequality with $\alpha = 1$, and setting $\beta_1 = 4$ and $\beta_2 = \frac{4}{3}$, on the third and last, on the right, we get

$$(\mathcal{w}_1\mathcal{w}_2\rho^2), p^k(q^k)^3) - ((\mathcal{w}_1\mathcal{w}_2\rho^2), q^k(p^k)^3) \le \frac{\mathcal{w}_1}{4}\int_\lambda |p^k|\,dx + \frac{3\mathcal{w}_1}{4}\int_\lambda |q^k|^4 dx + \frac{\mathcal{w}_1}{4}\int_\lambda |q^k|^4\,dx + \frac{3\mathcal{w}_1}{4}\int_\lambda |p^k|^4\,dx + \frac{\mathcal{w}_2}{4}\int_\lambda |p^k|^4\,dx + \frac{3\mathcal{w}_2}{4}\int_\lambda |\rho|^{\frac{8}{3}}|q^k|^4 dx + \frac{\mathcal{w}_2}{4}\int_\lambda |q^k|^4\,dx + \frac{3\mathcal{w}_2}{4}\int_\lambda |\rho|^{\frac{8}{3}}|p^k|^4 dx. \quad (60)$$

By subtracting the Equation (60) in (59) and multiplying the result by 4, gives

$$\frac{d}{dt}\|p^k\|_{0,4}^4 + \frac{d}{dt}\|q^k\|_{0,4}^4 + 12\|p^k\nabla p^k\|_0^2 + 12\|q^k\nabla q^k\|_0^2 + 4\gamma\|p^k\|_{L^4(\partial\lambda)}^4 + 4\gamma\|p^k\|_{L^4(\partial\lambda)}^4 + 4\|p^k\|_{0,6}^6 + 4\|q^k\|_{0,6}^6 \le \left(4 + \mathcal{w}_1 + \mathcal{w}_2 + 3\mathcal{w}_3 + 3\mathcal{w}_2\|\rho\|_{L^{\frac{8}{3}}(\lambda)}^{\frac{8}{3}}\right)\left[\|p^k\|_{0,4}^4 + \|q^k\|_{0,4}^4\right] \quad (61)$$

noting the injection of L^4 in to $L^{\frac{8}{3}}$ and integration over time $(0,T)$

$$\|p^k(T)\|_{0,4}^4 + \|q^k(T)\|_{0,4}^4 + 12\|p^k\nabla p^k\|_{L^2(\lambda T)}^2 + 12\|q^k\nabla q^k\|_{L^2(\lambda T)}^2 + 4\gamma\|p^k\|_{L^4(\partial\lambda T)}^4 + 4\gamma\|p^k\|_{L^4(\partial\lambda T)}^4 + 4\gamma\|p^k\|_{L^6(\lambda T)}^6 + 4\gamma\|q^k\|_{L^6(\lambda T)}^6 \le \exp\left[T\left(4 + \mathcal{w}_1 + \mathcal{w}_2 + 3\mathcal{w}_1 + 3\mathcal{w}_2\|\rho\|_{L^4(\lambda T)}^4\right)\right]\left[\|p^k(0)\|_{0,4}^4 + \|q^k(0)\|_{0,4}^4\right]. \quad (62)$$

Recalling $p_0, q_0 \in H^1(\lambda)$, and **Estimate II.**, we have $p^k, q^k$is uniformly bounded in $L^\infty(0,T;L^4(\lambda)) \cap L^6(\lambda_T)$.

Estimate V: Choosing $\varpi^k = -\Delta p^k, \varpi^k = -\Delta q^k$in the weak forms (21) and (22), respectively, integrating by parts leads to

$$\frac{1}{2}\frac{d}{dt}\|\nabla p^k\|_0^2 + \frac{1}{2}\frac{d}{dt}\|\nabla q^k\|_0^2 + \|\Delta p^k\|_0^2 + \|\Delta q^k\|_0^2 + \gamma\|p^k\|_{L^2(\partial T)}^2 + \gamma\|q^k\|_{L^2(\partial T)}^2 = -((1-\rho^2)p^k, \Delta p^k) - ((1-\rho^2)q^k, \Delta q^k) + ((\mathcal{w}_1 - \mathcal{w}_2\rho^2)q^k, \Delta p^k) - \left((\mathcal{w}_1 - \mathcal{w}_2\rho^2)p^k, \Delta q^k\right) \quad (63)$$

Next, by integration by parts, and utilizing Hölders inequality, and noting the injection of $H^1(\lambda)$ then using Young's inequality with $\alpha = 2$, and setting $\beta_1 = \beta_2 = 2$, we arrive at the following inequality:

$$-\Big((1-\rho^2)p^k\,,\Delta p^k\Big)\le\ \|\Delta p^k\|_0^2+\gamma\|p^k\|_{L^2(\partial T)}^2+\|\rho^k\|_{0,8}^2\|p^k\|_{0,4}\|\Delta p^k\|_0$$
$$\le\|\nabla p^k\|_0^2\ +\gamma\|p^k\|_{L^2(\partial T)}^2+\|\rho^k\|_1^2\|p^k\|_{0,4}\|\Delta p^k\|_0$$
$$\le\|\nabla p^k\|_0^2\ +\gamma\|p^k\|_{L^2(\partial T)}^2+\|\rho^k\|_1^4\ \|p^k\|_{0,4}^2+\frac{1}{4}\|\Delta p^k\|_0^2, \tag{64}$$

