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Strongly Primary Submodules

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Abstract

Let M be an R-module, where R is an integral domain with quotient field K .In this paper we introduce the concept of strongly primary submodule, where a submodule N of M is called strongly primary , if whenever $r \in K$, $x \in M$, $rx \in N$ implies $x \in N$ or $r^n \in (N:M)$ for some $n \in Z_+$. We study this concept and give some of its properties.

المستخلص

ليكن Mمقاسا على Rحيث ان Rساحة وحقل القسمة لها K. في هذا البحث نقدم مفهــوم الموديــول الجزئي الابتدائي بقوة، حيث انه اذا كان Nمقاسا جزئيا من M، فنقول ان N مقاسا جزئيا ابتدائيا بقــوة، اذا كان لكل، r ∋ K ، x ∋ M ، rx ∋ N يؤدي الى ان x ∋ N أو r^n ∋ (N:M) لبعض n ∋ Z_+ . لقــد درسنا هذا المفهوم و اعطينا بعض الخواص الاساسية له.

Introduction

Let R be an integral domain with quotient field K .Houston in [3] defined and studied strongly prime ideals, where an ideal P of a ring R is called strongly prime, (briefly s- prime) ideal if whenever $a,b \in K$, $a.b \in P$ implies either $a \in P$ or $b \in P$. In [5] the authors generalized this concept to submodules, where the concept of strongly prime submodules is introduced as follows : A submodule P of the R- module M is called strongly prime, (briefly s- prime) submodule, if whenever $r \in K$, $x \in M$, $rx \in P$ implies $x \in P$ or $r \in (P:M)$.

In this paper we introduce the concepts of strongly primary submodules and strongly primary ideals.

We study the basic properties of these concepts, and we study strongly primary submodules in multiplication modules. Moreover we study the relations between strongly primary submodules (ideals) and strongly prime submodules (ideals).

Finally, we remark that R in this paper stands for an integral domain with quotient field K, and M stands for a (left) unitary R-module.

S.1 strongly primary submodules

Basic Results

Recall that a proper submodule N of an R-module M is called prime, if whenever $r \in R, x \in M$, $rx \in N$ implies $x \in N$ or $r \in (N:M)$, [6], and N is called primary, if whenever $r \in R$, $x \in M$, $rx \in N$ implies $x \in N$ or $r^n \in (N:M)$ for some $n \in Z_+$, [4].

As an extension of primary submodules, we proceed as following:

Let R be an integral domain with quotient field K and let M be an R-module. Let N be an R-submodule of M. For each $r = \frac{a}{b} \in K \setminus R$, $x \in M$. We say that $rx \in N$, if there exists $y \in N$ such that $ax = by$, [1]. In this case we write $y = \frac{a}{b}x$.

Note that, if N is torsion free, then y is unique.

We define the following:

Definition 1.1

Let N be a proper submodule of an R-module M. N is called a strongly primary (briefly s-primary) submodule, if whenever $r \in K, x \in M$, $rx \in N$ implies $x \in N$ or $r^n \in (N:M)$, for some $n \in Z_+$, where $(N:M) = \{r : r \in R \text{ and } rM \subseteq N\}$.

An ideal I of a ring R is called a strongly primary, (briefly s- primary) ideal if I is an s-primary R- submodule of R, that is whenever $a,b\in K$, $ab\in I$ implies $a\in I$ or $b^n \in I$ for some $n \in Z_+$.

Remark 1.2. It is clear that every s-primary submodule is primary submodule, however the converse is not true as the following examples show:

Examples

(1) consider the ring Z as a Z-module. Any primary submodule of Z has the form p^nZ non zero for some prime number p, and $m\in Z_+$. However $\frac{p^m}{p+1}\bullet(p+1) = p^m \in p^m Z$, $\frac{p^m}{p+1} \notin p^m Z$ and $(p+1)^n \notin p^mZ$ for each $n\in Z_+$. Thus p^mZ is not an s- primary submodule.

(2) If M is the Z-module $Z \oplus Z$, $N= 8Z \oplus 8Z$. It is easy to see that N is primary submodule of M, but N is not s- primary.

The following gives a characterization of s- primary submodules:

Proposition 1.3.

Let M be an R-module and N a submodule of N. If N is s- primary, then for each $r\in K$, either $r^{-1}N \subseteq N$ or $r^n \in (N:M)$ for some $n\in Z_+$. The converse is true, if N is primary.

Proof. If N is an s-primary submodule. Suppose $r^{-1}N \not\subset N$ for some $r\in K$. So $\exists\, y \in r^{-1}N$ and $y \notin N$. Hence $y = r^{-1}x$ for some $x\in N$ and so $x = ry \in N$. But N is s- primary and $y\notin N$, hence $\exists\, n \in Z_+$ such that $r^n \in (N:M)$.

