Iraqi Journal of Science, 2026, Vol. xx, No. x, pp: xx
DOI: 10.24996/ijs.2026.67.6.31

ISSN: 0067-2904

On the Analysis of Extended Partial Metric Space Combined with Fuzzy Soft Structure

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Received: 9/ 3/2025 Accepted: 29/5/2025 Published: xx

Abstract

This note introduces a new extended concept into a fuzzy soft area that plays a prominent role in functional analysis, called the fuzzy soft b-partial metric space, which is an extension of the fuzzy soft metric space and the fuzzy soft partial metric space. The research discusses numerous analytical features based on this proposed space, including the F_{ss}-closed, F_{ss}-open ball, F_{ss}-Hausdorff space, and F_{ss}-converge sequence. Moreover, several essential theorems of convergence of this space are investigated. Furthermore, close connections between this newly defined space and b-metric spaces are also examined. In this regard, several instances are illustrated.

Keywords: F_{ss}-sets, F_{ss}-b-partial Metric spaces , F_{ss}-Hausdorff- b-PMS, F_{ss}-Cauchy.

حول تحليل الفضاء المتري الجزئي الموسع والمعرف على البنية الضبابية الناعمة

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الخلاصه:

تقدم هذه المقالة مفهوما جديدا موسعا في المجال الضبابي الناعم والذي يلعب دورا بارزا في التحليل الدالي، يدعى الفضاء المتري الجزئي-b الضبابي الناعم، وهو توسيع للفضاء المتري الضبابي الناعم والفضاء المتري الجزئي الضبابي الناعم. حيث يناقش البحث العديد من الميزات التحليلية بناء على الفضاء المقترح ويتضمن الكرات المغلقة $-F_{ss}$ والمفتوحة $-F_{ss}$ ، فضاء هاوسدورف $-F_{ss}$ و المتتابعة المتقاربة $-F_{ss}$ بالاضافة الى العديد من النظريات الاساسية للتقارب لهذا الفضاء اكتشفت علاوة على ذلك العلاقات بين هذا تعريف الفضاء الجديد والفضاء المتري-b كذلك برهنت وفي هذا الصدد تم توضيح عدة امثلة.

1. Introduction

In mathematics, the discipline of Fuzzy Soft sets (FSs) is pivotal in dealing with ambiguous and inaccurate data. FSs are of great importance because they make decisions on various real-world problems. Indeed, the applications of FSs are used in medicine, engineering, and optimization by developing new algorithms. In 1965, Zadeh [1] expanded the traditional concept of belonging in his Fuzzy Theory (FT), allowing some items to belong with degrees of membership ranging from zero to one. After Zadeh's study, many researchers developed FT and used their applications in diverse fields [2-7]. On the other hand, Molodtsov [8] in 1999 founded the Soft Set Theory (SST) to deal with imprecise and ambiguous data. Then, in 2003,

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Maji et al., [9] expanded the soft set by defining basic concepts such as equality, subset, superset, complement, null, absolute soft set, and De Morgan's law. Furthermore, Raghad I. Sabri, Buthainah A. A. Ahmed [10] in 2024, and Bayramov et al., [11] in 2023 worked in the same field with different research directions. In the context of mathematical analysis, Czerwik [12] in 1993 developed the b-metric space as a generalized traditional metric space. Mathews in 1994 [13] introduced partial -metric spaces as a generalization of classical metric spaces and showed that the self-distances are not necessarily zero. In 2020, Sarkar [14] presented a rectangular partial b-metric space and Cantor intersection theorem in this space, which has been proved. In the same year, Shaakir and Abdullah [15] generalized partial metric spaces by introducing general partial metric spaces and explained the relation between them and D-metric space. In 2022, Abu-Donia et al., [16] introduced partial 2-metric space and provided some fixed-point theorem under self-mapping in this space. In 2024, Gaba and Agyingi [17] proved that there is a unique fixed point in a partial b-metric space when sequential condition is satisfied. In the same year, Pandey et al., [18] studied some fixed-point theorems in complex valued partial b-metric space for the contractive mapping. In 2001, Maji et al., [19] discussed and introduced the principle of FSs as a generalization of soft sets and an extension of their application. Several researchers [20-27] have contributed to recent studies in the development of applied and theoretical sciences by generalizing the concept of metric space and its types to FSs. After that, Faried et al., [28-29] and Mohsen [30] provided, based on the FSs, general principles of inner product and Hilbert spaces. In this regard, FSs play a paramount part in topology theory by generalizing some of the separation axioms and their merits in terms of FSs, see [31-33]. Due to the importance of soft sets, in 2023, Mohsen and Mousa [34] posed the fuzzy soft Hilbert space in FSs and investigated various topological merits of this proposed space. Then, the area of soft sets has attracted numerous researchers [35-40] and is considered a continuation in the functional analysis field.

The present work introduces a novel concept in the fuzzy soft area that greatly impacts functional analysis, known as the fuzzy soft b-partial metric space. This space is an extension of fuzzy soft metric space and the fuzzy soft partial metric space. Based on this new space, several analytical advantages are discussed, including the F_{ss}-closed, F_{ss}-open ball, F_{ss}-Hausdorff space, and F_{ss}-converge sequence. Furthermore, some interesting theorems of convergence of this space are considered. It presents the close correlation between the newly defined space and b-metric spaces. In addition, several examples are also highlighted.

2-Preliminaries

This section reviews the relevant concepts that underlie the FSs and b-metric space.

Definition 2.1: [1] Let U be a universal set then the fuzzy set $\tilde{X}$ can be defined by ordered pair $\tilde{X}=\{(\ (u\ ,\mu_{\tilde{X}}(u))\ :u\ \in U\ ,\mu_{\tilde{X}}(u)\ \ \in I\ )\}$ where $\mu_{\tilde{X}}(u)$ is degree membership of u in $\tilde{X}$ and $I = [0,1]$.

Definition 2.2: [8] Let G:A→ $P(U)$ be mapping, $P(U)$ be a power set of U and $A\ \subseteq E$, where E is a collection of parameters. Then soft set of U can be defined as $G_A = \{(e, G_A(e): G_A(e) \in P(U), e \in E\}$ and denoted by ordered pair (G, A).

Definition 2.3: [19] Let G:A→ I^U be mapping and the fuzzy soft collection is $\{e \in A, G_A(e) \in I^U\}$, then (G, A) is said to be fuzzy soft of U and denoted by F_{ss} .

Definition 2.4: [31]

1- If $\mu G(e) = 1$ for all $e \in A$ then $F_{ss} - (G, A)$ is said to be absolute $-F_{ss}$, symbolized by $\widetilde{C_A}$.

2- If $\mu G(e) = \tilde{0}$ for all $e \in A$ then $F_{ss} - (G, A)$ is said to be null$-F_{ss}$, symbolized by $\tilde{\emptyset}$.

Definitions 2.5: [19] When U be common universal set of distinct $F_{ss} - (G_1, A)$ and (G_2, B)

1-If $A \subseteq B$, and $G_1(e) \subseteq G_2(e)$, $\mu G_1(e) \leq \mu G_2(e)$ for all $e \in A$, then we can write

$(G_1, A) \subseteq (G_2, B)$.
2-If $(G_1, A) \subseteq (G_2, B)$ and$(G_2, B) \subseteq (G_1, A)$ then $\mathrm{F}_{ss} - (G_1, A)$ and (G_2, B) are equal and denoted by $(G_1, A) = (G_2, B)$.
Definition 2.6: [28] If $\upsilon \in U$ and e$\in A$, such that:

$$\mu G(e) = \begin{cases} \rho \text{ if } \upsilon = \upsilon_0 \text{ and } e = e_0 \in A \\ 0 \ if\ u \in U - \{\upsilon_0\} \text{ or}, e \in A - \{e_0\} \text{ where } \rho \in (0,1] \end{cases}$$