similarly, we get

$$-\Big((1-\rho^2)q^k\,,\Delta q^k\Big)\le\ \|\nabla q^k\|_0^2+\gamma\|q^k\|_{L^2(\partial T)}^2+\|\rho^k\|_{0,8}^2\|q^k\|_{0,4}\|\Delta q^k\|_0$$
$$\le\|\nabla q^k\|_0^2\ +\gamma\|q^k\|_{L^2(\partial T)}^2+\|\rho^k\|_1^2\|q^k\|_{0,4}\|\Delta q^k\|_0$$
$$\le\|\nabla q^k\|_0^2\ +\gamma\|q^k\|_{L^2(\partial T)}^2+\|\rho^k\|_1^4\|q^k\|_{0,4}^2+\frac{1}{4}\left\|\Delta \mathrm{q}^k\right\|_0^2, \tag{65}$$

the same applies, we have that

$$\Big((w_1-w_2\rho^k)p^k,\Delta q^k\Big)$$
$$\le w_1(\nabla p^k,\nabla q^k)+w_1\gamma\int_{\partial\lambda}|p^kq^k|\,dA+w_2\|\rho^k\|_{0,6}^3\|q^k\|_{0,6}\|\Delta p^k\|_0$$
$$\le\ -w_1(\nabla p^k,\nabla q^k)+w_1\gamma\ \int_{\partial\lambda}|p^kq^k|\,dA+w_2^2\|\rho^k\|_{0,6}^6\|q^k\|_{0,6}^2+\frac{1}{4}\|\Delta q^k\|_0^2,$$
(66)

also, it follows that

$$-\Big((w_1-w_2\rho^2)p^k,\Delta q^k\Big)\le w_1(\nabla p^k,\nabla q^k)-\ w_1\gamma\int_{\partial\lambda}|p^kq^k|\,dA$$
$$+w_2\|\rho^k\|_{0,6}^3\|q^k\|_{0,6}\|\Delta p^k\|_0\le\ -w_1(\nabla p^k,\nabla q^k)-\ w_1\gamma\int_{\partial\lambda}|p^kq^k|\,dA$$
$$+w_2^2\|\rho^k\|_{0,6}^6\|q^k\|_{0,6}^2+\frac{1}{4}\|\Delta q^k\|_0^2. \tag{67}$$

By subtracting the results (64)-(67) in (63), and multiplying the result by 2, we get

$$\frac{d}{dt}\|\nabla p^k\|_0^2+\frac{d}{dt}\|\nabla q^k\|_0^2+2\|\Delta p^k\|_0^2+2\|\Delta q^k\|_0^2\le2\|\nabla p^k\|_0^2+2\|\nabla q^k\|_0^2$$
$$+2(1+w_2^2)\|\rho^k\|_{0,6}^6\big[\|p^k\|_{0,6}^2+\|q^k\|_{0,6}^2\big]. \tag{68}$$

Next, application Grönwall lemma, hence we obtain

$$\|\nabla p^k(T)\|_0^2+\|\nabla q^k(T)\|_0^2+\int_0^T\|\Delta p^k\|_0^2dt+\int_0^T\|\Delta q^k\|_0^2dt\le exp(2T)\ [\|\nabla p^k(0)\|_0^2+$$
$$\|\nabla q^k(T)\|_0^2]+\ exp\ [2T(1+w_2^2)]\ \int_0^T\|\rho^k\|_{0,6}^6\ \big[\|p^k\|_{0,6}^2+\|q^k\|_{0,6}^2\big]dt. \tag{69}$$

From **Estimate I**, **Estimate II**, and continuous injections $L^2\big(0,T;H^1(\lambda)\big)\hookrightarrow L^2\big(0,T;H^1(\lambda)\big)$, so we conclude that the right side of (29) is bounded. We conclude $p^k,q^k\in\ L^\infty\big(0,T;H^1(\lambda)\big)$. We now recall that $L^1\big(0,T;H^1(\lambda)\big)$, as the pre-dual of $L^\infty\big(0,T;H^1(\lambda)\big)$, is a separable bound in (69) that

$$\{p^k,q^k\}\rightharpoonup^*\ \{p,q\}\quad in\quad L^\infty\big(0,T;H^1(\lambda)\big). \tag{70}$$

Then, we have $\{p,q\}\in L^\infty\big(0,T;H^1(\lambda)\big)$. We apply some well-known elliptic regularity results to bounded, convex, open domains. In particular, we apply the elliptic regularity result for Robin boundary problems as stated in Grisvard (1985, Theorem 2.11), [23]. To ensure the validity of this result, we assume that the boundary $\partial\lambda$ is of class C^1. Under this smoothness condition, the weak solutions obtained can be shown to belong to $H^2(\lambda)$, thereby satisfying higher regularity properties [23, 27]. From the eigenvalue Equations (19) and (20) and we have for fixed finite k that $y_i\in\ H^2(\lambda)$, and hence $p^k(..t),q^k(..t)\in L^2(\lambda)$ (see [13] p. $149-151$), we have $\|p^k\|_2\le C\|\Delta p^k\|_0$, for $C>0$. We deduce, then, using the third and fourth bounds in (68), that $p^k,q^k\in\ L^2\big(0,T;H^2(\lambda)\big)$. Since $L^2\big(0,T;H^2(\lambda)\big)$ is a reflexive Banach space (see [17] p. 40), thus compactness arguments (see [18] p. 289), imply the existence of subsequences $\{p^k, q^k\} \in L^2\big(0,T;H^2(\lambda)\big)$ that satisfy

$$\{p^k, q^k\} \rightharpoonup \{p, q\} \quad \text{in} \quad L^2\big(0,T;H^2(\lambda)\big). \tag{71}$$

Thus, we arrive at $\{p, q\} \in L^2\big(0,T;H^2(\lambda)\big)$. Furthermore, since $\frac{\partial p}{\partial v} + \gamma p = 0$ and $\frac{\partial p}{\partial v} + \gamma p = 0$ on $\partial\lambda$ [28]. Indeed, using the weak convergence of $\{p^k, q^k\} \longrightarrow \{p, q\}$ in $H^2(\lambda)$, that $\frac{\partial p}{\partial v} + \gamma p = 0$ and $\frac{\partial p}{\partial v} + \gamma q = 0$ on $L^2(\partial\lambda)$.

