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Comutativity Results on Semiprime Rings

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Abstract

The purpose of this paper is to prove the following result:- Let R be a 2-torsion free semiprime ring. $Z(R)$ be the center of R U be a nonzero ideal of R, and d be a nonzero derivation on R, then R contains a nonzero central ideal if one of the following conditions holds:

i- $d^{n}[x, y] \mp [x, y] \in Z(R)$ for all $x, y \in U$, where n is a fixed positive integer.

ii- $d^{p}[x, y] \mp d^{m}[x, y] \mp [x, y] \in Z(R)$ for all $x, y \in U$. where p. m are fixed positive integers.

الخلاصه

أن هدف هذا البحث هو برهان النتيجة الاتية: لتكن R حلقة شبه أولية طليقة الالتواء من السنمط 2، U مثالي غير صفري من R ، $Z(R)$ مركز لـ R و $d: R \rightarrow R$ اشتقاق غير صفري فان R تحتوي على مثالي مركزي غير صفري إذا تحقق واحد من الشروط الاتية:

1- $d^{n}[x, y] \mp [x, y] \in Z(R)$ لكل x, y في U عندما n عدد صحيح موجب.

2- $d^{p}[x, y] \mp d^{m}[x, y] \mp [x, y] \in Z(R)$ لكل x, y في U عندما m ، p عددان صحيحان موجبان.

Introduction:

Many studies were done on a derivations and commutativity in prime and semiprime rings see [1-6]. Daif and Bell [4] proved that, a simeprime ring must be commutative if it admits a derivation d such that

$d[x, y]=[x, y]$ for all $x, y \in R$. or

$d[x, y]+[x, y=0$ for all $x, y \in R$,

Motoshi Hongan [5] generalized the above result by proving that: Let R be a 2-torsion free semiprime ring. $Z(R)$ be the center of R and $d: R \rightarrow R$ be a derivation. If $d[x, y]+[x, y] \in Z(R)$ or $d[x, y]-[x, y] \in Z(R)$, for all $x, y \in U$, where U is a nonzero ideal of R, then R contains a nonzero central ideal, and the following question was raised : let R be a 2-torsion free semiprime ring, $d: R \rightarrow R$ be a nonzero derivation, and U a nonzero ideal of R, and let n be a fixed positive integer. Does the condition that $d^{n}[x, y]+[x, y] \in Z(R)$ or $d^{n}[x, y]-[x, y] \in Z(R)$, for all $x, y \in U$ imply that $U \subseteq Z(R)$, the purpose of this paper is to answer this question, and give extension for it by proving that R must be contains a nonzero central ideal, when d is satisfying the following condition;

$d^{p}[x, y]+d^{m}[x, y]+[x, y] \in Z(R)$ or

$d^{p}[x, y]-d^{m}[x, y]-[x, y] \in Z(R)$, for all $x, y \in U$. Where p and m are fixed positive integers.

Preliminaries

Throughout this paper, a ring R is semiprime if $aRa=0$, for $a \in R$, then $a=0$. A ring R is 2-torsion free if, $2x=0$, forces $x=0$, for $x \in R$.

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An additive map d from R to R is called a derivation if $d(xy)=d(x)y+xd(y)$, for all $x,y\in R$, we write $[x,y]=xy-yx$ and note that the important identity $[x,yz]=y[x,y]+[x,y]z$. Given a subset S of R, we put

$V_R(S)=\{x\in R\,|\,[x,s]=0 \text{ for all } s\in S\}$.

We need to state the following results.

Lemma 1: [5, lemma 1]
Let R be a simeprime ring. U be a nonzero ideal of R, and $a\in R$.

1- Let $b\in U$, if $[b,x]=0$ for all $x\in U$, then $b\in Z(R)$ therefore, if U is commutative, then $U\subseteq Z(R)$.
2- If $[a,x]\in Z(R)$ for all $x\in U$, then $a\in V_R(U)$.
3- Let R be a 2-torsion free ring and $[a,[x,y]]\in Z(R)$, for all $x,y\in U$, then $a\in V_R(U)$.

Lemma 2:
Let R be a simeprime ring. U be a nonzero ideal of R, and $d:R\to R$ be a nonzero derivation such that $d^n[x,y]\mp[x,y]\in Z(R)$, for all $x,y\in U$, if $d^n(U)\subseteq V_R(U)$ then U is a commutative and $U\subseteq Z(R)$, where n is a fixed positive integer.

Proof:

Let $a\in U$, since $d^n[x,y]\mp[x,y]\in Z(R)$ for any $x,y\in U$, we have

$$0=[a,d^n[x,y]\mp[x,y]]$$
$$0=[a,d^n[x,y]]\mp[a,[x,y]]$$
$$0=[a,d^n(xy)-d^n(yx)]\mp[a,[x,y]]$$
$$0=[a,d^n(xy)]-[a,d^n(yx)]\mp[a,[x,y]]$$

Since $d^n(U)\subseteq V_R(U)$, i. e. $[a,d^n(U)]=0$, for all $x,y\in U$ then $0=\mp[a,[x,y]]$ and we get $a\in V_R(U)$ by [4, lemma 1].

Therefore, U is commutative, and so we obtain that $U\subseteq Z(R)$ by lemma 1.

Lemma 3:
Let R be a simeprime ring. U be a nonzero ideal of R, and $d:R\to R$ be a nonzero derivation such that $d^p[x,y]\mp d^m[x,y]\mp[x,y]\in Z(R)$, for all $x,y\in U$, if $d^p(U)\mp d^m(U)\subseteq V_R(U)$ then U is a commutative and so $U\subseteq Z(R)$, where p and m are positive integers.

