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Action of S^3 on Homotopy 15-Spheres

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Abstract

In this paper, we establish a relation between differential S^3 actions on homotopy 15-spheres and (12, 7) sphere pairs.

المستخلص

في هذا البحث نجد علاقة بين فعل الزمرة S^3 على السطوح الكروية الهوموتوبيــة ذات البعــد 15وبــين أزواج السطوح الكروية من النمط (12, 7).

Introduction

Let R^n be the Euclidean real n-space, and $D^n = \{x\in R^n \mid \| x \| \leq 1\}$ and $S^{n-1} = \{x\in R^n \mid \| x \| = 1\}$. If n=2m, then R^{2m} can be identified with C^m, the Euclidean complex m-space, and $S^1 = \{z\in C \mid |z| = 1\}$. If n=4m, R^{4m} can be identified with Q^m, where Q is the algebra of quaternions, and $S^3 = \{w\in Q \mid \| w \| = 1\}$. There exists a standard free differentiable action of S^1 on S^{2n+1} whose orbit space is the complex projective space CP(n), and any differentiable manifold having the homotopy type of CP(n) is called a homotopy complex projective space HCP(n). Similarly, there exits a standard free differentiable action of S^3 on S^{4m+3} whose orbit space is the quaternion projective m-space, QP(m), and any differentiable manifold having the homotopy type of QP(m) is called a homotopy quaternion projective space HQP(m). [4], [5]

In [12], [13], Montgomery and Yang established a relation between S^1 actions on homotopy 7-spheres and (6, 3) sphere pairs. In this paper, we try to establish similar relation between S^3 actions on homotopy 15-spheres and (12, 7) sphere pairs. However, our results are not conclusive as the results of Montgomery and Yang.

We remark that an action of group G on a manifold M is free if gx=x for some $g \in G$, $x \in M$, implies g=e. And the action is semi-free if it is free outside the fixed point set. Finally, we note that all actions considered in this work are differentiable. For general references for the basic concepts and results, see [1], [3], [6], [7], [8], [10], [14].

1. Differentiable S^1 action on Homotopy 7-Spheres.

In their papers [3] and [4], Montgomery and Yang gave two different standard examples of differentiable actions of S^1 on S^7 such that one of these actions is free and the other is semi-free with fixed point set diffeomorphic to S^3 and orbit space diffeomorphic to S^6. From the free action they gave two characterizations of HCP(3), the first one is known, the second characterization is anew one. They prove also [4] that there are exactly six unoriented homotopy 7-spheres, not diffeomorphic to one another, such that on each of them there is a free differentiable action of S^1. Moreover, on any homotopy 7-sphere, if there exists a free differentiable action of S^1, then there are infinitely many topologically distinct actions [4]. Hence they proved the existence of infinitely many manifolds of the homotopy type of CP(3) but not diffeomorphic to CP(3).

On the other hand Montgomery and Yang used the standard semi-free action to obtain a sphere pair (6, 3), where in a pair (P, Q), P is a differentiable 6-manifold diffeomorphic to S^6 and Q is the image of an embedding of S^3 into P. Then they proved that; if S^1 acts differentiably on a homotopy 7-sphere X such that the action has fixed point set F diffeomorphic to S3 and is free otherwise, then $(X/S^1, F)$ is a (6, 3)- sphere pair. Conversely for any (6, 3)- sphere pair (P, Q) there exists such action with $(X/S^1, F)$ diffeomorphic to (P, Q). [14]

Montgomery and Yang proved also that; on any homotopy 7-sphere, there are infinitely many distinct differentiable actions of S^1.

2. Standard Action of S^3 on S^{15}

There exist standard actions of S^3 on S^{15} defined as follows;

(1) φ_1: $S^3 \times S^{15} \to S^{15}$ such that $\varphi_1(g,(u, v)=(gu, v)$, $g \in S^3$, $u, v \in Q^2$.
φ_1 is free differentiable action of S^3 on S^{15} whose orbit space is QP(3).

(2) φ_2: $S^3 \times S^{15} \to S^{15}$ such that $\varphi_2(g,(u, v))=(gu, gv)$, $g \in S^3$, $u, v \in Q^2$.
φ_2 is a differentiable action of S^3 on S^{15}, however, it is not free, it is semi-free. Under this action, S^3 leaves all points $\{0\} \times S^7$ fixed and acts freely otherwise.
In order to find the orbit space of φ_2, we give the following proposition.

2.1 Proposition: If the group S^3 acts deferentially on a homotopy 15-sphere X such that the action φ has a fixed point set diffeomorphic to S^7 and free otherwise, then the orbit space is diffeomprphic to S^{12}.