Estimate VI: Set $\varpi^k = \frac{\partial p^k}{\partial t}$, $\varpi^k = \frac{\partial q^k}{\partial t}$ in the weak form (21)-(22) the results, when combined, yield

$$\left\|\frac{\partial p^k}{\partial t}\right\|_0^2 + \left\|\frac{\partial p^k}{\partial t}\right\|_0^2 + \frac{1}{2}\frac{d}{dt}\|\nabla p^k\|_0^2 + \frac{1}{2}\frac{d}{dt}\|\nabla q^k\|_0^2 + \frac{\gamma}{2}\frac{d}{dt}\|p^k\|_{L^2(\partial\lambda)}^2 + \frac{\gamma}{2}\frac{d}{dt}\|q^k\|_{L^2(\partial\lambda)}^2 = \left((1-\rho^2)p^k, \frac{\partial p^k}{\partial t}\right) + \left((1-\rho^2)q^k, \frac{\partial q^k}{\partial t}\right) - \left((w_1 - w_2\rho^2)q^k, \frac{\partial p^k}{\partial t}\right) + \left((w_1 - w_2\rho^2)p^k, \frac{\partial q^k}{\partial t}\right). \tag{72}$$

Using Hölders inequality on the first and second right-hand side, we have that

$$\left((1-\rho^2)p^k, \frac{\partial p^k}{\partial t}\right) \le \frac{1}{2}\frac{d}{dt}\|p^k\|_0^2 + \frac{1}{2}\|p^k\|_1^2\|\rho\|_{0,4}^2, \tag{73}$$

$$\left((1-\rho^2)q^k, \frac{\partial q^k}{\partial t}\right) \le \frac{1}{2}\frac{d}{dt}\|q^k\|_0^2 + \frac{1}{2}\|q^k\|_1^2\|\rho\|_{0,4}^2. \tag{74}$$

Now, using Hölder's inequality, and noting the injection of $H^1(\lambda)$ into $L^8(\lambda)$ then Young's inequality with $\alpha = 2$ and setting $\beta_1 = \beta_2 = 2$, on the third and last the right-hand side, we get

$$-\left((w_1 - w_2\rho^2)q^k, \frac{\partial p^k}{\partial t}\right) \le w_1\|q^k\|_0\left\|\frac{\partial p^k}{\partial t}\right\|_0 - w_2\|\rho\|_{0,8}^2\|q^k\|_{0,4} + \left\|\frac{\partial p^k}{\partial t}\right\|_0$$
$$\le w_1^2\|q^k\|_0^2 + \frac{1}{4}\left\|\frac{\partial p^k}{\partial t}\right\|_0^2 + w_2^2\|\rho\|_1^4\|q^k\|_{0,4}^2 + \frac{1}{4}\left\|\frac{\partial p^k}{\partial t}\right\|_0^2, \tag{75}$$

$$\left((w_1 - w_2\rho^2)p^k, \frac{\partial q^k}{\partial t}\right) \le w_1\|p^k\|_0\left\|\frac{\partial q^k}{\partial t}\right\|_0 - w_2\|\rho\|_{0,8}^2\|p^k\|_{0,4} + \left\|\frac{\partial q^k}{\partial t}\right\|_0$$
$$\le w_1^2\|p^k\|_0^2 + \frac{1}{4}\left\|\frac{\partial q^k}{\partial t}\right\|_0^2 + w_2^2\|\rho\|_1^4\|p^k\|_{0,4}^2 + \frac{1}{4}\left\|\frac{\partial p^k}{\partial t}\right\|_0^2. \tag{76}$$

By subtracting the results (73)-(76) in (72) and multiplying the result by 2, gives

$$\left\|\frac{\partial p^k}{\partial t}\right\|_0^2 + \left\|\frac{\partial q^k}{\partial t}\right\|_0^2 + \frac{d}{dt}\|\nabla p^k\|_0^2 + \gamma\frac{d}{dt}\|\nabla q^k\|_0^2 + \gamma\frac{d}{dt}\|q^k\|_{L^2(\partial\lambda)}^2$$
$$\le \frac{d}{dt}\|p^k\|_0^2 + \frac{d}{dt}\|q^k\|_0^2 + \|p^k\|_1^2\|\rho\|_{0,4}^2 + \|q^k\|_1^2\|\rho\|_{0,4}^2 + 2w_1^2\|p^k\|_0^2 + 2w_1^2\|q^k\|_0^2$$
$$+2w_2^2\|\rho\|_1^4\|q^k\|_{0,4}^2 + 2w_1^2\|\rho\|_1^4\|p^k\|_{0,4}^2. \tag{77}$$

Integrating over time $(0,T)$ and by substituting $\rho^2 = p^2 + q^2$ in the right-hand side and noting that
$L^4(\lambda) \hookrightarrow L^2(\lambda)$, leads to

$$\int_0^T \left\|\frac{\partial p^k}{\partial t}\right\|_0^2 dt + \int_0^T \left\|\frac{\partial q^k}{\partial t}\right\|_0^2 dt + \|\nabla p^k(T)\|_0^2 + \|\nabla q^k(T)\|_0^2 + \gamma\|q^k\|_{L^2(\partial\lambda)}^2$$
$$+\gamma\|q^k(T)\|_{L^2(\partial\lambda)}^2 + \|\rho^k(0)\|_0^2 \leq \|\rho^k(T)\|_0^2 + \int_0^T \|\rho^k\|_{0,4}^2 [\,\|\rho^k\|_1^2 + w_2^2\|\rho\|_1^4 + 2w_1^2]dt$$
$$\|\nabla p^k(0)\|_0^2 + \|\nabla q^k(0)\|_0^2 + \gamma\|q^k(0)\|_{L^2(\partial\lambda)}^2 \quad (78)$$