then $\mathrm{F}_{ss} - (G, A)$ is said to be F_{ss} −point and symbolized by $\upsilon_{\mu G(e)}$.
Definition 2.7: [29] 1. If for all $\in A, G(e) \subseteq G_1(e)$ then F_{ss} −point $\upsilon_{\mu G(e)}$ is said to be belong to (G_1, A) and denoted by $\upsilon_{\mu G(e)}$.
2. If $\upsilon^1 = \upsilon^2$, $\mu G_1(e_1) = \mu G_2(e_2)$, then F_{ss} −point ${\upsilon^1}_{\mu G(e_1)}$is equal to ${\upsilon^2}_{\mu G(e_2)}$.
Definition 2.8: [35] The intersection of $\mathrm{F}_{ss} - (G_1, A)$ and (G_2, B) is $\mathrm{F}_{ss} - (G_3, A)$ such that:
$\mu G_{3\,(\,e)}(\upsilon)$={$\mu G_{1\,(\,e)}(\upsilon)$, if $e \in A - B, \upsilon \in U$ $\mu G_{2\,(\,e)}(\upsilon)$, if $e \in B - A, \upsilon \in$
U $min[\mu G_{1\,(\,e)}(\upsilon), \mu G_{2\,(\,e)}(\upsilon)]$ if $e \in A \cap B, \upsilon \in U$, where $\upsilon \in U$, B$\cup$ A =C and e$\in C$.
Definition 2.9: [20] Let $\tilde{d} : \mathrm{F}_{ss}(U) \times \mathrm{F}_{ss}(U) \to R^+(A)$ be function where $\mathrm{F}_{ss}(U)$ is a non-empty F_{ss} of U then $\tilde{d}$ said to fuzzy soft metric on $\mathrm{F}_{ss}(U)$ if for all ${\upsilon^1}_{\mu G(e_1)}$, ${\upsilon^2}_{\mu G(e_2)}$, ${\upsilon^3}_{\mu G(e_3)} \in \mathrm{F}_{ss}(U)$, the following conditions are satisfy:
1-$\tilde{d}({\upsilon^1}_{\mu G(e_1)}, {\upsilon^2}_{\mu G(e_2)}) \cong \tilde{d}({\upsilon^2}_{\mu G(e_2)}, {\upsilon^1}_{\mu G(e_1)})$.
2-$\tilde{d}({\upsilon^1}_{\mu G(e_1)}, {\upsilon^2}_{\mu G(e_2)}) \geqq 0$.
3- $\tilde{d}({\upsilon^1}_{\mu G(e_1)}, {\upsilon^2}_{\mu G(e_2)}) \cong 0 \leftrightarrow {\upsilon^1}_{\mu G(e_1)} \cong {\upsilon^2}_{\mu G(e_2)}$.
4- $\tilde{d}({\upsilon^1}_{\mu G(e_1)}, {\upsilon^3}_{\mu G(e_3)}) \leqq [\tilde{d}({\upsilon^1}_{\mu G(e_1)}, {\upsilon^2}_{\mu G(e_2)}) + \tilde{d}({\upsilon^2}_{\mu G(e_2)}, {\upsilon^3}_{\mu G(e_3)})]$,
and $(\mathrm{F}_{ss}(U), \tilde{d})$ is fuzzy soft metric space and denoted by $\mathrm{F}_{ss} - MS$.
Definition 2.10: [34] Let $\tilde{d} : \mathrm{F}_{ss}(U) \times \mathrm{F}_{ss}(U) \to R^+(A)$ be function where $\mathrm{F}_{ss}(U)$ is non-empty F_{ss} of U and S≥ 1, then $\tilde{d}$ said to fuzzy soft b- metric on $\mathrm{F}_{ss}(U)$ if for all ${\upsilon^1}_{\mu G(e_1)}$, ${\upsilon^2}_{\mu G(e_2)}$, ${\upsilon^3}_{\mu G(e_3)} \in \mathrm{F}_{ss}(U)$, the following conditions are satisfy:
1-$\tilde{d}^b({\upsilon^1}_{\mu G(e_1)}, {\upsilon^2}_{\mu G(e_2)}) \cong \tilde{d}^b({\upsilon^2}_{\mu G(e_2)}, {\upsilon^1}_{\mu G(e_1)})$.
2-$\tilde{d}^b({\upsilon^1}_{\mu G(e_1)}, {\upsilon^2}_{\mu G(e_2)})$.
3- $\tilde{d}^b({\upsilon^1}_{\mu G(e_1)}, {\upsilon^2}_{\mu G(e_2)}) \cong 0$ if and only if ${\upsilon^1}_{\mu G(e_1)} \cong {\upsilon^2}_{\mu G(e_2)}$.
4- $\tilde{d}^b({\upsilon^1}_{\mu G(e_1)}, {\upsilon^3}_{\mu G(e_3)}) \leqq S\,[\tilde{d}^b({\upsilon^1}_{\mu G(e_1)}, {\upsilon^2}_{\mu G(e_2)}) + \tilde{d}^b({\upsilon^2}_{\mu G(e_2)}, {\upsilon^3}_{\mu G(e_3)})]$,
and $(\mathrm{F}_{ss}(U), \tilde{d}^b)$ fuzzy soft b- metric space and denoted by $\mathrm{F}_{ss} - bMS$.
Definition 2.11: [34] Let $\tilde{P} : \mathrm{F}_{ss}(U) \times \mathrm{F}_{ss}(U) \to R^+(A)$ be function where $\mathrm{F}_{ss}(U)$ is non-empty F_{ss} of U and S real number such that S ≥ 0, then for all ${\upsilon^1}_{\mu G(e_1)}$, ${\upsilon^2}_{\mu G(e_2)}$, ${\upsilon^3}_{\mu G(e_3)} \in \mathrm{F}_{ss}(U)$, $\tilde{P}$ is called fuzzy soft partial on $\mathrm{F}_{ss}(U)$ if the following conditions are hold:
i)$\tilde{P}({\upsilon^1}_{\mu G(e_1)}, {\upsilon^2}_{\mu G(e_2)}) \cong \tilde{P}({\upsilon^2}_{\mu G(e_2)}, {\upsilon^1}_{\mu G(e_1)})$.
ii)$\tilde{P}({\upsilon^2}_{\mu G(e_2)}, {\upsilon^2}_{\mu G(e_2)}) \cong \tilde{P}({\upsilon^1}_{\mu G(e_1)}, {\upsilon^1}_{\mu G(e_1)}) \cong \tilde{P}({\upsilon^1}_{\mu G(e_1)}, {\upsilon^2}_{\mu G(e_2)})$
if and only if ${\upsilon^1}_{\mu G(e_1)} \cong {\upsilon^2}_{\mu G(e_2)}$.
iii) $\tilde{P}({\upsilon^1}_{\mu G(e_1)}, {\upsilon^2}_{\mu G(e_2)}) \geqq \tilde{P}({\upsilon^1}_{\mu G(e_1)}, {\upsilon^1}_{\mu G(e_1)})$.
v)$\tilde{P}({\upsilon^1}_{\mu G(e_1)}, {\upsilon^2}_{\mu G(e_2)}) \leqq$
$[\tilde{P}({\upsilon^1}_{\mu G(e_1)}, {\upsilon^3}_{\mu G(e_3)}) + \tilde{P}({\upsilon^3}_{\mu G(e_3)}, {\upsilon^2}_{\mu G(e_2)}) - \tilde{P}({\upsilon^3}_{\mu G(e_3)}, {\upsilon^3}_{\mu G(e_3)})]$,
and $(\mathrm{F}_{ss}(U), \tilde{P})$ is a fuzzy soft partial metric space and denoted by $\mathrm{F}_{ss} - PMS$.

3- Analytic attributes of fuzzy soft b-partial metric space

This part provides the concept of b-partial metric subjected to FSs and discusses several analytic advantages. Some illustrating examples are described in this posed space, which are crucial to comprehend some characteristics.

Definition 3.1: Let $\tilde{d}_p^b\colon \mathrm{F}_{ss}(U) \times \mathrm{F}_{ss}(U) \to R^+(A)$ be function where $\mathrm{F}_{ss}(U)$ is non-empty F_{ss} of U then for all $\upsilon^1{}_{\mu G(e_1)}, \upsilon^2{}_{\mu G(e_2)}, \upsilon^3{}_{\mu G(e_3)} \in \mathrm{F}_{ss}(U)$, $\tilde{d}_p^b$ is called fuzzy soft b- partial metric on $\mathrm{F}_{ss}(U)$ if the following conditions are hold:
i) $\tilde{d}_p^b(\upsilon^1{}_{\mu G(e_1)}, \upsilon^2{}_{\mu G(e_2)}) \cong \tilde{d}_p^b(\upsilon^2{}_{\mu G(e_2)}, \upsilon^1{}_{\mu G(e_1)})$.
ii) $\tilde{d}_p^b(\upsilon^2{}_{\mu G(e_2)}, \upsilon^2{}_{\mu G(e_2)}) \cong \tilde{d}_p^b(\upsilon^1{}_{\mu G(e_1)}, \upsilon^1{}_{\mu G(e_1)}) \cong \tilde{d}_p^b(\upsilon^1{}_{\mu G(e_1)}, \upsilon^2{}_{\mu G(e_2)})$ if and only if $\upsilon^1{}_{\mu G(e_1)} \cong \upsilon^2{}_{\mu G(e_2)}$.
iii) $\tilde{d}_p^b(\upsilon^1{}_{\mu G(e_1)}, \upsilon^2{}_{\mu G(e_2)}) \gtrapprox \tilde{d}_p^b(\upsilon^1{}_{\mu G(e_1)}, \upsilon^1{}_{\mu G(e_1)})$.
v) $\tilde{d}_p^b(\upsilon^1{}_{\mu G(e_1)}, \upsilon^2{}_{\mu G(e_2)}) \lessapprox S \left[\tilde{d}_p^b(\upsilon^1{}_{\mu G(e_1)}, \upsilon^3{}_{\mu G(e_3)}) + \tilde{d}_p^b(\upsilon^3{}_{\mu G(e_3)}, \upsilon^2{}_{\mu G(e_2)}) - \tilde{d}_p^b(\upsilon^3{}_{\mu G(e_3)}, \upsilon^3{}_{\mu G(e_3)})\right]$,
and $(\mathrm{F}_{ss}(U), \tilde{d}_p^b)$ is fuzzy soft b- partial metric space and denoted by $\mathrm{F}_{ss} - bPMS$.

Example 3.2: Let $\mathrm{F}_{ss}(U) = R^+(A)$ and $\tilde{d}_p^b\colon \mathrm{F}_{ss}(U) \times \mathrm{F}_{ss}(U) \to R^+(A)$ defined by:
$\tilde{d}_p^b(\upsilon^1{}_{\mu G(e_1)}, \upsilon^2{}_{\mu G(e_2)}) = (max\{\upsilon^1{}_{\mu G(e_1)}, \upsilon^2{}_{\mu G(e_2)}\})^2 + 2$, then $(\mathrm{F}_{ss}(U), \tilde{d}_p^b)$ is $\mathrm{F}_{ss} - bPMS$ with S=4.

Remarks 3.3:
1) $\mathrm{F}_{ss} - bPMS$ is $\mathrm{F}_{ss} - PMS$ when S=1, but the convers is not necessary be true. To see an example that illustrates this when S=1, and $\tilde{d}_p^b\colon \mathrm{F}_{ss}(U) \times \mathrm{F}_{ss}(U) \to R^+(A)$ defined by,
$\tilde{d}_p^b(\upsilon^1{}_{\mu G(k_1)}, \upsilon^2{}_{\mu G(k_2)}) = \widetilde{0.7}$, $\tilde{d}_p^b(\upsilon^1{}_{\mu G(k_1)}, \upsilon^3{}_{\mu G(k_3)}) = \widetilde{0.8}$
$\tilde{d}_p^b(\upsilon^2{}_{\mu G(k_2)}, \upsilon^3{}_{\mu G(k_3)}) = \widetilde{0.9}$, $\tilde{d}_p^b(\upsilon^1{}_{\mu G(k_1)}, \upsilon^1{}_{\mu G(k_1)}) = \widetilde{0.2}$, $\tilde{d}_p^b(\upsilon^2{}_{\mu G(k_2)}, \upsilon^2{}_{\mu G(k_2)}) = \widetilde{0.5}$
$\tilde{d}_p^b(\upsilon^3{}_{\mu G(k_3)}, \upsilon^3{}_{\mu G(k_3)}) = \widetilde{0.6}$, for all $\upsilon^1{}_{\mu G(k_1)}, \upsilon^2{}_{\mu G(k_2)}, \upsilon^3{}_{\mu G(k_3)}$ in $\mathrm{F}_{ss}(U)$.
2) $\mathrm{F}_{ss} - bPMS$ is not $\mathrm{F}_{ss} - bMS$, see the example above.
3) $\mathrm{F}_{ss} - bMS$ is $\mathrm{F}_{ss} - bPMS$ with (self distance) but the convers is not necessary be true, that means b-partial is not b- metric, since $d(\upsilon^1{}_{\mu G(k_1)}, \upsilon^1{}_{\mu G(k_1)}) = \tilde{0}$ in $\mathrm{F}_{ss} - bMS$, but $d(\upsilon^1{}_{\mu G(k_1)}, \upsilon^1{}_{\mu G(k_1)}) \ncong \tilde{0}$.
4) If $\upsilon^1{}_{\mu G(k_1)}, \upsilon^2{}_{\mu G(k_2)}$ be F_{ss} −point in $\mathrm{F}_{ss}(U)$ such that $\tilde{d}_p^b(\upsilon^1{}_{\mu G(k_1)}, \upsilon^2{}_{\mu G(k_2)}) = \tilde{0}$.
We have $\upsilon^1{}_{\mu G(k_1)} = \upsilon^2{}_{\mu G(k_2)}$, but the convers is not necessary be true, since, $\upsilon^1{}_{\mu G(k_1)} = \upsilon^2{}_{\mu G(k_2)}$, $\tilde{d}_p^b(\upsilon^1{}_{\mu G(k_1)}, \upsilon^1{}_{\mu G(k_1)}) \neq \tilde{0}$ and $\tilde{d}_p^b(\upsilon^2{}_{\mu G(k_2)}, \upsilon^2{}_{\mu G(k_2)}) \neq \tilde{0}$.