Proof: let N be a closed tubular neighborhood of the fix point set S^7 in X. The vector normal bundle defined by N characterized by an element. $\alpha \in \prod_6 (S_o(8))=0$. Hence N is trivial bundle. That N is diffeomorphic to $S^7 \times D^8$. Moreover, X-S^7 has the homotopy type of X-int $(S^7 x D^8)$. But X-int $(S^7 x D^8)$ has the homotopy type of $D^8 x S^7$. See [1].

Now X-int $(S^7xD^8) / S^3$ is a 12-dimensional manifold with boundary, and the boundary has the homotopy type of $(S^7xS^7) / S^3$.it can be seen that $(S^7xS^7) / S^3$ is actually S^4xS^7. Now by assumption we have S^3 acts differentially on X with fixed point set diffeomorphic to S^7, that is S^3 acts freely on X-S^7 and X-S^7 has the homotopy X-int (S^7xD^8) as we just said, and X-int (S^7xD^8) has the homotopy type of S^4xD^8. So $(X-S^7) / S^3$ has the homotopy type of S^4xD^4. On the other hand S^7 has the homotopy type of D^5xS^7. Hence X / S^3 has the homotopy type of $S^4xD^8 \bigcup_{id} D^7xS^7$.

And this manifold is of the homotopy type of $\sum^{12}$, see [1]. But $\theta_{12}=0$ [9], hence $\sum^{12}$ is diffeomorphic to S^{12}.

Now, let us recall the following definition [3]:

2.2 Definition: By a (12, 7) sphere pair, we mean a pair (P, Q) in which P is differentiable 12-manifold diffeomorphic to S^{12} and Q is the image of an embedding of S^7 into P.

For example, there is a natural embedding of R^8 in R^{13}, this gives a natural embedding of S^7 into S^{12}, and hence a (12, 7) pair.

2.3 Corollary: If the group S^3 acts differentiably on a homotopy 15- sphere X such that the action has a fixed point set F diffeomorphic to S^7 and is free otherwise, then $(X/S^3, F)$ is a (12, 7) –sphere pair.

Proof: the result follows immediately from (2.2) because by (2.1), X/S^3 is a differentiable manifold of dimension 12, which is diffeomorphic to S^{12}.

The pair $(M/S^3, F(S^3, S^{15}))$ given earlier in the action φ_2 is another example of a (12, 7)-sphere pair, where M/S^3 is diffeomorphic to S^{12} and $F(S^3, S^{12})=\{0\}xS^7$ which is deffeomorphic to S^7.

Denote by G the set of diffeomorphism classes of (12, 7) -sphere pairs [That is (P, Q) is diffeomorphic to (P′, Q′) if there is a diffeomorphism f: P→P′ such that f(Q)=Q′]. Then G can be made an abelian group by the same way in [13] with a binary operation induced by the connected sum operation.

3. Two Characterizations of HQP(3)'s

Recall that if M is an HOP(3), then $H_i(M, Z)=Z$ if $i \equiv 0 \pmod 4$ $i \leq 12$ and 0 otherwise. Next we give a definition.

3. Definition: Let M be an HQP(3). By a primary embedding of S^4 into M, we mean an embedding $j:S^4 \to M$ such that $j^*\beta=\beta_M$, where ß is a generator of $H_4(S^4)$.

3.1 Lemma: Whenever M is an HQP(3), there is a primary embedding
$j:S^4 \to M$.

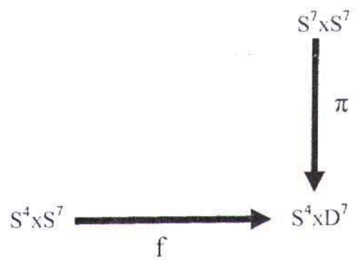
Proof: Since M is connected, it follows from the Hurewicz isomorphism theorem that $\prod_i(M)=0$, $I<4$ and $\prod_4(M)$ is isomorphic to $H_4(M)=Z$. Let α be a generator of $\prod_4(M)$, and let $j:S^4 \to M$ be a representative of α. By Whitney embedding theorm, one may assume that j is a smooth embedding. But $H_4(M)=Z$. Thus if ß is a generator of $H_4(M)$ and β_M is a generator of $H_4(M)$, then it is clear that $j^* \beta= \beta_M$.

Next, we give a necessary condition under which one can get an HQP(3) by pasting together two copies of S^4xD^8 along their boundaries.