Using Hölders inequality on second term in (78) on the right-hand side, leads to

$$\int_0^T \left\|\frac{\partial p^k}{\partial t}\right\|_0^2 dt + \int_0^T \left\|\frac{\partial q^k}{\partial t}\right\|_0^2 dt + \|\nabla p^k(T)\|_0^2 + \|\nabla q^k(T)\|_0^2 \leq \|\rho^k(T)\|_0^2$$
$$+\left(\int_0^T \|\rho^k\|_{0,4}^4 dt\right)^{\frac{1}{2}}\left(\int_0^T \|\rho^k\|_1^4 dt\right)^{\frac{1}{2}} + w_2^2\left(\int_0^T \|\rho^k\|_{0,4}^4 dt\right)^{\frac{1}{2}}\left(\int_0^T \|\rho^k\|_1^8 dt\right)^{\frac{1}{2}}$$
$$+ w_1^2\left(\int_0^T \|\rho^k\|_{0,4}^4 dt\right)^{\frac{1}{2}} + \|\nabla p^k(0)\|_0^2 + \|\nabla q^k(0)\|_0^2 + \gamma\|p^k(0)\|_{L^2(\partial\lambda)}^2 + \gamma\|q^k\|_{L^2(\partial\lambda)}^2. \quad (79)$$

Now, on noting (79), and from bounds in Estimate I and Estimate II, therefore, the right-hand side of (78) is bounded by a positive constant. As a result, $\frac{\partial p^k}{\partial t}$, $\frac{\partial p^k}{\partial t}$ $\in L^2(\lambda_T)$. As $L^2(\lambda_T)$ is a reflexive Banach space, as a result of compactness arguments, we conclude the existence of subsequences $\{p^k, q^k\} \in L^2(\lambda_T)$ such that

$$\left\{\frac{\partial p^k}{\partial t}, \frac{\partial q^k}{\partial t}\right\} \rightharpoonup \left\{\frac{\partial p}{\partial t}, \frac{\partial q}{\partial t}\right\} \quad \text{in} \quad L^2(\lambda_T), \quad (80)$$

we have that $\frac{\partial p}{\partial t}, \frac{\partial q}{\partial t} \in L^2(\lambda_T)$ and also in $L^\infty\big(0,T;H^1(\lambda)\big)$.

Lemma 4.1.1 For some $\varsigma \geq 0$, suppose that $p \in L^2\big(0,T;H^{\varsigma+1}(\lambda)\big), \frac{\partial \rho}{\partial t} \in L^2\big(0,T;H^1(\lambda)\big)$. It follows that $p \in C\big([0,T];H^1(\lambda)\big)$, ([12] p. 191_194). Here, in our case, $\varsigma = 1$, $H^{\varsigma+1}(\lambda)$= $H^2(\lambda)$, $H^{\varsigma}(\lambda)$=$H^1(\lambda)$, $H^{k-1}(\lambda)$=$L^2(\lambda)$. Therefore, from Lemma 4.1.1, we have that $p, q \in C\big([0,T];H^1(\lambda)\big)$. This completes the proof of Theorem 4.1.

5. Numerical Simulation

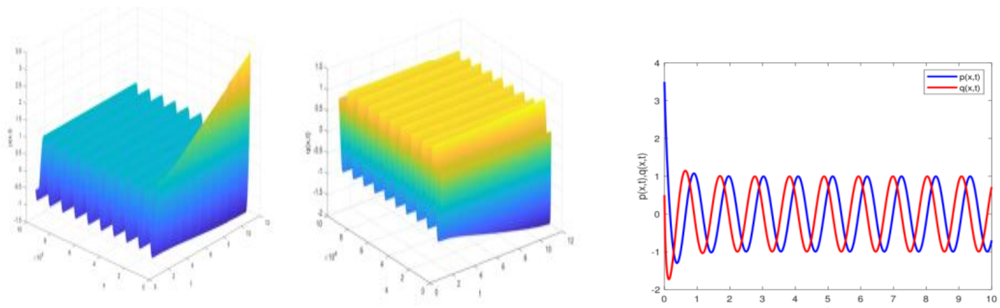
5.1 One Dimension

The numerical simulation was conducted in one dimension with $\lambda = [0, L]$, for $0 \leq t \leq T$ with mesh points $x_j = jh,\ j = 0,\ldots,J$ where $h = L/J$. The solution to the differential equation was calculated in the interval $\lambda = [0, L] = [0,1]$ with $h = 0.5$, as well as the adoption of $J = 10$, if $\gamma = 0$, then the system with Neumann boundary conditions, and if $\gamma = 1$, the system with Robin boundary conditions [29]. Here, we adopt the following examples:

Example 5.1.1 We will investigate the numerical solutions of (7)-(8) with the following initial conditions:

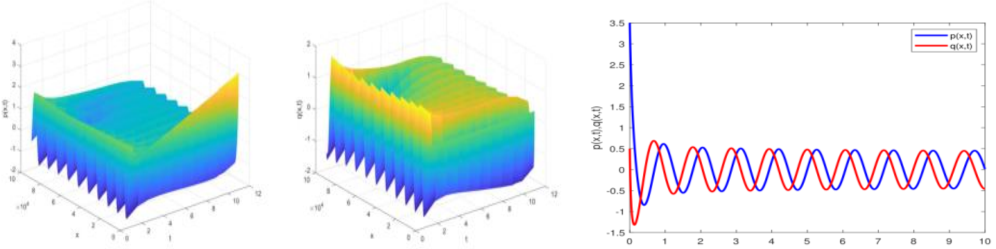
$$p(x, 0) = 1 + 0.5x, \qquad q(x, 0) = 0.5. \quad (81)$$

For $w_1 = -5, w_2 = 1$, and $\Delta t = 0.0001$, the numerical solution was calculated. Figure 1 and 2 shows the numerical solution for the time level $T = 10$. These figures show that the solutions have a positive property and do not produce any negative values. It was established in [22] that the system is attracted to the equilibrium point (q^*, p^*). If $\gamma = 0$, then we have Neumann boundary conditions; however, if $\gamma = 1$, then we have Robin boundary conditions. The solutions not only maintain positive but also converge to the system's equilibrium point.