Conversely, to prove N is s-primary. Let $r\in K$, $x\in M$ such that $rx\in N$. If $r\in R$, then there is nothing to prove since N is primary. If $r\notin R$, then $x= r^{-1}y$ for some $y\in N$. Hence either $x\in N$ or $r^n \in (N:M)$ for some $n\in Z_+$. ♦

The following is characterization of s-primary ideals:

Proposition 1.4.

Let R be any ring and I an ideal of R. Consider the following statements:

(i) I is an s- primary ideal of R.

(ii) $\forall r\in K$,either $r^{-1}I\subseteq I$ or $r^n \in I$ for some $n\in Z_+$.

(iii) $\forall\, r\in K\backslash R$, $r^{-1}I\subseteq \sqrt{I}$.

Then (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii). Moreover, if I is primary, then (iii) $\Rightarrow$ (i).

Proof. (i) $\Rightarrow$ (ii)

The proof follows from proposition 1.3.

(ii) $\Rightarrow$ (iii):

Let $r\in K\backslash R$. By (ii), either $r^{-1}I\subseteq I$ or $r^n \in I$ for some $n\in Z_+$.

If $r^n \in I$, then $y\in \sqrt{I}$ and so, $y\in R$, which is a contradiction. Thus $r^{-1}I\subseteq I$, and hence, $r^{-1}I\subseteq \sqrt{I}$

(iii) $\Rightarrow$ (i) :

Let $a,b\in K$ such that $ab\in I$. If $a,b\in R$, then there is nothing to prove, since I is a primary ideal. If $a\notin R$, then $a\in K\backslash R$, and $b = a^{-1}(ab)\in a^{-1}I$. Hence $b\in\sqrt{I}$ by (3); that is $b^n\in I$ for some $n\in Z_+$. Thus I is s- primary.♦

Corollary1.5. Let M be an R-module, and I be an s-primary ideal of R. If IM is a primary submodule of M, then IM is an s-primary submodule of M.

Proof. Let $r\in K$. Then either $r^{-1}I \subseteq I$ or $r^n \in I$ for some $n\in Z_+$. By proposition 1.4. Hence either $(r^{-1}I)\, M \subseteq IM$ or $r^nM\subseteq IM$ for some $n\in Z_+$. It follows that either $r^{-1}\,(IM) \subseteq IM$ or $r^n \in (IM.M)$ for some $n\in Z_+$ Thus IM is an s-primary submodule of M. ♦

Proposition 1.6. If N is an s-primary submodule of an R-module M, then (N:M) is an s-primary ideal of R.

Proof. It is easy, so it is omitted.

Remark 1.7 The converse of proposition 1.6. is not true as the following example shows.

Example. Let M be the Z-module $Z \oplus Z$, and let $N = (0) \oplus 8Z$. N is not primary submodule of M, since $4\,(0,2)\in N$, $(0,2)\notin N$, and $4^n \notin (N:M)= (0)$, $\forall\, n\in Z_+$. Hence N is not s-primary. But (N:M) =(0) is an s- primary ideal of Z.

Recall that an R-module M is called a multiplication module, if for each submodule N of M, there exists an ideal I of R such that N=IM, [2].

Now we give a condition under which, the converse of proposition 1.6. is true.

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Theorem 1.8. Let M be a multiplication R-module, and N be a primary submodule of M. If (N:M) is an s-primary ideal of R, then N is an s-primary submodule.

Proof. Let $r\in K$, and let $I = (N:M)$. Hence $N = IM$. Since I is an s-primary ideal, and N is a primary submodule, then N is an s-primary submodule, by corollary 1.5. ♦

By combining proposition 1.6 and theorem 1.8., we get the following:

Corollary 1.9. Let M be amultiplication R-module and let N be a primary submodule of M. Then N is an s-primary submodule iff (N:M) is an s-primary ideal of R. Next we have:

Proposition 1.10. Let N be a submodule of an R-module M. Then N is an s-primary R-submodule iff N is an s-primary $\overline{R}$-submodule, where $\overline{R} = R/ann_R M$, provided $ann_R M$ is a prime ideal of R.

Proof. Let N be an s- primary R-submodule of M. Let $\bar{r}\in\overline{K}$ (quotient field of $\overline{R}$), $x\in M$ and $\bar{r}x\in N$. Assume $x\notin N$, $\bar{r} = \frac{\bar{a}}{\bar{b}}$, where $\bar{a} = a + ann_R$ M, $\bar{b} = b + ann_R M$, $\bar{b} \neq ann_R M$, for some $a,b \in R$.

Now $\frac{\bar{a}}{\bar{b}}x\in N$, implies $\exists$ $y \in N$ such that $\bar{a}\,\bar{x} = \bar{b}y$. Hence $ax = by$; that is $\frac{a}{b}x\in N$. But N is an s-primary, R-submodule, and $\frac{a}{b}\in K$ (quotient field of R), hence $\exists n \in Z_+$, such that $\left(\frac{a}{b}\right)^n \in (N:M)$.