Remark 3.4: Let $(G, A), (G, B) \subseteq \mathrm{F}_{ss}(U)$ and $\mathrm{F}_{ss} - bPMS$ is $(\mathrm{F}_{ss}(U), \tilde{d}_p^b)$ then:
$\tilde{d}_p^b\,((G, A), (G, B))$ is symbolized of the distance between (G, A) and (G, B) such that defined as $\tilde{d}_p^b\,((G, A), (G, B)) = \inf\{\tilde{d}_p^b\,(\upsilon_{\mu G(a)}, \upsilon_{\mu G(b)}) \mid \upsilon_{\mu G(a)} \tilde{\in} (G, A), \ \upsilon_{\mu G(a)} \tilde{\in} (G, B)\}$.

Definition 3.5: Let $(\mathrm{F}_{ss}(U), \tilde{d}_p^b)$ be $\mathrm{F}_{ss} - bPMS$ and $\langle \upsilon^n{}_{\mu G(e_n)} \rangle$ be any sequence in $(\mathrm{F}_{ss}(U), \tilde{d}_p^b)$ then $\langle \upsilon^n{}_{\mu G(e_n)} \rangle$ is said to be convergent to $\upsilon_{\mu G(e)}$ in $\mathrm{F}_{ss}(U)$ if $(\upsilon^n{}_{\mu G(e_n)}, \upsilon_{\mu G(e)}) \cong \tilde{d}_p^b(\upsilon_{\mu G(e)}, \upsilon_{\mu G(e)}) \cong \tilde{d}_p^b(\upsilon^n{}_{\mu G(e_n)}, \upsilon^n{}_{\mu G(e_n)})$.
Definition 3.6: In $(\mathrm{F}_{ss}(U), \tilde{d}_p^b)$ a sequence $\langle \upsilon^n{}_{\mu G(e_n)} \rangle$ is said to F_{ss} − Cauchy sequence if $\tilde{d}_p^b(\upsilon^n{}_{\mu G(e_n)}, \upsilon^m{}_{\mu G(e_m)})$ exists and finite.

Definition 3.7: A $\mathrm{F}_{ss} - bPMS$ is said to complete if every F_{ss} −Cauchy sequence in $(\mathrm{F}_{ss}(U), \tilde{d}_p^b)$ is convergent.

Theorem 3.8: Every F_{ss} − convergent sequences in $(\mathrm{F}_{ss}U)$ is F_{ss} − Cauchy sequences where $(\mathrm{F}_{ss}(U), \tilde{d}_p^b)$ is $\mathrm{F}_{ss} - bPMS$.

Proof: In $(F_{ss}U)$, suppose$\langle v^{n}{}_{\mu G(e_n)}\rangle$ F_{ss} $-$ convergent to $v_{\mu G(e)}$ since $\langle v^{n}{}_{\mu G(e_n)}\rangle \to v_{\mu G(e)}$ then for any $\tilde{\xi} \in R^{+}(A)$, $\tilde{\xi} > 0$ there is K is non negative integer , $K = K\ (\epsilon)$ such that for all $n > K$ $\tilde{d}_p^b\left(v^{n}{}_{\mu G(e_n)}, v_{\mu G(e)}\right) < \frac{\tilde{\xi}}{2S}$ and $|\mu G(e_n)(S) - \mu G(e_m)(S)| < \frac{\tilde{\xi}}{2}$.
So, for every $n, m > K$ we have;
$\tilde{d}_p^b\left(v^{n}{}_{\mu G(e_n)}, v_{\mu G(e)}\right) < \frac{\tilde{\xi}}{2S} < S\left[\tilde{d}_p^b\left(v^{n}{}_{\mu G(e_n)}, v_{\mu G(e)}\right) + \tilde{d}_p^b\left(v_{\mu G(e)}, v^{m}{}_{\mu G(e_m)}\right)\right]$=S$\left[\frac{\tilde{\xi}}{2S} + \frac{\tilde{\xi}}{2S}\right] = \tilde{\xi}$ and $|\mu G(e_n)(S) - \mu G(e_m)(S)| < |\mu G(e_n)(S) - \mu G(e)(S)|$+$|\mu G(e)(S) - \mu G(e_m)(S)| < \frac{\tilde{\epsilon}}{2} + \frac{\tilde{\epsilon}}{2}$.

Definition 3.9: Let $\bar{\zeta}$ be a F_{ss} real number and $\left(F_{ss}(U), \tilde{d}_p^b\right)$ be $F_{ss} - bPMS.$ We can define F_{ss} $-$open ball in $\left(F_{ss}(U), \tilde{d}_p^b\right)$ by $\left\{v_{\mu G(a)} \in F_{ss}(U): \tilde{d}_p^b\left(v_{\mu G(a)}, v_{\mu G(e)}\right) \tilde{<} \bar{\zeta}\right\}$ which is denoted by $\tilde{d}_{p\ \bar{\zeta}}^{b}\left(v_{\mu G(e)}\right)$, and F_{ss} $-$closed ball is defined by $\left\{v_{\mu G(a)} \in F_{ss}(U): \tilde{d}_p^b\left(v_{\mu G(a)}, v_{\mu G(e)}\right) \tilde{\leq} \bar{\zeta}\right\}$ and denoted by $\tilde{d}_{p\ \bar{\zeta}}^{b}\left[v_{\mu G(e)}\right]$.

Definition 3.10: In a $F_{ss} - bPMS$ a $F_{ss} - (G, A)$ is said to be F_{ss} $-$open if for each F_{ss} $-$point $v_{\mu G(e)}$ of (G, A) $\exists$ F_{ss} $-$open ball $\tilde{d}_{p\ \bar{\zeta}}^{b}\left(v_{\mu G(e)}\right) \tilde{\subseteq} (G, A)$.

Definition 3.11: The F_{ss} $-$point $v_{\mu G(e)}$ is F_{ss} $-$ limit point of (G, A) in$F_{ss} - bPMS$ if and only if for every $\tilde{0} < \tilde{r} \in R^{+}$ (A) we have $\tilde{d}_{p\ \bar{\zeta}}^{b}\left(v_{\mu G(e)}\right) \tilde{\cap} (G, A) \neq \tilde{\emptyset}$.

Definition 3.12: If $\tilde{d}_p^b\left(v^{1}{}_{\mu G(e_1)}, v^{2}{}_{\mu G(e_2)}\right) \tilde{>} \tilde{0}$ for any $v^{1}{}_{\mu G(e_1)}$, $v^{2}{}_{\mu G(k_2)}$ in $F_{ss}(U)$ then there exists $\tilde{d}_{p\ \bar{\zeta}}^{b}\left(v^{1}{}_{\mu G(e_1)}\right)$ and $\tilde{d}_{p\ \bar{\zeta}}^{b}\left(v^{2}{}_{\mu G(e_2)}\right)$ be two F_{ss} $-$open ball such that $\tilde{d}_{p\ \bar{\zeta}}^{b}\left(v^{1}{}_{\mu G(e_1)}\right) \tilde{\cap}$ $\tilde{d}_{p\ \bar{\zeta}}^{b}\left(v^{2}{}_{\mu G(e_2)}\right) = \tilde{\emptyset}$ where the radius $\bar{\zeta}$ and center $v^{1}{}_{\mu G(e_1)}$ and $v^{2}{}_{\mu G(e_2)}$ then $F_{ss} - bPMS$ is said to be Hausdorff space.

Definition 3.13: In a $F_{ss} - bPMS$ a sequence $\langle v^{n}{}_{\mu G(e_n)}\rangle$ is said to be bounded if there exists F_{ss} real number$\bar{\zeta}$ such that $v^{n}{}_{\mu G(e_n)} \tilde{\in} \tilde{d}_{p\ \bar{\zeta}}^{b}\left(v_{\mu G(e)}\right)$ and $\left|\mu G\left(v^{n}{}_{\mu G(e_n)}\right)(S) - \mu G(e)(S)\right| < \tilde{\varepsilon}$ where $\tilde{\varepsilon} \tilde{\in} \left(\tilde{0}, \tilde{1}\right)$ for all $n \in N$.