(a) Mesh graph of p. (b) Mesh graph of q. (c) Combined plot of p and q, of Example 5.1.1, when $x = 1$.

Figure 1: Graphs for p and q at $w_1 = -5, w_2 = 1, \gamma = 0$, and $\Delta t = 0.0001$.



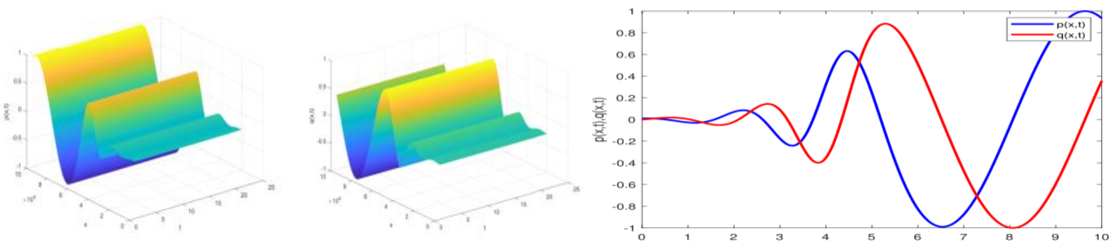
(a) Mesh graph of p. (b) Mesh graph of q. (c) Combined plot of p and q, of Example 5.1.1, when $x = 1$.

Figure 2: Graphs for p and q at $w_1 = -5, w_2 = 1, \gamma = 1$, and $\Delta t = 0.0001$.

Example 5.1.2 The second example adopted the following initial conditions:

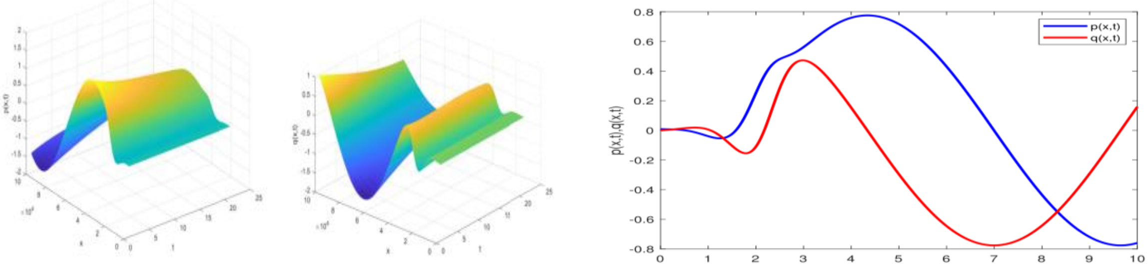
$$p(x,0) = 0.01, \qquad q(x,0) = 0. \tag{82}$$

The constants that are used to calculate the numerical solutions were as follows: $w_1 = 3, w_2 = 2$, and $\Delta t = 0.0001$. Figure 3 and 4 show the numerical solution for this example at a time level $T = 10$. It has been proven in [22] that the system's equilibrium point (q^*, p^*) is stable



(a) Mesh graph of p. (b) Mesh graph of q. (c) Combined plot of p and q, of Example 5.1.2, when $x = 1$.

Figure 3: Graphs for p and q at $w_1 = 3, w_2 = 2, \gamma = 0$, and $\Delta t = 0.0001$.



(a) Mesh graph of p. (b) Mesh graph of q. (c) Combined plot of p and q, of Example 5.1.2, when $x = 1$.

Figure 4: Graphs for p and q at $w_1 = 3, w_2 = 2, \gamma = 1$, and $\Delta t = 0.0001$.

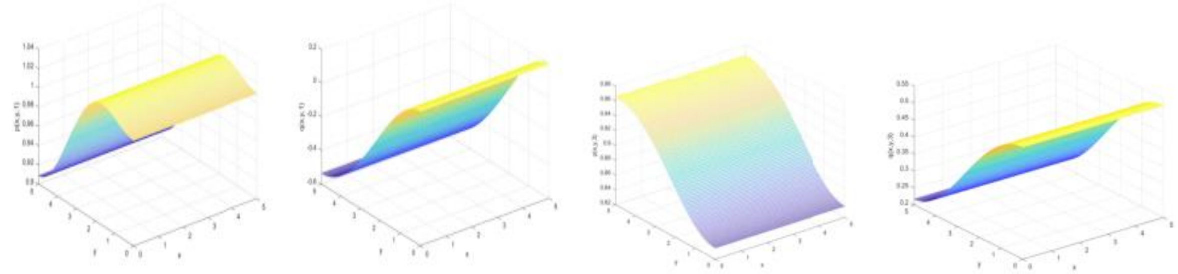
5.2 Two-Dimensional

We used $\lambda = [0, L] \times [0, L]$ and a square uniform mesh with vertices $(x_i, y_j) = (ih, jh)$, where $i, j = 0, \ldots, J$. Note $h = L/J$, the same space step was used in both directions of x and y. A 'right-angled' triangulation was used, in which each square is divided by a diagonal running from the top-right corner to the bottom-left corner [30]. The 'natural way' is utilised to order the nodes. From left to right, the nodes were numbered starting with the bottom row.