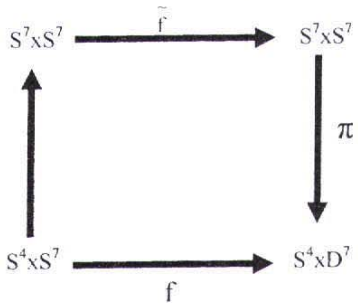
"Let $P:S^4xS^7 \to S^4$, P': $S^4xS^7 \to S^7$ be projections and let $q:S^7 \to S^4xS^7$ be defined by $q(v)=(S^3u, v)$, where S^3u is a preassignd point of S^4.....(*)

3.2 Lemma: If there exists a diffeomorphism f: $S^4xS^7 \to S^4xS^7$ such that pfq: $S^7 \to S^4$ represents a generator of $\prod_4(S^4)$ and pfq: $S^7 \to S^7$ is of degree–1, then $M=(S^4xD^8)U_f(S^4xD^8)$ is an HQP(3). (Note that $\prod_7(S^4)=Z_{12}\oplus Z$, [1, p.329].

Proof: Let π: $S^7xS^7 \to S^4XS^7$ be the projection defined by $\pi(x, y)=(\pi'x, y)$, where π' is the Hopf map $\pi':S^7 \to S^4$. Now consider the following diagram;



Thus f induces a bundle over S^7xS^4 and there exists a diffeomorphism $\tilde{f}$ such that the following diagram commutes;



Let $X=(S^7 \times D^8 = \cup_{\tilde{f}} (S^7 \times D^8)$.

To show that X is simply connected we use Van Kampen theorem. Thus, let $u_1=S^7xD^8=u_2$. Note that $u_1 \cap u_2=S^7xS^7$, then $\prod_1(u_1)=\prod_1(u_2)=\{e\}=\prod_1(u_1\cap u_2)$ because it has the homotopy type of S^7, then $\prod_1(u_1 \cap u_2)=\{e\}*\{e\}=\{e\}$, where * stands for the free product of groups. Hence $\prod_1(X)=\prod_1(u_1\cap u_2)=\{e\}$. Thus X is simply connected. And there is a natural S^3 fibration g: X→M, that is g: $(S^7xD^8 \cup_{\tilde{f}} (S^7xD^8) \to (S^4xD^8) \cup_{\tilde{f}} (S^4xD^8)$.

We denote by α and ß the elements of $H_7(S^7xS^7)$ represented by $S^7x\{y\}$ and $\{x\}xS^7$ respectively. Since pfq is a generator of $\prod_7(S^4)$ and p'fq is of degree –1 then $\tilde{f}_*(\alpha)=\alpha$ and $\tilde{f}_*(\beta)=\alpha-\beta$. Making use of Mayer-Vietoris sequence for the third (X, u_1, u_2), we have;

$$\ldots \to H_i(u_1\cap u_2) \to H_i(u_1)+H_i(u_2) \to H_i(X) \to H_{i-1}(u_1\cap u_2) \to \ldots$$

Which implies $H_i(X)=H_i(S^{15})$ for all i. Hence X is a homotopy 15-sphere $\sum^{15}$ and consequently M is an HQP(3).

3.3 Lemma: If (P, Q) is a (12, 7)-sphere pair, P,P′ and q are the maps defined in(*), then there are orientation- preserving embedding k: $S^4xD^8 \to P$, k′: $D^5xS^7 \to P$ such that;

(i) $p(k'^{-1} k)q$: $S^7 \to S^4$ is homotopic to constant map and $p(k'^{-1} k)q$: $S^7 \to S^7$ is of degree –1 and;

(ii) $k'(\{0\}xS^7)=Q$

Proof: Since (P, Q) is a (12, 7)-sphere pair, then by [3, Th. 2.2], the normal bundle of Q in P is trivial. So there is an orientation-preserving embedding k′: $D^5xS^7 \to$ P which defines an orientation-preserving diffeomorphism of $\{0\}xS^7$ onto Q.

Using transversality argument, it can be shown that P-int(D^5xS^7) is 3-connected, hence H_i(P-int$(D^5xS^7))=0$ for $1\leq i \leq 3$. Moreover H_4(P-int$(D^5xS^7))=Z=\prod_4$ (P-int(D^5xS^7)).

Let j: $S^4 \to P$ be an embedding with $j(S^4)\subseteq$ P-k′(D^5xS^7), and j represents a generator of $\prod_4$ (P-int(D^5xS^7)), and such that $j(S^4)$ has linking number 1 with Q.