Theorem 3.14: Every $\left(F_{ss}(U), \tilde{d}_p^b\right)$ is Hausdorff space.
Proof: Let$\left(F_{ss}(U), \tilde{d}_p^b\right)$ be $F_{ss} - bPMS$ at least have two F_{ss} $-$points, and let $v^{1}{}_{\mu G(e_1)}$, $v^{2}{}_{\mu G(e_2)}$, $v^{3}{}_{\mu G(e_3)}$ be F_{ss} $-$points in $F_{ss}(U)$ such that
$\tilde{d}_p^b\left(v^{1}{}_{\mu G(e_1)}, v^{2}{}_{\mu G(e_2)}\right) > 0$ and $\tilde{d}_p^b\left(v^{3}{}_{\mu G(e_1)}, v^{3}{}_{\mu G(e_2)}\right) > 0$. Take any $\bar{\zeta}$ be a F_{ss} real number
$\tilde{d}_p^b\left(v^{1}{}_{\mu G(e_1)}, v^{2}{}_{\mu G(e_2)}\right) > 0$ and
$\tilde{d}_p^b\left(v^{3}{}_{\mu G(e_3)}, v^{3}{}_{\mu G(e_3)}\right) > 0$
$0 < \bar{\zeta} < \frac{1}{2S}\ \tilde{d}_p^b\left(v^{1}{}_{\mu G(e_1)}, v^{2}{}_{\mu G(e_2)}\right)$. Consider F_{ss} $-$ open balls
$\tilde{d}_{p\ \bar{\zeta}}^{b}\left(v^{1}{}_{\mu G(e_1)}\right) = \left\{v^{1}{}_{\mu G(a_1)} \in F_{ss}(U): \tilde{d}_p^b\left(v^{1}{}_{\mu G(a_1)}, v^{1}{}_{\mu G(e_1)}\right) \leq \bar{\zeta}\right\}$
$\tilde{d}_{p\ \bar{\zeta}}^{b}\left(v^{2}{}_{\mu G(e_2)}\right) = \left\{v^{2}{}_{\mu G(a_2)} \in F_{ss}(U): \tilde{d}_p^b\left(v^{2}{}_{\mu G(a_2)}, v^{2}{}_{\mu G(e_2)}\right) \leq \bar{\zeta}\right\}$
and
$\tilde{d}_{p\ \bar{\zeta}}^{b}\left(v^{3}{}_{\mu G(e_3)}\right) = \left\{v^{3}{}_{\mu G(a_3)} \in F_{ss}(U): \tilde{d}_p^b\left(v^{3}{}_{\mu G(a_3)}, v^{3}{}_{\mu G(e_3)}\right) \leq \bar{\zeta}\right\}$.
Let $v^{3}{}_{\mu G(e_3)} \in \tilde{d}_{p\ \bar{\zeta}}^{b}\left(v^{1}{}_{\mu G(e_1)}\right) \cap \tilde{d}_{p\ \bar{\zeta}}^{b}\left(v^{2}{}_{\mu G(e_2)}\right)$

$\upsilon^3{}_{\mu G(e_3)} \in \tilde{d}^b_{p\,\bar{\zeta}}(\upsilon^1{}_{\mu G(e_1)})$ then $\tilde{d}^b_p(\upsilon^3{}_{\mu G(e_3)}, \upsilon^1{}_{\mu G(e_1)}) \leq \bar{\zeta}$

$\upsilon^3{}_{\mu G(e_3)} \in \tilde{d}^b_{p\,\bar{\zeta}}(\upsilon^2{}_{\mu G(e_2)})$ then $\tilde{d}^b_p(\upsilon^3{}_{\mu G(e_3)}, \upsilon^2{}_{\mu G(e_2)}) \leq \bar{\zeta}$

since $\upsilon^3{}_{\mu G(e_3)} \in \tilde{d}^b_{p\,\bar{\zeta}}(\upsilon^3{}_{\mu G(e_3)})$ then $\tilde{d}^b_p(\upsilon^3{}_{\mu G(e_3)}, \upsilon^3{}_{\mu G(e_3)}) \leq \bar{\zeta}$

by the Definition 3.1

$\tilde{d}^b_p(\upsilon^1{}_{\mu G(e_1)}, \upsilon^2{}_{\mu G(e_2)}) \leq S[\bar{\zeta} + \bar{\zeta} - \bar{\zeta}] = S\bar{\zeta}$.

Therefore, $\bar{\zeta} > \frac{1}{S}\tilde{d}^b_p(\upsilon^1{}_{\mu G(e_1)}, \upsilon^2{}_{\mu G(e_2)})$ which contradict the hypothesis. So, clearly $\tilde{d}^b_{p\,\bar{\zeta}}(\upsilon^1{}_{\mu G(e_1)}) \cap \tilde{d}^b_{p\,\bar{\zeta}}(\upsilon^2{}_{\mu G(e_2)}) = \tilde{\emptyset}$ and hence $(\mathrm{F}_{ss}(U), \tilde{d}^b_p)$ is Hausdorff space.

Theorem 3.15: If every $\mathrm{F}_{ss} -$ Cauchy sequences in $(\mathrm{F}_{ss}U)$ has convergent sub sequences, then $\mathrm{F}_{ss} - bPMS$ is complete.

Proof: To prove $\mathrm{F}_{ss} - bPMS$ is complete we must prove that if $\langle \upsilon^n{}_{\mu G(e_n)} \rangle$ has sub sequences $\langle \upsilon^{nk}{}_{\mu G(e_{nk})} \rangle \to \upsilon_{\mu G(e)}$ in $\mathrm{F}_{ss}(U)$, then $\langle \upsilon^n{}_{\mu G(e_n)} \rangle \to \upsilon_{\mu G(e)}$ where $\langle \upsilon^n{}_{\mu G(e_n)} \rangle$ is $\mathrm{F}_{ss} -$ Cauchy sequences in $(\mathrm{F}_{ss}U)$. Given $\tilde{\epsilon} > 0$, there exists a $\xi > 0$ and take N=N(U) large enough such that $\tilde{d}^b_p(\upsilon^n{}_{\mu G(K_n)}, \upsilon^m{}_{\mu G(K_m)}) < \xi$ for all $n, m \geq N$ implies

$|\mu G(K_n)(S) - \mu G(K_m)(S)| < \tilde{\epsilon}$, then choose $n_i \geq N$ and $\tilde{d}^b_p\left(\upsilon^{n_i}{}_{\mu G(K_{n_i})}, \upsilon_{\mu G(k)}\right) < \xi$ with $|\mu G(K_{ni})(S) - \mu G(k)(S)| < \tilde{\epsilon}$ and by using $n_1 < n_2 \dots\dots$ is an increasing sequences of integers and $\langle \upsilon^n{}_{\mu G(k_n)} \rangle \to \upsilon_{\mu G(k)}$, therefore $n \geq N$ we have

$$\tilde{d}^b_p(\upsilon^n{}_{\mu G(K_n)}, \upsilon_{\mu G(k)}) \leq S[\tilde{d}^b_p\left(\upsilon^n{}_{\mu G(K_n)}, \upsilon^{n_i}{}_{\mu G(K_{n_i})}\right) + \tilde{d}^b_p\left(\upsilon^{n_i}{}_{\mu G(K_{n_i})}, \upsilon_{\mu G(k)}\right)$$
$$-\tilde{d}^b_p\left(\upsilon^{n_i}{}_{\mu G(K_{n_i})}, \upsilon^{n_i}{}_{\mu G(K_{n_i})}\right)] < S(\xi + \xi - \xi) = \tilde{\xi}$$

and $|\mu G(K_n)(S) - \mu G(K)(S)| \leq S[|\mu G(K_n)(S) - \mu G(K_{n_i})(S)|$
$+|\mu G(K_{n_i})(S) - \mu G(k)(S)| - |\mu G(K_{n_i})(S) - \mu G(K_{n_i})(S)|] < \mathrm{S}(\tilde{\epsilon} + \tilde{\epsilon} - \tilde{\epsilon}) = \tilde{\epsilon}$.

Theorem 3.16: The F_{ss} $-$open ball $\tilde{d}^b_{p\,\bar{\zeta}}(\upsilon_{\mu G(e)})$ is open set in $\mathrm{F}_{ss} - bPMS$.

Proof: Let $\tilde{d}^b_{p\,\bar{\zeta}}(\upsilon_{\mu G(e)})$ be F_{ss} $-$open ball such that

$\upsilon_{\mu G(e_k)} \in \tilde{d}^b_{p\,\bar{\zeta}}(\upsilon_{\mu G(e)})$ that means $\tilde{d}^b_p(\upsilon_{\mu G(e)}, \upsilon_{\mu G(e_k)}) < \tilde{\zeta}$ with $|\mu G(e_k)(S) - \mu G(e)(S)| < \tilde{\epsilon}$ and choose $|\mu G(e_k)(S) - \mu G(e)(S)| < \frac{\tilde{\epsilon}_1}{S}$.

Then consider F_{ss} $-$real number $\frac{\tilde{\zeta}_1}{S} < -\tilde{d}^b_p(\upsilon_{\mu G(e)}, \upsilon_{\mu G(e_k)}) + \frac{\tilde{\zeta}}{S}$, where $\tilde{\epsilon}_2 \in (\tilde{0}, \tilde{1})$ such that $\tilde{\epsilon}_2 < \tilde{\epsilon} - \tilde{\epsilon}_1$. Consider $\upsilon_{\mu G(e_w)} \in \tilde{d}^b_{p\,\frac{\tilde{\zeta}_1}{S}}(\upsilon_{\mu G(e)})$ that means $\tilde{d}^b_p(\upsilon_{\mu G(e_w)}, \upsilon_{\mu G(e_k)}) < \frac{\tilde{\zeta}_1}{S}$, with $|\mu G(e_w)(S) - \mu G(e_k)(S)| < \frac{\tilde{\epsilon}_2}{S}$.

$\tilde{d}^b_p(\upsilon_{\mu G(e)}, \upsilon_{\mu G(e_w)}) \leq S[\tilde{d}^b_p(\upsilon_{\mu G(e)}, \upsilon_{\mu G(e_k)}) + \tilde{d}^b_p(\upsilon_{\mu G(e_k)}, \upsilon_{\mu G(e_w)}) - \tilde{d}^b_p(\upsilon_{\mu G(e_k)}, \upsilon_{\mu G(e_k)})] <$ $S[\tilde{d}^b_p(\upsilon_{\mu G(e)}, \upsilon_{\mu G(e_k)}) + \frac{\tilde{\zeta}_1}{S} - \tilde{d}^b_p(\upsilon_{\mu G(e_k)}, \upsilon_{\mu G(e_k)})] < S\left[\frac{\tilde{\zeta}}{S}\right] = \tilde{\zeta}$ and $|\mu G(e)(S) - \mu G(e_w(S)| \leq$ $S[|\mu G(e)(S) - \mu G(e_k)(S)| + |\mu G(e_k)(S) - \mu G(e_w(S)| - |\mu G(e_w(S) - \mu G(e_k)(S)|] <$ $S[\frac{\tilde{\epsilon}_1}{S} + \frac{\tilde{\epsilon}_2}{S}] < \tilde{\epsilon}$, thus $\tilde{d}^b_{p\,\frac{\tilde{\zeta}_1}{S}}(\upsilon_{\mu G(e)}) \subseteq \tilde{d}^b_{p\,\bar{\zeta}}(\upsilon_{\mu G(e)})$ since $\upsilon_{\mu G(e_k)}$ is an arbitrary element of $\tilde{d}^b_{p\,\bar{\zeta}}(\upsilon_{\mu G(e)})$ then $\tilde{d}^b_{p\,\bar{\zeta}}(\upsilon_{\mu G(e)})$ is F_{ss} $-$open set.