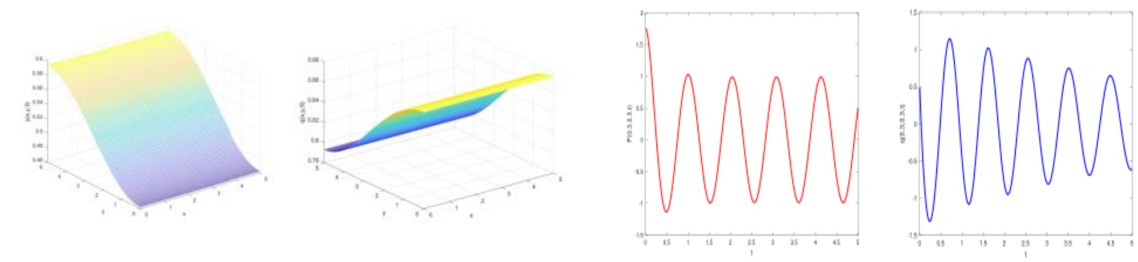
Example 5.2.1 In this example, the parameters are chosen such that $T = 5, w_1 = -5, w_2 = 1, \gamma = 1$, and the problem is solved over the domain $\lambda = [0,1] \times [0,1]$ with $h = 0.05, \Delta t = 0.001$ and $J = 100$. The initial conditions are:

$$p(x, y, 0) = 1 + 0.5x, \qquad q(x, y, 0) = 0.5. \tag{83}$$

The numerical solutions of p and q, for the time level $T = 5$, are shown in Figures 5 and 6.

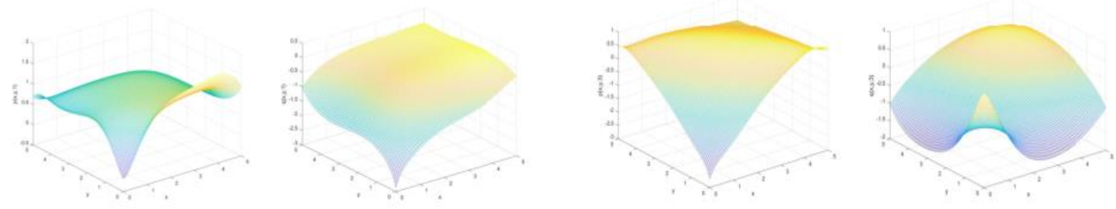


(a) solution of p at $T = 1$ (b) solution of q at $T = 1$ (c) solution of p at $T = 3$ (d) solution of q at $T = 3$

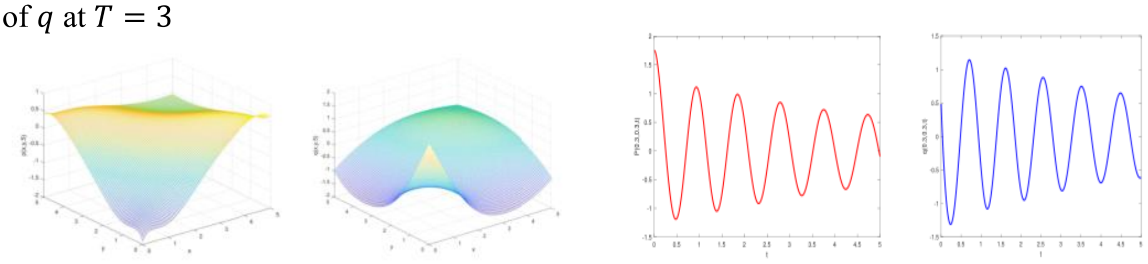


(e) solution of p at $T = 5$ (f) solution of q at $T = 5$(g)solution of p at $x = 0.3, y = 0.3$ (h)solution of q at $x = 0.3, y = 0.3$

Figure 5: Numerical solution of p , of Example 5.2.1, at $w_1 = -5, w_2 = 1, \gamma = 0$



(a) solution of p at $T = 1$ (b) solution of q at $T = 1$ (c) solution of p at $T = 3$ (d) solution of q at $T = 3$



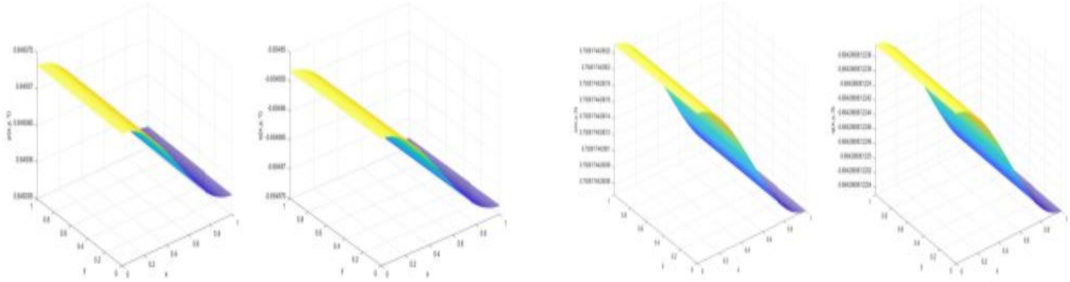
(e) solution of p at $T = 5$ (f) solution of q at $T = 5$ (g)solution of p at $x = 0.3, y = 0.3$ (h)solution of q at $x = 0.3, y = 0.3$

Figure 6: Numerical solution of q , of Example 5.2.1, at $w_1 = -5, w_2 = 1, \gamma = 1$.