Proof:

Let $a\in U$, since $d^p[x,y]\mp d^m[x,y]\mp[x,y]\in Z(R)$ for all $x,y\in U$, we have

$$0=[a,d^p[x,y]\mp d^m[x,y]\mp[x,y]]$$
$$0=[a,d^p[x,y]\mp d^m[x,y]]\mp[a,[x,y]]$$
$$0=[a,d^p(xy)-d^p(yx)\mp d^m(xy)-d^m(yx)]\mp[a,[x,y]]$$
$$0=[a,d^p(xy)\mp d^m(xy)]-[a,d^p(yx)\mp d^m(yx)]\mp[a,[x,y]]$$

Since $d^p(U)\mp d^m(U)\subseteq V_R(U)$, then $[a,d^p(U)\mp d^m(U)]=0$, thus

$$[a,d^p(xy)+d^m(xy)]=[a,d^p(yx)+d^m(yx)]=0$$

for all $x,y\in U$ then $0=\mp[a,[x,y]]$ and we get $a\in V_R(U)$ by [4, lemma 1].

Therefore, U is a commutative ideal, we obtain $U\subseteq Z(R)$ by lemma 1.

The main results

Theorem 1:
Let R be a 2-torsion free semiprime ring. U be a nonzero ideal of R, if R admits a nonzero nonzero derivation d satisfying $d^n[x,y]\mp[x,y]\in Z(R)$, for all $x,y\in U$, then R contains a nonzero central ideal, where n is a fixed positive integer.

Proof:
Suppose that $d\neq 0$, for any $x,y,z\in U$ we have $d^n[x,[y,z]]\mp[x,[y,z]]\in Z(R)$, for all $x,y,z\in U$

$$=d^n(x[y,z])-d^n([y,z]x)\mp[x,[y,z]]$$
$$=d^n(x)[y,z]+xd^n[y,z]-d^n[y,z]x-[y,z]d^n(x)\mp[x,[y,z]]$$
$$=[d^n(x),[y,z]]+[x,d^n[y,z]]\mp[x,[y,z]]$$
$$=[d^n(x),[y,z]]+[x,d^n[y,z]\mp[y,z]]$$

Since $d^n[y,z]\mp[y,z]\in Z(R)$, then we have $[x,d^n[y,z]\mp[y,z]]=0$, for all $y,z\in U$.

Thus $[d^n(x),[y,z]] \in Z(R)$, since R is a 2-torsion free semiprime ring and $[d^n(x),[y,z]] \in Z(R)$, for all $y,z \in U$ then by Lemma 1, we obtain $d^n(x) \in V_R(U)$, that is $d^n(U) \subseteq V_R(U)$, therefore by lemma 2 we have $U \subseteq Z(R)$.

So we can get the following corollary.

Corollary 1:

Let R be a 2-torsion free semiprime ring, and $d: R \to R$ be a nonzero derivation on R. If $d^n([x,y]) \mp [x,y] \in Z(R)$, for all $x,y \in R$, then R is commutative, where n is a fixed positive integer.

Remark

In theorem 1 and corollary 1, we can not exclude the condition "2-torsion free" as below.

Example:

We denote by Z the integer system Let

$$R = \begin{pmatrix} \frac{Z}{2Z} & \frac{Z}{2Z} \\ \frac{Z}{2Z} & \frac{Z}{2Z} \end{pmatrix}, \quad a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and d be the inner derivation induced by a, that is, $d(x) = [a,x]$ for all $x \in R$, then R is a non-commutative prime ring with Char. $R = 2$, when $n = 2$ then $d^2[x,y] \mp [x,y] \in Z(R)$, for all $x,y \in R$. And by the same way we can prove:

Theorem 2:

Let R be a 2-torsion free semiprime ring, and U be a nonzero ideal of R, if R admits a nonzero derivation d satisfying $d^p[x,y] \mp d^m[x,y] \mp [x,y] \in Z(R)$, for all $x,y \in U$, then R contains a nonzero central ideal, where p and m are fixed positive integers.

Proof:

We suppose that $d \neq 0$, for any $x,y,z \in U$ we have

$$d^p[x,[y,z]] \mp d^m[x,[y,z]] \mp [x,[y,z]] \in Z(R),$$

Then

$$= d^p(x[y,z]) - d^p([y,z]x) \mp d^m(x[y,z]) - d^m([y,z]) \mp [x,[y,z]]$$

$$= [d^p(x),[y,z]] \mp [x, d^p[y,z]] \mp [d^m(x),[y,z]] \mp [x, d^m[y,z]] \mp [x,[y,z]]$$

$$= [d^p(x) \mp d^m(x),[y,z]] \mp [x, d^p[y,z] \mp d^m[y,z] \mp [y,z]]$$

Since $d^p[y,z] \mp d^m[y,z] \mp [y,z] \in Z(R)$, for all $y,z \in U$, then $[x, d^p[y,z] \mp d^m[y,z] \mp [y,z]] = 0$, for all $y,z \in U$. Thus we obtain $[d^p(x) + d^m(x),[y,z]] \in Z(R)$. by lemma 1, we have

$d^p(x) + d^m(x) \in V_R(U)$, that is $d^p(U) + d^m(U) \subseteq V_R(U)$, therefore, by lemma 3 we have $U \subseteq Z(R)$.

Corollary 2:

Let R be a 2-torsion free semiprime ring, and $d: R \to R$ be a nonzero derivation, if $d^p[x,y] \mp d^m[x,y] \mp [x,y] \in Z(R)$, for all $x,y \in R$, then R is commutative, where p and m are fixed positive integers.

Proof:

From precedence theorem 2, we obtain $U \subseteq Z(R)$, U is a nonzero ideal of R, so if $d^p[x,y] \mp d^m[x,y] \mp [x,y] \in Z(R)$ for all $x,y \in R$, we get $R \subseteq Z(R)$, then R is commutative.

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