Theorem 3.17: Let$(\mathrm{F}_{ss}(U), \tilde{d}^b_p)$ be $\mathrm{F}_{ss} - bPMS$with coefficient S≥ 1, then $\tilde{\tilde{d}}_b(\upsilon^1{}_{\mu G(e_1)}, \upsilon^2{}_{\mu G(e_2)})$=$2\tilde{d}^b_p(\upsilon^1{}_{\mu G(e_1)}, \upsilon^2{}_{\mu G(e_2)}) - \tilde{d}^b_p(\upsilon^1{}_{\mu G(e_1)}, \upsilon^1{}_{\mu G(e_1)}) -$

$\tilde{d}_p^b(v^2{}_{\mu G(e_2)}, v^2{}_{\mu G(e_2)})$ for each $v^1{}_{\mu G(e_1)}, v^2{}_{\mu G(e_2)}, v^3{}_{\mu G(e_3)}$ in $F_{ss}(U)$ is$F_{ss}-bMS$ on $F_{ss}(U)$ with the same coefficient S.

Proof: For all $v^1{}_{\mu G(e_1)}, v^2{}_{\mu G(e_2)}, v^3{}_{\mu G(e_3)} \in F_{ss}(U)$ and by definition of $\tilde{\tilde{d}}_b$ and $\tilde{d}_p^b$, it is easy to prove that $\tilde{\tilde{d}}$ satisfies (1), (2), and (3) of Definition 2.10

Now, to prove

$\tilde{d}^b(v^1{}_{\mu G(e_1)}, v^3{}_{\mu G(e_3)}) \leq S\,[\tilde{d}^b(v^1{}_{\mu G(e_1)}, v^2{}_{\mu G(e_2)}) + \tilde{d}^b(v^2{}_{\mu G(e_2)}, v^3{}_{\mu G(e_3)})]$. …..(1)

Since $\tilde{\tilde{d}}_b\,(v^1{}_{\mu G(e_1)}, v^2{}_{\mu G(e_2)})$=

$2\tilde{d}_p^b(v^1{}_{\mu G(e_1)}, v^2{}_{\mu G(e_2)}) - \tilde{d}_p^b(v^1{}_{\mu G(e_1)}, v^1{}_{\mu G(e_1)}) - \tilde{d}_p^b(v^2{}_{\mu G(e_2)}, v^2{}_{\mu G(e_2)})$. …..(2)

$\tilde{\tilde{d}}_b\,(v^1{}_{\mu G(e_1)}, v^3{}_{\mu G(e_3)})$=

$2\tilde{d}_p^b(v^1{}_{\mu G(e_1)}, v^3{}_{\mu G(e_3)}) - \tilde{d}_p^b(v^1{}_{\mu G(e_1)}, v^1{}_{\mu G(e_1)}) - \tilde{d}_p^b(v^3{}_{\mu G(e_3)}, v^3{}_{\mu G(e_3)})$. …..(3)

$\tilde{\tilde{d}}_b(v^2{}_{\mu G(e_2)}, v^3{}_{\mu G(e_3)})$=

$2\tilde{d}_p^b(v^2{}_{\mu G(e_2)}, v^3{}_{\mu G(e_3)}) - \tilde{d}_p^b(v^2{}_{\mu G(e_2)}, v^2{}_{\mu G(e_2)}) - \tilde{d}_p^b(v^3{}_{\mu G(e_3)}, v^3{}_{\mu G(e_3)})$. …..(4)

Substituting (3) in the left side of the inequality (1) and (2), (4) in the right side , we get:

$2\tilde{d}_p^b(v^1{}_{\mu G(e_1)}, v^3{}_{\mu G(e_3)}) - \tilde{d}_p^b(v^1{}_{\mu G(e_1)}, v^1{}_{\mu G(e_1)}) - \tilde{d}_p^b(v^3{}_{\mu G(e_3)}, v^3{}_{\mu G(e_3)})$

$\leq S[\,2(\tilde{d}_p^b(v^1{}_{\mu G(e_1)}, v^2{}_{\mu G(e_2)}) + \tilde{d}_p^b(v^2{}_{\mu G(e_2)}, v^3{}_{\mu G(e_3)}) - \tilde{d}_p^b(v^2{}_{\mu G(e_2)}, v^2{}_{\mu G(e_2)}))$

$-\tilde{d}_p^b(v^1{}_{\mu G(e_1)}, v^1{}_{\mu G(e_1)}) - \tilde{d}_p^b(v^3{}_{\mu G(e_3)}, v^3{}_{\mu G(e_3)})\,]$

$\leq S[\,2\tilde{d}_p^b(v^1{}_{\mu G(e_1)}, v^3{}_{\mu G(e_3)}) - \tilde{d}_p^b(v^1{}_{\mu G(e_1)}, v^1{}_{\mu G(e_1)}) - \tilde{d}_p^b(v^3{}_{\mu G(e_3)}, v^3{}_{\mu G(e_3)})]$.

Then,

$\tilde{\tilde{d}}_b(v^1{}_{\mu G(e_1)}, v^3{}_{\mu G(e_3)}) \leq S\,\left[\tilde{\tilde{d}}_b(v^1{}_{\mu G(e_1)}, v^2{}_{\mu G(e_2)}) + \tilde{\tilde{d}}_b(v^2{}_{\mu G(e_2)}, v^3{}_{\mu G(e_3)})\right]$.

Theorem 3.18: Let $(F_{ss}(U), \tilde{d}_p^b)$ be F_{ss} −PbMS and let $\langle v^n{}_{\mu G(e_n)}\rangle$ be F_{ss} − sequence in $F_{ss}(U)$ then $\langle v^n{}_{\mu G(e_n)}\rangle$ is converges to $v_{\mu G(e)}$in $(F_{ss}(U), \tilde{d}^b)$ hat is $\tilde{d}^b(v^n{}_{\mu G(e_n)}, v_{\mu G(e)})$ =0 if and only if $(v^n{}_{\mu G(e_n)}, v_{\mu G(e)})$ =$\tilde{d}_p^b(v^n{}_{\mu G(e_n)}, v^n{}_{\mu G(e_n)})$ =$\tilde{d}_p^b(v_{\mu G(e)}, v_{\mu G(e)})$

Proof: Let$\langle v^n{}_{\mu G(e_n)}\rangle$ converges to $v_{\mu G(e)}$in $(F_{ss}(U), \tilde{d}^b)$ that is mean $\tilde{d}^b(v^n{}_{\mu G(e_n)}, v_{\mu G(e)})$ =0 by Theorem 3.17 $\tilde{d}^b(v^n{}_{\mu G(e_n)}, v_{\mu G(e)})$=

$2\tilde{d}_p^b(v^n{}_{\mu G(e_n)}, v_{\mu G(e)}) - \tilde{d}_p^b(v^n{}_{\mu G(e_n)}, v^n{}_{\mu G(e_n)}) - \tilde{d}_p^b(v_{\mu G(e)}, v_{\mu G(e)})$.

By take the limit as $n \to \infty$ then

$lim\,[_{n\to\infty} 2\tilde{d}_p^b(v^n{}_{\mu G(e_n)}, v_{\mu G(e)}) - \tilde{d}_p^b(v^n{}_{\mu G(e_n)}, v^n{}_{\mu G(e_n)}) - \tilde{d}_p^b(v_{\mu G(e)}, v_{\mu G(e)})] = 0$.

Since $\tilde{d}_p^b(v^n{}_{\mu G(e_n)}, v^n{}_{\mu G(e_n)})$ converges as $n \to \infty$ then $lim_{\,n\to\infty}\, \tilde{d}_p^b(v^n{}_{\mu G(e_n)}, v_{\mu G(e)}) = \tilde{d}_p^b(v_{\mu G(e)}, v_{\mu G(e)})$

Conversely, suppose that $\tilde{d}_p^b(v^n{}_{\mu G(e_n)}, v^n{}_{\mu G(e_n)})$ =$\tilde{d}_p^b(v_{\mu G(e)}, v_{\mu G(e)})$ to prove $\tilde{d}^b(v^n{}_{\mu G(e_n)}, v_{\mu G(e)})$ =0 by Theorem 3.17, $\tilde{d}^b(v^n{}_{\mu G(e_n)}, v_{\mu G(e)})$=

$2\tilde{d}_p^b(v^n{}_{\mu G(e_n)}, v_{\mu G(e)}) - \tilde{d}_p^b(v^n{}_{\mu G(e_n)}, v^n{}_{\mu G(e_n)}) - \tilde{d}_p^b(v_{\mu G(e)}, v_{\mu G(e)})$

Since $\tilde{d}_p^b(v^n{}_{\mu G(e_n)}, v_{\mu G(e)})$ =$\tilde{d}_p^b(v_{\mu G(e)}, v_{\mu G(e)})$ then

$lim\,[2_{n\to\infty}\tilde{d}_p^b(v_{\mu G(e)}, v_{\mu G(e)}) - \tilde{d}_p^b(v^n{}_{\mu G(e_n)}, v^n{}_{\mu G(e_n)}) - \tilde{d}_p^b(v_{\mu G(e)}, v_{\mu G(e)})\,]$

$=lim\,[_{n\to\infty}\tilde{d}_p^b\left(\tilde{d}_p^b(v_{\mu G(e)}, v_{\mu G(e)})\right) - \tilde{d}_p^b(v^n{}_{\mu G(e_n)}, v^n{}_{\mu G(e_n)})]$

then $\tilde{d}_p^b(v^n{}_{\mu G(e_n)}, v^n{}_{\mu G(e_n)})$ =$\tilde{d}_p^b(v_{\mu G(e)}, v_{\mu G(e)})$ and $\tilde{d}^b(v^n{}_{\mu G(e_n)}, v_{\mu G(e)})$ =0.

Theorem 3.19: (1) In $F_{ss}-bPMS$, a sequences $\langle v^n{}_{\mu G(e_n)}\rangle$ is F_{ss} − Cauchy sequences if and only if is F_{ss} − Cauchy sequences in $F_{ss}-bMS$.

(2) $F_{ss}-bMS$ is complete if and only if $F_{ss}-bPMS$ is complete.