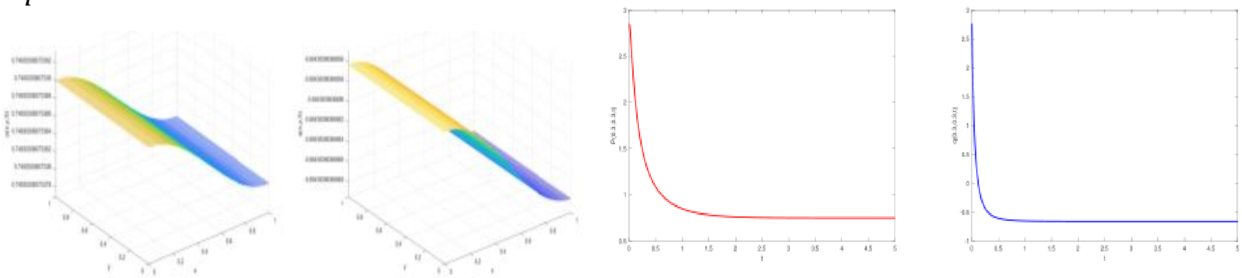
Example 5.2.2 In this example, the parameters are chosen such that $T = 5, w_1 = w_2 = 1, \gamma = 0$, and the problem is solved over the domain $\lambda = [0,1] \times [0,1]$ with $h = 0.01, \Delta t = 0.001$ and $J = 100$. The initial conditions are:

$$p(x, y, 0) = exp(1 + 0.5y^2), \qquad q(x, y, 0) = exp(1 + 0.5x^2). \qquad (84)$$

The numerical solutions of p and q, for the time level $T = 5$, are shown in Figures 7 and 8.

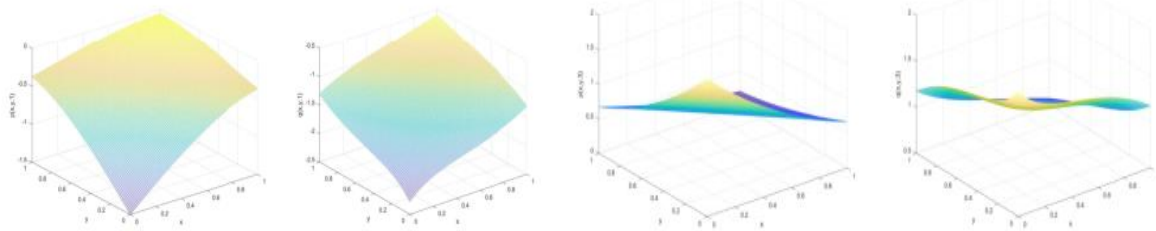


(a) solution of p at $T = 1$ (b) solution of q at $T = 1$ (c) solution of p at $T = 3$ (d) solution of q at $T = 3$

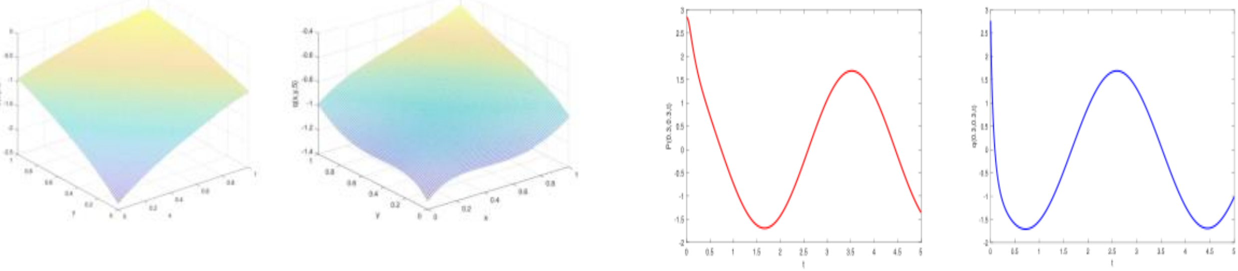


(e) solution of p at $T = 5$ (f) solution of q at $T = 5$ (g)solution of p at $x = 0.3, y = 0.3$ (h)solution of q at $x = 0.3, y = 0.3$

Figure 7: Numerical solution of p , of Example 5.2.2, at $w_1 = w_2 = 1, \gamma = 0$.



(a) solution of p at $T = 1$ (b) solution of q at $T = 1$ (c) solution of p at $T = 3$ (d) solution of q at $T = 3$



(e) solution of p at $T = 5$ (f) solution of q at $T = 5$ (g) solution of p at $x = 0.3, y = 0.3$ (h)solution of q at $x = 0.3, y = 0.3$

Figure 8: Numerical solution of q , of Example 5.2.2, at $w_1 = w_2 = 1, \gamma = 1$.

6. Conclusions

In this study, we investigated a nonlinear predator-prey system under Robin boundary conditions within a convex bounded domain. By employing the Faedo-Galerkin approach alongside compactness arguments, we rigorously established the existence and uniqueness of both weak and strong solutions. Furthermore, higher regularity the solutions was demonstrated, and continuous dependence on initial data was confirmed. Through careful construction of approximations and detailed energy estimates, we converted the infinite-dimensional problem into a finite-dimensional one, facilitating both theoretical analysis and practical computation. To validate and illustrate the theoretical results, numerical simulations in one and two spatial dimensions were performed. These simulations confirmed the theoretical predictions and revealed the dynamic patterns and stability behavior of the system. These results contribute significantly to the mathematical understanding of reaction-diffusion systems with nonlinear interactions and complex boundary behavior, laying the groundwork for future research in related fields, such as chemical kinetics, biological pattern formation, and ecological modeling.

References

[1] A. N. Jasim, "On the Maximum Diameter of Graphs," Passer Journal of Basic and Applied Sciences, vol. 7, no. 2, 2025, doi: 10.24271/psr.2025.499398.1897.