Now, since i: $j(S^4) \to$P- k′(E^5xS^7) is a homotopy equivalence, where $E^5=D^5-S^4$, it follows that the induced homomorphism i_* : $H_i(j(S^4)) \to H_i$(P-k′(E^5xS^7)) is an isomorphism for every integer i. Consider the homology exact sequence of the pair $(j(S^4))$, (P- k′(E^5xS^7)):

$$\ldots \to H_i(j(S^4))^{i^*} \to H_i(P- k'(E^5xS^7)) \to H_i(P- k'(E^5xS^7)), j(S^4) \to \ldots \to H_{i-1}(j(S^4))^{i^*} \to H_{i-1}(P- k'(E^5xS^7)) \to \ldots$$

Since the two homomorphisms i_* shown above are isomorphisms, we have;

H_i(P- k′(E^5xS^7)), $j(S^4)=0$ for all i. Therefore $j(S^4)$ is a deformation retract of P- k′(E^5xS^7).

By Smale's theorem, there is an orientation-preserving embedding k: $S^4xD^8 \to P$ with $k(S^4xD^8)=$ P- k′(E^5xS^7) and such that;

(i) k(x, 0)=j(x) and;

(ii) The orientation of $k(S^4x\{0\})$ defined by x→ k(x, 0) is orientation-preserving and $k(S^4x\{0\})$ has linking number 1 with Q.

By our choice of k and k′ above, it is easily seen that $P'(k'^{-1}k)q$: $S^7 \to S^7$ is of degree –1. However, $P'(k'^{-1}k)q$: $S^7 \to S^4$ may represent a non zero element σ of $=\prod_7(S^4)$.

In order to have our assertion, we replace k′ by an embedding;

k'_1: $D^5xS^7 \to P$ constructed as follows;

Let $\zeta:S^7 \to S_o(5)$ be differentiable map such that the composite map,

$S^7 \xrightarrow{\xi} S_o(5) \rightarrow \frac{S_0(5)}{S_0(4)} = S^4$ represented - σ in $\prod_7(S^4)$. Then for any $(x, y) \in S^4xS^7$, we let

$k_1'(x, y) = k'(\zeta(y), x, y)$

Clearly $p(k_1'^{-1}k)q$: $S^7 \rightarrow S^4$ is homotopic to a constant map.

3.4 Lemma: If k and k′ are as in lemma (3.3) and f is the diffeomorphism of lemma (3.2) and λ: $k(S^4xS^7) \rightarrow S^4xS^7$ is defined by;

$\lambda = f\,k'^{-1}$, then $M = k(S^4xD^8) \cup_\lambda (S^4xD^8)$ is an HQP(3).

Proof: By lemma (3.2) M is an HQP(3).

References

1. Browder, W., (**1967**), "*Surgery and the theory of differentiable transformation groups*", Proc. Of the Conf. On transformation groups, New Oprleans, 1-64.
2. Guillemin, V. and Pollack, A., (**1974**) "*differentiable topology*", Prentice Hall.
3. Haefliger, A., (**1962**), "*Knotted (4k-1) spheres in 6k-space*", Ann. Math. 75.
4. Hirzebruch, F., (**1962**), "*Topological methods in algebraic topology*", Springer-Verlag.
5. Hsiang, W. C. and Hsiang , W. Y., (**1965**), "*Some free differentiable actions of S^1 and S^3 on 11-spheres*", Quart. J. Math. 2.
6. Hsiang, W. C., (**1966**), "*A note on free differentiable actions of S^1 and S^3 on homotopy spheres*", Ann. Math. 83.
7. Hu, S. T., (**1959**), "*Homotopy theory*", Academic Press.
8. Husemoller, D., (**1966**), "*Fibre bundles*", McGraw-Hill.
9. Kervaire, M. and Milnor, J., (**1963**), "*Groups of homotopy spheres*", Ann. Math. 77.
10. Ku, H. T. and Ku, M. C., (**1970**), "*Free differentiable actions of S^1 and S^3 on homotopy 11-spheres*", Proc. Amer. Math. Soc. 25.
11. Maunder, C. R. F., (**1970**), "*Algebraic topology*", Van Nostrand.
12. Milnor, J. and Stasheff, J. D., (**1974**), "*Characteristics Classes*", University of Tokyo Press.
13. Montgomery, D. and Yang, C. T., (**1966**), "*Differentiable actions on homotopy seven spheres*", Trans. Amer. Math. Soc. 122.
14. Montgomery, D. and Yang, C. T., (**1968**), "*Differentiable actions on homotopy seven spheres II*", Proc. Conf. On transformation groups.
15. Spanier, E. H., (**1966**), "*Algebraic topology*", McGraw-Hill.