Proof: (1) Let $\langle \upsilon^n{}_{\mu G(e_n)}\rangle$ be $F_{ss}-$ Cauchy sequences in$F_{ss}-bPMS$.
So, $\tilde{d}^b\left(\upsilon^n{}_{\mu G(e_n)},\upsilon^m{}_{\mu G(e_m)}\right)=\left(\upsilon_{\mu G(e)},\upsilon_{\mu G(e)}\right)$
Let $\tilde{\epsilon}>0$, then there exists a natural number N_0 such that:
$\left|\tilde{d}_p^b\left(\upsilon^n{}_{\mu G(e_n)},\upsilon^m{}_{\mu G(e_m)}\right)-\left(\upsilon_{\mu G(e)},\upsilon_{\mu G(e)}\right)\right|<\frac{\tilde{\epsilon}}{4}$ for all $n,m\geq N$
$\left|\tilde{d}_p^b\left(\upsilon^n{}_{\mu G(e_n)},\upsilon^m{}_{\mu G(e_m)}\right)\right|=$
$\left|2\tilde{d}_p^b\left(\upsilon^n{}_{\mu G(e_n)},\upsilon^m{}_{\mu G(e_m)}\right)-\tilde{d}_p^b\left(\upsilon^n{}_{\mu G(e_n)},\upsilon^n{}_{\mu G(e_n)}\right)-\tilde{d}_p^b\left(\upsilon^m{}_{\mu G(e_m)},\upsilon^m{}_{\mu G(e_m)}\right)\right|$
$=\left|2\tilde{d}_p^b\left(\upsilon^n{}_{\mu G(e_n)},\upsilon^m{}_{\mu G(e_m)}\right)-2\left(\upsilon_{\mu G(e)},\upsilon_{\mu G(e)}\right)-\tilde{d}_p^b\left(\upsilon^n{}_{\mu G(e_n)},\upsilon^n{}_{\mu G(e_n)}\right)+\right.$
$\left.\left(\upsilon_{\mu G(e)},\upsilon_{\mu G(e)}\right)-\tilde{d}_p^b\left(\upsilon^m{}_{\mu G(e_m)},\upsilon^m{}_{\mu G(e_m)}\right)+\left(\upsilon_{\mu G(e)},\upsilon_{\mu G(e)}\right)\right|$
$\leq\left|\tilde{d}_p^b\left(\upsilon^n{}_{\mu G(e_n)},\upsilon^m{}_{\mu G(e_m)}\right)-\left(\upsilon_{\mu G(e)},\upsilon_{\mu G(e)}\right)\right|$
$+\left|\tilde{d}_p^b\left(\upsilon^n{}_{\mu G(e_n)},\upsilon^m{}_{\mu G(e_m)}\right)-\left(\upsilon_{\mu G(e)},\upsilon_{\mu G(e)}\right)\right|<\tilde{\epsilon}$ for all $n,m\geq N$
so $\langle \upsilon^n{}_{\mu G(e_n)}\rangle$ is $F_{ss}-$ Cauchy sequences in $F_{ss}-bMS$.
Conversely, let $\langle \upsilon^n{}_{\mu G(e_n)}\rangle$ be $F_{ss}-$ Cauchy sequences in $F_{ss}-bMS$. And let $\tilde{\epsilon}=\frac{\tilde{1}}{2}$,
then n_0 in N such that $\tilde{d}^b\left(\upsilon^n{}_{\mu G(e_n)},\upsilon^m{}_{\mu G(e_m)}\right)<\frac{\tilde{1}}{2}$ for all $n,m\geq n_0$, implies $2\tilde{d}_p^b\left(\upsilon^n{}_{\mu G(e_n)},\upsilon^{n_0}{}_{\mu G(e_{n_0})}\right)-\tilde{d}_p^b\left(\upsilon^{n_0}{}_{\mu G(e_{n_0})},\upsilon^{n_0}{}_{\mu G(e_{n_0})}\right)-\tilde{d}_p^b\left(\upsilon^n{}_{\mu G(e_n)},\upsilon^n{}_{\mu G(e_n)}\right)\frac{\tilde{1}}{2}$.
Implies $\tilde{d}_p^b\left(\upsilon^n{}_{\mu G(e_n)},\upsilon^{n_0}{}_{\mu G(e_{n_0})}\right)-\tilde{d}_p^b\left(\upsilon^{n_0}{}_{\mu G(e_{n_0})},\upsilon^{n_0}{}_{\mu G(e_{n_0})}\right)<\frac{\tilde{1}}{2}$
$\tilde{d}_p^b\left(\upsilon^n{}_{\mu G(e_n)},\upsilon^n{}_{\mu G(e_n)}\right)<\tilde{d}_p^b\left(\upsilon^n{}_{\mu G(e_n)},\upsilon^{n_0}{}_{\mu G(e_{n_0})}\right)<\tilde{d}_p^b\left(\upsilon^{n_0}{}_{\mu G(e_{n_0})},\upsilon^{n_0}{}_{\mu G(e_{n_0})}\right)+\frac{\tilde{1}}{2}$
So, $\{\tilde{d}_p^b\left(\upsilon^n{}_{\mu G(e_n)},\upsilon^n{}_{\mu G(e_n)}\right)\}$ is abounded sequences in R(A).
Hence, $\tilde{d}_p^b\left(\upsilon^{n_k}{}_{\mu G(e_{n_k})},\upsilon^{n_k}{}_{\mu G(e_{n_k})}\right)=\left(\upsilon^{n_1}{}_{\mu G(e_{n_1})},\upsilon^{n_1}{}_{\mu G(e_{n_1})}\right)$
Since $\langle \upsilon^n{}_{\mu G(e_n)}\rangle$ is $F_{ss}-$ Cauchy sequences in $F_{ss}-bMS$ for $\epsilon>0$ there exists $n_{\tilde{\epsilon}}$ such that $\tilde{d}_p^b\left(\upsilon^n{}_{\mu G(e_n)},\upsilon^m{}_{\mu G(e_m)}\right)<\epsilon$ for all $n,m\geq n_{\tilde{\epsilon}}$.
Then, $\tilde{d}_p^b\left(\upsilon^n{}_{\mu G(e_n)},\upsilon^n{}_{\mu G(e_n)}\right)-\tilde{d}_p^b\left(\upsilon^m{}_{\mu G(e_m)},\upsilon^m{}_{\mu G(e_m)}\right)$
$\leq\tilde{d}_p^b\left(\upsilon^n{}_{\mu G(e_n)},\upsilon^m{}_{\mu G(e_m)}\right)-\tilde{d}_p^b\left(\upsilon^m{}_{\mu G(e_m)},\upsilon^m{}_{\mu G(e_m)}\right)<\tilde{d}_p^b\left(\upsilon^n{}_{\mu G(e_n)},\upsilon^m{}_{\mu G(e_m)}\right)<\tilde{\epsilon}$.
So, $\{\tilde{d}_p^b\left(\upsilon^n{}_{\mu G(e_n)},\upsilon^n{}_{\mu G(e_n)}\right)\}$ is Cauchy sequences in R(A).
Implies $\tilde{d}_p^b\left(\upsilon^n{}_{\mu G(e_n)},\upsilon^n{}_{\mu G(e_n)}\right)=\left(\upsilon^{n_1}{}_{\mu G(e_{n_1})},\upsilon^{n_1}{}_{\mu G(e_{n_1})}\right)$
Now, $\left|\tilde{d}_p^b\left(\upsilon^n{}_{\mu G(e_n)},\upsilon^m{}_{\mu G(e_m)}\right)-\tilde{d}_p^b\left(\upsilon^{n_1}{}_{\mu G(e_{n_1})},\upsilon^{n_1}{}_{\mu G(e_{n_1})}\right)\right|$
$=\left|\tilde{d}_p^b\left(\upsilon^n{}_{\mu G(e_n)},\upsilon^m{}_{\mu G(e_m)}\right)-\tilde{d}_p^b\left(\upsilon^n{}_{\mu G(e_n)},\upsilon^n{}_{\mu G(e_n)}\right)+\tilde{d}_p^b\left(\upsilon^n{}_{\mu G(e_n)},\upsilon^n{}_{\mu G(e_n)}\right)-\right.$
$\left.\left(\upsilon^{n_1}{}_{\mu G(e_{n_1})},\upsilon^{n_1}{}_{\mu G(e_{n_1})}\right)\right|\leq\tilde{d}_p^b\left(\upsilon^n{}_{\mu G(e_n)},\upsilon^m{}_{\mu G(e_m)}\right)+\left|\tilde{d}_p^b\left(\upsilon^n{}_{\mu G(e_n)},\upsilon^n{}_{\mu G(e_n)}\right)-\right.$
$\left.\left(\upsilon^{n_1}{}_{\mu G(e_{n_1})},\upsilon^{n_1}{}_{\mu G(e_{n_1})}\right)\right|$ implies $\tilde{d}_p^b\left(\upsilon^n{}_{\mu G(e_n)},\upsilon^m{}_{\mu G(e_m)}\right)=\left(\upsilon^{n_1}{}_{\mu G(e_{n_1})},\upsilon^{n_1}{}_{\mu G(e_{n_1})}\right)$.
Therefore, $\langle \upsilon^n{}_{\mu G(e_n)}\rangle$ is $F_{ss}-$ Cauchy sequences in $F_{ss}-bPMS$.
(2) Let $\langle \upsilon^n{}_{\mu G(e_n)}\rangle$ be any $F_{ss}-$ Cauchy sequences in $F_{ss}-bPMS$,
then $\langle \upsilon^n{}_{\mu G(e_n)}\rangle$ is any $F_{ss}-$ Cauchy sequences in $F_{ss}-bMS$.
Since $F_{ss}-bMS$ is complete there exists a $F_{ss}-$point$\upsilon_{\mu G(e)}$ in $F_{ss}(U)$ such that and by Theorem 3.18, $F_{ss}-bPMS$ is complete.
Conversely , let $F_{ss}-bPMS$ is complete and let $\langle \upsilon^n{}_{\mu G(e_n)}\rangle$ be any $F_{ss}-$ Cauchy sequences in $F_{ss}-bMS$ then $\langle \upsilon^n{}_{\mu G(e_n)}\rangle$ is $F_{ss}-$ Cauchy sequences in $F_{ss}-bPMS$, since $F_{ss}-bPMS$ is complete, there exists a $F_{ss}-$point$\upsilon^1{}_{\mu G(e_1)}$ in $F_{ss}(U)$ such that
$\tilde{d}_p^b\left(\upsilon^n{}_{\mu G(e_n)},\upsilon^1{}_{\mu G(e_1)}\right)=\tilde{d}_p^b\left(\upsilon^1{}_{\mu G(e_1)},\upsilon^1{}_{\mu G(e_1)}\right)=\tilde{d}_p^b\left(\upsilon^n{}_{\mu G(e_n)},\upsilon^n{}_{\mu G(e_n)}\right)$
Now, by Theorem 3.18, $\tilde{d}_p^b\left(\upsilon^n{}_{\mu G(e_n)},\upsilon^1{}_{\mu G(e_1)}\right)=0$ hence $F_{ss}-bMS$ is complete.