[2] A. Al-Haboobi and A. H. Al-Muslimawi, "Simulation of Newtonian axisymmetric thermal channel flow by using a developed Taylor–Galerkin/pressure correction method," Int. J. Appl. Comput. Math., vol. 11, no. 5, Art. no. 172, 2025.

[3] A. Kherd, A. Saaban and I. E. I. Fadhel, "Wang-Ball polynomials for the numerical solution of singular ordinary differential equations", *Iraqi Journal of Science*, vol. 62, no. 3, pp. 941-949, 2021.

[4] H. Hashim, M. H. Homod, and A. J. Harfash, "Finite element analysis of attraction-repulsion chemotaxis system. Part II: Time convergence, error analysis and numerical results, " *Communications on Applied Mathematics and Computation*, vol. 4, no. 3, pp. 1057–1104, 2022.

[5] I. Prigogine and G. Nicolis, *Self-organisation in nonequilibrium systems: towards a dynamic of complexity*, Bifurcation analysis, pp. 3–12, Springer, 1985.

[6] F. Piazza, "The physics of boundary conditions in reaction–diffusion problems," *The Journal of Chemical Physics*, vol. 157, no. 23, 2022.

[7] K.Y. Lam and Y. Lou, *Introduction to Reaction-Diffusion Equations: Theory and Applications to Spatial Ecology and Evolutionary Biology*. Springer Nature, 2022.

[8] K. M. Agbavon, A. R. Appadu, and B. İnan, "Comparative study of some numerical methods for the standard FitzHugh-Nagumo equation," *Computational Mathematics and Applications*, pp. 95–127, 2020.

[9] S. Haq, I. Ali, and K. S. Nisar, "A computational study of two-dimensional reaction diffusion Brusselator system with applications in chemical processes," *Alexandria Engineering Journal*, vol. 60, no. 5, pp. 4381–4392, 2021.

[10] G. A. Al-Juaifri and A. J. Harfash, "Numerical analysis of the Brusselator model with Robin boundary conditions," SeMA Journal, vol. 82, no. 3, pp. 351–397, 2025.

[11] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Paris: Dunod, 1969.

[12] S.A. Narawade, L. Pati, B. Mishra, S.K. Tripathy, "Dynamical system analysis for accelerating models in non-metricity f(Q) gravity," *Physics of the Dark Universe*, vol. 36, pp. 101020, 2022.

[13] C. Ye, J. Cui, and H. Dong, "Asymptotic analysis of nonlinear Robin-type boundary value problems with small periodic structure," *Multiscale Modeling and Simulation*, vol. 19, no. 2, pp. 830–845, 2021.

[14] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, vol. 68, Springer

Science and Business Media, 2012.

[15] T. Cazenave, *Semilinear Schrodinger Equations*, vol. 10, American Mathematical society, 2003.

[16] P. G. Ciarlet, *The finite element method for elliptic problems*, Society for Industrial and Applied Mathematics (SIAM), 2002.

[17] E. Pişkin and B. Okutmuştur, *An Introduction to Sobolev Spaces*, Bentham Science Publishers, 2021.

[18] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*. Philadelphia, PA, USA: Society for Industrial and Applied Mathematics, 2002.

[19] C.W. Ha, "Weak compactness and the minimax equality" *Nonlinear and Convex Analysis*, CRC Press, pp. 77–82 ,2023.

[20] W. Hu, Z. Wang, Y. Zhao, and Z. Deng, "Symmetry breaking of infinite-dimensional dynamic system," *Applied Mathematics Letters*, vol. 103, pp. 106207, 2020.

[21] P.C. Huang and B.Y. Pan, "The Robin problems for the coupled system of reaction–diffusion equations," *Boundary Value Problems*, vol. 2024, no. 1, pp. 29, 2024.

[22] B. İnan, K. K. Ali, A. Saha, and T. Ak, "Analytical and numerical solutions of the FitzHugh–Nagumo equation and their multistability behavior," *Numerical Methods for Partial Differential Equations*, vol. 37, no. 1, pp. 7–23, 2021.

[23] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Monographs and Studies in Mathematics, vol. 24, Pitman, Boston, 1985.

[24] G. A. Al-Musawi and A. J. Harfash, "Analysis of a coupled pair of Cahn-Hilliard equations with nondegenerate mobility," *Journal of Mathematical Modeling*, vol. 12, no. 1, pp. 51–70, 2024.

[25] S. M. Hassan and A. J. Harfash, "Finite element approximation of a Keller–Segel model with additional self-and cross-diffusion terms and a logistic source," *Communications in Nonlinear Science and Numerical Simulation*, vol. 104, 2022.

[26] H. Lhachemi and R. Shorten, "Boundary output feedback stabilization of state delayed reaction–diffusion PDEs," *Automatica*, vol. 156, pp. 111188, 2023.

[27] G. A. Al-Musawi and A. J. Harfash. "Finite element analysis of extended Fisher-Kolmogorov equation with Neumann boundary conditions", *Applied Numerical Mathematics*, vol. 201, 2024.

[28] S. Kumar and H. Ramos, "Parameter-uniform approximation on equidistributed meshes for singularly perturbed parabolic reaction-diffusion problems with Robin boundary conditions," *Applied Mathematics and Computation*, vol. 392, 125677, 2021.

[29] M. H. Hashim and A. J. Harfash, "Finite element analysis of attraction-repulsion chemotaxis system. Part II: Time convergence, error analysis and numerical results," *Communications on Applied Mathematics and Computation*, vol. 4, no. 3, pp. 1057-1104, 2022.

[30] P. V. L. Em, "Comparison of synchronization speed of ring networks of reaction-diffusion equations of FitzHugh-Nagumo type with unidirectional and bidirectional coupling," *Advanced Mathematical Models and Applications*, vol. 9, no. 3, 2024.