Thus $\left(\frac{a}{b}\right)^n m\in N$ for any $m\in M$. It follows that $a^n m = b^n c$, for some $c\in N$, which implies $\bar{a}^n m = \overline{b}^{n} c$.

Thus $\left(\frac{\bar{a}}{\bar{b}}\right)^n \in (N:M)$, and so N is an s-primary $\overline{R}$-submodule.

The proof of the converse is similar.

S.2. S – primary submodules (Ideals) and S- Prime Submodules (Ideals)

In this section, we give some relations between s-primary submodules (ideals) and s-prime submodules (ideals).

First, we have;

Remark 2.1.

Every s-prime submodule (ideal) is an s-primary submodule (ideal).

Proof. It follows directly, so it is omitted.

Note: We believe that an s-primary submodule need not be an s-prime submodule, however, we can't find an example to explain this.

It is known that if I is primary ideal of a ring R, then $\sqrt{I}$ is a prime ideal of R.

The following is a generalization of this result:

Theorem 2.2. Let I be an s-primary ideal of R. Then $\sqrt{I}$ is an s-prime ideal of R.

Proof. Since I is an s- primary ideal, I is a primary ideal by remark 1.2., and hence $\sqrt{I}$ is an s-prime ideal of R. To prove $\sqrt{I}$ is a prime ideal of R, it is enough to show that $r^{-1}I\subseteq \sqrt{I}$ for each $r\in K\backslash R$, by [3, Prop. 1.2], so let $y\in r^{-1}\sqrt{I}$, $r\in K\backslash R$. Then $y = r^{-1}x$, for some $x\in\sqrt{I}$. But $x \in \sqrt{I}$ implies $\exists$ $n \in Z_+$, such that $x^n \in I$. It follows that $y^n = r^{-n}x^n = (r^n)^{-1} x^n$. If $r^n \in K\backslash R$, then $y^n = r^{-1} x^n$, where $r_1 = r^n$. Hence $y^n \in r_1^{-1}I$. But $r_1^{-1}I \subseteq \sqrt{I}$ by proposition 1.6. So that $y^n \in \sqrt{I}$. Thus $y \in \sqrt{\sqrt{I}}$. If $r^n \in R$, then $x^n = r^n y^n \in I \subseteq \sqrt{I}$. But $\sqrt{I}$ is a prime ideal of R, so either $r^n \in\sqrt{I}$ or $y^n \in \sqrt{I}$. However, $r^n \in \sqrt{I}$ implies $r\in R$, which is a contradiction. Thus $y^n \in \sqrt{I}$, which implies $y\in\sqrt{I}$. Therefore $r^{-1}\sqrt{I} \subseteq\sqrt{I}$ and $\sqrt{I}$ is an s-prime ideal.

Corollary 2.3. Let I, J be two s- primary ideals of a ring R then either $\sqrt{I} \subseteq\sqrt{J}$ or $\sqrt{J} \subseteq\sqrt{I}$.

Proof. Since I, J are s-primary ideals of R, $\sqrt{I}$ and $\sqrt{J}$ are s-prime ideals of R, by Theorem 2.2, hence by [3, Cor. 1.3] ,either $\sqrt{I} \subseteq\sqrt{J}$ or $\sqrt{J} \subseteq\sqrt{I}$.

Corollary 2.4. Let M be a multiplication faithful R-module. If N and W are s-primary submodules, then either rad N $\subseteq$ rad W or rad W $\subseteq$ rad N.

Proof. By proposition 1.6, (N:M) and (W:M) are s-primary ideals of R. Hence by Cor. 2.3 either $\sqrt{(N:M)} \subseteq \sqrt{(W:M)}$ or $\sqrt{(W:M)} \subseteq \sqrt{(N:M)}$. It follows that either rad N = √(N:M) M ⊆ √(W:M) M = rad W or rad W = √(W:M) M ⊆ √(N:M) M = rad N.

Proposition 2.5.. Let M be a multiplication R-module and N is an s-primary submodule of M such that N ⊇ ann_R M or M is faithful. Then rad N is an s-prime submodule.

Proof. Since N is an s-primary submodule, then (N:M) is an s-primary ideal of R by proposition 1.6. Hence √(N:M) is an s-prime ideal of R by theorem 2.2. It follows that √(N:M) M is an s-prime submodule of M by [5, Pro. 1.14(1)]. Thus rad N is an s-prime submodule of M, Since rad N = √(N:M) M. Next we have:

Proposition 2.6.. Let M be a multiplication R-module, if every primary ideal of R is an s-primary ideal of R, then every primary submodule of M is an s-primary submodule of M.

Proof. Let N be a primary submodule of M. Then (N:M) is a primary ideal of R. Hence (N:M) is an s-primary ideal of R. Thus N = (N:M) M is an s-primary submodule by theorem 1.8

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