Theorem 3.20: Let $\left(\mathrm{F}_{ss}(U), \tilde{d}_p^b\right)$ be a $\mathrm{F}_{ss} - bPMS$ and A be any subset of $\mathrm{F}_{ss}(U)$ then $\tilde{d}_p^b(\bar{A}) \le S\tilde{d}_p^b(A)$ where $\tilde{d}_p^b(A) = sup\,\{\tilde{d}_p^b\left(v^1{}_{\mu G(e_1)}, v^2{}_{\mu G(e_2)}\right) - \tilde{d}_p^b\left(v^1{}_{\mu G(e_1)}, v^1{}_{\mu G(e_1)}\right)$ $\forall v^1{}_{\mu G(e_1)}, v^2{}_{\mu G(e_2)} \in A$.

Proof: Let $v^1{}_{\mu G(e_1)}, v^2{}_{\mu G(e_2)} \in \bar{A}$ then there exists $\{v^n{}_{\mu G(e_n)}\}, \{v^m{}_{\mu G(e_m)}\}$ in A

such that $\{v^n{}_{\mu G(e_n)}\} \to v^1{}_{\mu G(e_1)}$ and $\{v^m{}_{\mu G(e_m)}\} \to v^2{}_{\mu G(e_2)}$.
That is mean,

$$\tilde{d}_p^b\left(v^n{}_{\mu G(e_n)}, v^1{}_{\mu G(e_1)}\right) = \tilde{d}_p^b\left(v^1{}_{\mu G(e_1)}, v^1{}_{\mu G(e_1)}\right) = \tilde{d}_p^b\left(v^n{}_{\mu G(e_n)}, v^n{}_{\mu G(e_n)}\right).$$

And $\tilde{d}_p^b\left(v^m{}_{\mu G(e_m)}, v^2{}_{\mu G(e_2)}\right) = \tilde{d}_p^b\left(v^2{}_{\mu G(e_2)}, v^2{}_{\mu G(e_2)}\right)$
$= \tilde{d}_p^b\left(v^m{}_{\mu G(e_m)}, v^m{}_{\mu G(e_m)}\right)$.

Now, by triangular property of $\mathrm{F}_{ss} - bPMS$ then:
$\tilde{d}_p^b\left(v^1{}_{\mu G(e_1)}, v^2{}_{\mu G(e_2)}\right) \le \mathrm{S}[\tilde{d}_p^b\left(v^1{}_{\mu G(e_1)}, v^n{}_{\mu G(e_n)}\right) + \tilde{d}_p^b\left(v^n{}_{\mu G(e_n)}, v^2{}_{\mu G(e_2)}\right) -$
$\tilde{d}_p^b\left(v^n{}_{\mu G(e_n)}, v^n{}_{\mu G(e_n)}\right)]$. …….(5)
And re-applying this feature again then:
$\tilde{d}_p^b\left(v^n{}_{\mu G(e_n)}, v^2{}_{\mu G(e_2)}\right) \le S[\tilde{d}_p^b\left(v^n{}_{\mu G(e_n)}, v^m{}_{\mu G(e_m)}\right) + \tilde{d}_p^b\left(v^m{}_{\mu G(e_m)}, v^2{}_{\mu G(e_2)}\right) -$
$\tilde{d}_p^b\left(v^m{}_{\mu G(e_m)}, v^m{}_{\mu G(e_m)}\right)]$. …..(6)
By compensation (6) in (5) then:
$\tilde{d}_p^b\left(v^1{}_{\mu G(e_1)}, v^2{}_{\mu G(e_2)}\right) \le \mathrm{S}[\tilde{d}_p^b\left(v^1{}_{\mu G(e_1)}, v^n{}_{\mu G(e_n)}\right) + S\left(\tilde{d}_p^b\left(v^n{}_{\mu G(e_n)}, v^m{}_{\mu G(e_m)}\right) +\right.$
$\left.\tilde{d}_p^b\left(v^m{}_{\mu G(e_m)}, v^2{}_{\mu G(e_2)}\right) - \tilde{d}_p^b\left(v^m{}_{\mu G(e_m)}, v^m{}_{\mu G(e_m)}\right)\right) - \tilde{d}_p^b\left(v^n{}_{\mu G(e_n)}, v^n{}_{\mu G(e_n)}\right)]$.
Taking limit $m \to \infty, n \to \infty$ in the inequality above then:
$\tilde{d}_p^b\left(v^1{}_{\mu G(e_1)}, v^2{}_{\mu G(e_2)}\right) \le \mathrm{S}[\tilde{d}_p^b\left(v^1{}_{\mu G(e_1)}, v^1{}_{\mu G(e_1)}\right) + S\left(\tilde{d}_p^b\left(v^1{}_{\mu G(e_1)}, v^2{}_{\mu G(e_2)}\right) +\right.$
$\left.\tilde{d}_p^b\left(v^2{}_{\mu G(e_2)}, v^2{}_{\mu G(e_2)}\right) - \tilde{d}_p^b\left(v^2{}_{\mu G(e_2)}, v^2{}_{\mu G(e_2)}\right)\right) - \tilde{d}_p^b\left(v^1{}_{\mu G(e_1)}, v^1{}_{\mu G(e_1)}\right)]$
$\tilde{d}_p^b\left(v^1{}_{\mu G(e_1)}, v^2{}_{\mu G(e_2)}\right) \le S\left[S\tilde{d}_p^b\left(v^1{}_{\mu G(e_1)}, v^2{}_{\mu G(e_2)}\right)\right]$.
Adding and subtracting $\tilde{d}_p^b\left(v^1{}_{\mu G(e_1)}, v^1{}_{\mu G(e_1)}\right)$ to inequality above then:
$\tilde{d}_p^b\left(v^1{}_{\mu G(e_1)}, v^2{}_{\mu G(e_2)}\right) - \tilde{d}_p^b\left(v^1{}_{\mu G(e_1)}, v^1{}_{\mu G(e_1)}\right) \le S\left[S\tilde{d}_p^b\left(v^1{}_{\mu G(e_1)}, v^2{}_{\mu G(e_2)}\right)\right]$
$-\tilde{d}_p^b\left(v^1{}_{\mu G(e_1)}, v^1{}_{\mu G(e_1)}\right)$.
Implies $\sup\{\tilde{d}_p^b\left(v^1{}_{\mu G(e_1)}, v^2{}_{\mu G(e_2)}\right) - \tilde{d}_p^b\left(v^1{}_{\mu G(e_1)}, v^1{}_{\mu G(e_1)}\right) \forall v^1{}_{\mu G(e_1)}, v^2{}_{\mu G(e_2)} \in \bar{A}\}$
$\le \sup\{S^2\tilde{d}_p^b\left(v^1{}_{\mu G(e_1)}, v^2{}_{\mu G(e_2)}\right) - \tilde{d}_p^b\left(v^1{}_{\mu G(e_1)}, v^1{}_{\mu G(e_1)}\right)\}$
$\le S\tilde{d}_p^b(A)$. Implies $\tilde{d}_p^b(\bar{A}) \le S\tilde{d}_p^b(A)$.

Theorem 3.21: A $\mathrm{F}_{ss} - bPMS$ is complete if and only if every
F_{ss} − sequences $\langle C^n{}_{\mu G(e_n)}\rangle$ of F_{ss} − closed sets in$\mathrm{F}_{ss} - bPMS$ satisfying:
(1) $C^{n+1}{}_{\mu G(e_{n+1})} \subseteq C^n{}_{\mu G(e_n)}$ $\forall n \in N$ and;
(2) $\tilde{d}_p^b(C^n{}_{\mu G(e_n)}) \to \tilde{0}$ as $n \to \infty$ then $\bigcap_{n=1}^{\infty} C^n{}_{\mu G(e_n)}$ is singleton.
Proof: Let $\{v^n{}_{\mu G(e_n)}\}$ be a F_{ss} −cauchy sequences in $\mathrm{F}_{ss} - bPMS$ then $\tilde{d}_p^b\left(v^{n+p}{}_{\mu G(e_1)}, v^n{}_{\mu G(e_n)}\right) = v^*{}_{\mu G(e_*)}$.
For arbitrary $\tilde{\epsilon} > 0$ $\exists$ natural number κ such that
$\left|\tilde{d}_p^b(v^{n+p}{}_{\mu G(e_{n+p})}, v^n{}_{\mu G(e_n)}) - v^*{}_{\mu G(e_*)}\right| < \frac{\tilde{\epsilon}}{2}\, \forall n \ge \kappa$.
Let $C^n{}_{\mu G(e_n)} = \{v^{n+p-1}{}_{\mu G(e_{n+p-1})} : p \in N\}$ then $C^{n+1}{}_{\mu G(e_{n+1})} \tilde{\subseteq} C^n{}_{\mu G(e_n)}$

Implies $C^{n+1}{}_{\mu G(e_{n+1})} \tilde{\subseteq} C^{n}{}_{\mu G(e_n)}$.
Now, $\left|\tilde{d}_p^b(v^{n+p}{}_{\mu G(e_{n+p})}, v^{n}{}_{\mu G(e_n)}) - \tilde{d}_p^b(v^{n}{}_{\mu G(e_n)}, v^{n}{}_{\mu G(e_n)})\right| \leq$
$\left|\tilde{d}_p^b(v^{n+p}{}_{\mu G(e_{n+p})}, v^{n}{}_{\mu G(e_n)}) - v^{*}{}_{\mu G(e_*)}\right| + \left|\tilde{d}_p^b(v^{n}{}_{\mu G(e_n)}, v^{n}{}_{\mu G(e_n)}) - v^{*}{}_{\mu G(e_*)}\right| < \frac{\tilde{\epsilon}}{2} + \frac{\tilde{\epsilon}}{2}$
$=\tilde{\epsilon}\ \forall n \geq \kappa$. Also,
$\left|\tilde{d}_p^b(v^{n+p}{}_{\mu G(e_{n+p})}, v^{n}{}_{\mu G(e_n)}) - \tilde{d}_p^b(v^{n+p}{}_{\mu G(e_{n+p})}, v^{n+p}{}_{\mu G(e_{n+p})})\right| \leq$
$\left|\tilde{d}_p^b(v^{n+p}{}_{\mu G(e_{n+p})}, v^{n}{}_{\mu G(e_n)}) - v^{*}{}_{\mu G(e_*)}\right| + \left|\tilde{d}_p^b(v^{n+p}{}_{\mu G(e_{n+p})}, v^{n+p}{}_{\mu G(e_{n+p})}) - v^{*}{}_{\mu G(e_*)}\right| < \frac{\tilde{\epsilon}}{2} + \frac{\tilde{\epsilon}}{2} = \tilde{\epsilon}\ \forall n \geq \kappa$. Implies
$\left[\tilde{d}_p^b\left(v^{n+p}{}_{\mu G(e_{n+p})}, v^{n}{}_{\mu G(e_n)}\right) - \left\{\tilde{d}_p^b\left(v^{n}{}_{\mu G(e_n)}, v^{n}{}_{\mu G(e_n)}\right), \tilde{d}_p^b\left(v^{n+p}{}_{\mu G(e_{n+p})}, v^{n+p}{}_{\mu G(e_{n+p})}\right)\right\} : \forall v^{n+p}{}_{\mu G(e_{n+p})}, v^{n}{}_{\mu G(e_n)} \in C^{n}{}_{\mu G(e_n)}\right] = \tilde{0}.$

Implies $\tilde{d}_p^b(C^{n}{}_{\mu G(e_n)}) \to \tilde{0}$ as $n \to \infty$
And by using Lemma 3.3, then $\tilde{d}_p^b(\overline{C^{n}}{}_{\mu G(e_n)}) \to \tilde{0}$ as $n \to \infty$.
So, $\bigcap_{n=1}^{\infty} \overline{C^{n}}{}_{\mu G(e_n)} \neq \tilde{\emptyset}$.
Let $v^{1}{}_{\mu G(e_1)} \in \bigcap_{n=1}^{\infty} \overline{C^{n}}{}_{\mu G(e_n)} \to v^{1}{}_{\mu G(e_1)} \in \overline{C^{n}}{}_{\mu G(e_n)} \forall n \in N$
also $v^{n}{}_{\mu G(e_n)} \in C^{n}{}_{\mu G(e_n)} \subset \overline{C^{n}}{}_{\mu G(e_n)}$,
then $\tilde{0} \leq \tilde{d}_p^b\left(v^{n}{}_{\mu G(e_n)}, v^{n}{}_{\mu G(e_n)}\right) - \tilde{d}_p^b\left(v^{1}{}_{\mu G(e_1)}, v^{1}{}_{\mu G(e_1)}\right) \leq \tilde{d}_p^b(\overline{C^{n}}{}_{\mu G(e_n)})$
taking the limit and using the Sandwich Theorem then
$\lim_{n\to\infty} \tilde{d}_p^b\left(v^{n}{}_{\mu G(e_n)}, v^{1}{}_{\mu G(e_1)}\right) = \tilde{d}_p^b\left(v^{1}{}_{\mu G(e_1)}, v^{1}{}_{\mu G(e_1)}\right)$. ….(7)
By the same way that have $\tilde{0} \leq \tilde{d}_p^b\left(v^{1}{}_{\mu G(e_1)}, v^{n}{}_{\mu G(e_n)}\right) - \tilde{d}_p^b\left(v^{n}{}_{\mu G(e_n)}, v^{n}{}_{\mu G(e_n)}\right) \leq \tilde{d}_p^b(\overline{C^{n}}{}_{\mu G(e_n)})$ and also using the sandwich theorem then
$\lim_{n\to\infty} \tilde{d}_p^b\left(v^{n}{}_{\mu G(e_n)}, v^{n}{}_{\mu G(e_n)}\right) = \tilde{d}_p^b\left(v^{1}{}_{\mu G(e_1)}, v^{1}{}_{\mu G(e_1)}\right)$. ….(8)
By (7) and (8) get on
$\lim_{n\to\infty} \tilde{d}_p^b\left(v^{n}{}_{\mu G(e_n)}, v^{1}{}_{\mu G(e_1)}\right) = \tilde{d}_p^b\left(v^{1}{}_{\mu G(e_1)}, v^{1}{}_{\mu G(e_1)}\right) = \lim_{n\to\infty} \tilde{d}_p^b\left(v^{n}{}_{\mu G(e_n)}, v^{n}{}_{\mu G(e_n)}\right)$
Hence, $\mathrm{F}_{ss} - bPMS$ be complete and the condition (1) and (2) are satisfying.
Now, to prove $\bigcap_{n=1}^{\infty} C^{n}{}_{\mu G(e_n)}$ is singleton consider
$C^{n}{}_{\mu G(e_n)} \in C^{n}{}_{\mu G(e_n)}$ for all $n \in N$. Since $C^{n+1}{}_{\mu G(e_{n+1})} \tilde{\subseteq} C^{n}{}_{\mu G(e_n)}$.
Implies $v^{m}{}_{\mu G(e_m)} \in C^{n}{}_{\mu G(e_n)}$ for all $m \geq n$ that implies
$\tilde{0} \leq \tilde{d}_p^b\left(v^{n}{}_{\mu G(e_n)}, v^{m}{}_{\mu G(e_m)}\right) - \tilde{d}_p^b\left(v^{n}{}_{\mu G(e_n)}, v^{n}{}_{\mu G(e_n)}\right) \leq \tilde{d}_p^b(C^{n}{}_{\mu G(e_n)})$ that implies
$\tilde{0} \leq \tilde{d}_p^b\left(v^{m}{}_{\mu G(e_m)}, v^{n}{}_{\mu G(e_n)}\right) - \tilde{d}_p^b\left(v^{m}{}_{\mu G(e_m)}, v^{m}{}_{\mu G(e_m)}\right) \leq \tilde{d}_p^b(C^{n}{}_{\mu G(e_n)})$ which gives
$\tilde{0} \leq 2\tilde{d}_p^b\left(v^{n}{}_{\mu G(e_n)}, v^{m}{}_{\mu G(e_m)}\right) - \tilde{d}_p^b\left(v^{n}{}_{\mu G(e_n)}, v^{n}{}_{\mu G(e_n)}\right) - \tilde{d}_p^b\left(v^{m}{}_{\mu G(e_m)}, v^{m}{}_{\mu G(e_m)}\right) \leq 2\tilde{d}_p^b(C^{n}{}_{\mu G(e_n)})$ implies $\tilde{0} \leq 2\tilde{d}_p^b\left(v^{n}{}_{\mu G(e_n)}, v^{m}{}_{\mu G(e_m)}\right) \leq 2\tilde{d}_p^b\left(C^{n}{}_{\mu G(e_n)}\right)$.
Using (2) and by using Sandwich Theorem obtaining $v^{n}{}_{\mu G(e_n)}$ be a F_{ss} −Cauchy sequences in $\mathrm{F}_{ss} - bPMS$ and since $\mathrm{F}_{ss} - bPMS$ is complete by Theorem 3.20 then $\mathrm{F}_{ss} - bMS$ is complete.
Hence, $\exists v^{1}{}_{\mu G(e_1)} \in \mathrm{F}_{ss}(U)$ such that $\tilde{d}^b\left(v^{n}{}_{\mu G(e_n)}, v^{1}{}_{\mu G(e_1)}\right) \to \tilde{0}$ as $n \to \infty$ which implies $v^{n}{}_{\mu G(e_n)}$ converges to $v^{1}{}_{\mu G(e_1)}$ in $\mathrm{F}_{ss} - bPMS$. This gives $v^{1}{}_{\mu G(e_1)} \in C^{n}{}_{\mu G(e_n)}$ is F_{ss} − closed in $\left(\mathrm{F}_{ss}(U), \tilde{d}_p^b\right)$, thus $v^{1}{}_{\mu G(e_1)} \in C^{n}{}_{\mu G(e_n)} \forall nN$.
Let $v^{2}{}_{\mu G(e_2)} \in \bigcap_{n=1}^{\infty} C^{n}{}_{\mu G(e_n)} \to v^{2}{}_{\mu G(e_2)} \in C^{n}{}_{\mu G(e_n)}$ for all $n \in N$,
$\tilde{0} \leq \tilde{d}_p^b\left(v^{1}{}_{\mu G(e_1)}, v^{2}{}_{\mu G(e_2)}\right) - \tilde{d}_p^b\left(v^{1}{}_{\mu G(e_1)}, v^{1}{}_{\mu G(e_1)}\right) \leq \tilde{d}_p^b\left(C^{n}{}_{\mu G(e_n)}\right)$.
Taking the limit and using the Sandwich Theorem we get
$\tilde{d}_p^b\left(v^{1}{}_{\mu G(e_1)}, v^{2}{}_{\mu G(e_2)}\right) = \tilde{d}_p^b\left(v^{1}{}_{\mu G(e_1)}, v^{1}{}_{\mu G(e_1)}\right)$.
Similarly, get on $\tilde{d}_p^b\left(v^{1}{}_{\mu G(e_1)}, v^{2}{}_{\mu G(e_2)}\right) = \tilde{d}_p^b\left(v^{2}{}_{\mu G(e_2)}, v^{2}{}_{\mu G(e_2)}\right)$.

Therefore, $\tilde{d}_p^b\left(v^1{}_{\mu G(e_1)}, v^2{}_{\mu G(e_2)}\right) = \tilde{d}_p^b\left(v^1{}_{\mu G(e_1)}, v^1{}_{\mu G(e_1)}\right) = \tilde{d}_p^b\left(v^2{}_{\mu G(e_2)}, v^2{}_{\mu G(e_2)}\right)$ this implies $v^1{}_{\mu G(e_1)} = v^2{}_{\mu G(e_2)}$ and $\bigcap_{n=1}^{\infty} C^n{}_{\mu G(e_n)}$ is singleton then the proof is complete.

4. Conclusions

In this research, the newly posed space combines the FSs and partial metric space principles, namely the fuzzy soft b-partial metric space. Then, several analytic merits are successfully investigated, such as the F_{ss}-closed, F_{ss}-open ball, F_{ss}-Harsdorff space, and F_{ss}-converge sequence. Finally, the relevance between this newly designated space and b-metric spaces is also indicated. The appropriate examples are illustrated to characterize the various merits of posed space.

Acknowledgement

This work was supported by Mustansiriyah University Thanks for the help provided to successfully complete this research.

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