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On multiplication modules and their generalization

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Abstract

R. Jain studied multiplication modules and their generalizations. The aim of this paper is to give various properties for these classes of modules. In particular, we study $M \otimes N$ and Hom (M, N), where M and N belongs to these classes.

الخلاصة

درس جين المقاسات الجدائية وأعماماتها. الغرض الرئيسي من هذا البحث هو تطوير خواص المقاسات الجدائية وأعماماتها. تم البرهنة على أن المقاس M يكون مقاساً جدائياً أذا وفقط أذا كان كل مقاس جزئي جوهري منه مقاساً جزئيا جدائياً. أيضا تم البرهنة على أن المقاس$M \otimes N$ يكون جدائياً تقريباً أذا كان كل من M و N مقاساً جدائياً تقريباً.

Introduction:

R.Jain in [6] studied multiplication modules and their generalizations.In this paper, we add some results on multiplication modules and their generalization.

In section 1, we give a characterization of multiplication modules in terms of essential submodules. We prove that a module M is a multiplication module if and only if every essential submodule of M is a multiplication submodule. Also we prove that an epimorphic image of an almost (semi) multiplication module is an almost (semi) multiplication module.

In section 2, we prove that the tensor product of two almost (weak) multiplication modules is an almost (weak) multiplication module. In section 3, we prove that if M is a finitely generated weak multiplication module and N is a multiplication submodule of N such that ann $M \subseteq$ ann N, then Hom (M, N) is a weak multiplication.

Finally we remark that all rings considered in this paper are commutative with identity, and all modules are unitary.

§1: Multiplication modules and their generalizations.

Let R be a commutative ring with identity and let M be a unitary R-module. In this section we study multiplication, almost multiplication, weak multiplication and semi- multiplication modules.

Recall that a submodule N of an R-module M is called a multiplication submodule if for each submodule K of N, there exists an ideal I such that K=IN, [6]. And a module M is called a multiplication module if every submodule of M is a multiplication submodule.

A module M is called faithful if ann M=0, [7]. It is known that if R is a noetherian ring and M is an R-module such that M is a multiplication submodule of M, then M is a noetherian module, [6].

The converse is true when M is faithful, finitely generated and multiplication submodule of M, [9].

The following proposition gives another condition under which the converse is true.

Proposition1.1: Let M be a noetherian R-module. If M is a multiplication submodule of M and there exists an element $x \in M$ such that ann x=0, then R is a notherian ring.

Proof: Let $I_1 \subseteq I_2 \subseteq \ldots\ldots$ be an ascending sequence of ideals in R .It is clear that $I_1x \subseteq I_2x \subseteq \ldots\ldots$is an ascending chain of submodules of M. But M is noetherian, so there exists a positive integer n such that $I_nx=I_kx$ for each $k \geq n$. Now, let $a \in I_k$, then $ax \in I_kx=I_nx$ and hence there exists $b \in I_n$ such that ax=bx. Thus (a-b) x=0. But ann

x=0, therefore a=b and hence $a \in I_n$ and $I_k = I_n$ for each $k \geq n$. Thus R is a noetherian ring.
Now, let R be an integral domain. An R-module M is called torsion free module if whenever rm=0, for some $r \in R$ and $m \in M$, then either r=0 or m=0,[7].

Proposition 1.2: Let R be an integral domain and let M be an R-module such that there exists an element $m \in M$ with ann(m)=0.If M is a multiplication submodule of M, then M is a torsion free module.

Proof: Let $0 \neq x \in M$ and $r \in R$ such that rx=0, since $Rx \subseteq M$ and M is a multiplication submodule of M, then there exists an ideal I in R such that Rx=IM and hence R rx = r I M=0.
Now, let $0 \neq w \in I$ then rwm=0.But ann(m)=0,therefore rw=0.Since R is an integral domain ,then r=0 and hence M is a torsion free module .

It was proved in [5], that if M is a multiplication submodule of M and $f:M \rightarrow \overline{M}$ is an epimorphism, then $\overline{M}$ is a multiplication submodule of $\overline{M}$ We prove the following :

Proposition 1.3: Let $f:M \rightarrow \overline{M}$ be an epimorphism, if M is a multiplication module, then $\overline{M}$ is a multiplication moduleation module.

Proof: Let N and K be submodules of $\overline{M}$ such that $K \subseteq N$. It is clear that $f^{-1}(K) \subseteq f^{-1}(N) \subseteq M$. Since M is a multiplication module, then there exists an ideal I of R such that $f^{-1}(K) = I f^{-1}(N)$ and hence $f(f^{-1}(K))=f(I f^{-1}(N))=If(f^{-1}(N))$.But f is an epimorphism, therefore K=IN. Thus N is a multiplication submodule of $\overline{M}$ and $\overline{M}$ is a multiplication R-module.

Proposition 1.4: Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be a short exact sequence. If B is a multiplication module, then each of A and C is a multiplication module.

Proof: Since $g:B \longrightarrow C$ is an epimorphism and B is a multiplication module. Then C is a multiplication module, by (1.3). Now, to show that A is a multiplication module, let K and N be submodules of A such that $K \subseteq N$. It is clear that $f(K) \subseteq f(N) \subseteq B$. Since B is a multiplication module, then there exists an ideal I in R such that f(K) =If(N) = f(IN). But f is a monomorphism, therefore K=IN and hence N is a multiplication submodule of A. Thus A is a multiplication module.

Remark 1.5: Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be a short exact sequence. If each of A and C is a multiplication R-module, then B may not be amultiplication R-module as the following example shows, Consider the following short exact sequence $0 \longrightarrow Z \xrightarrow{i} Z \oplus Z \xrightarrow{\pi} Z \longrightarrow 0$

Where i is the inclusion map and π is the projection map. It is clear that Z as Z module is amultipliction module. Now, Z(0,1) is a submodule of $Z \oplus Z$ and $Z(0,1) \neq (n)(Z \oplus Z)$, for each $n \in Z$. Thus $Z \oplus Z$ is not a multiplication submodule of $Z \oplus Z$ and hence $Z \oplus Z$ is not amultiplication module.

Recall that a non zero submodule N of an R-module M is called an essential submodule if $N \cap K \neq 0$ for every non zero submodule K of M, [4].The following theorem is characterization of multiplication modules.

Theorem 1.6: Let M be an R-module. M is a multiplication module if and oniy if every essential submodule of M is a multiplication submodule.

Proof: Let N be a submodule of M, then there exists a submodule K of M such that $N \oplus K$ is essential in M [7], and hence $N \oplus K$ is a multiplication submodule of M. But $(N \oplus K)/K \approx N$ and $(N \oplus K)/K$ is a multiplication submodule of M/K [1.4], therefore N is a multiplication submodule of M. Thus M is a multiplication module.

An R-module M is called an almost multiplication module if the R_P-module M_P is a multiplication module for each prime ideal P in R, [6]. And a ring R is called an almost multiplication ring if R_P is a multiplication ring for each prime ideal P in R.

Recall that a ring R is called a local ring (semi–local ring) if it has a unique maximal ideal (a finite number of maximal ideals), [7].

It is clear that every multiplication module is an almost multiplication module. It was proved in [6] that the converse is true when M is noetherian or R is a semi–local ring.

A submodule N of an R-module M is called a prime submodule if for each $r \in R$ and $m \in M$ such that $rx \in N$ and $x \notin N$, then $rM \subseteq N$, [3].

Let M be an R- module. We say that the dimension of M is n if there exists a proper prime submodules $P_0, P_1, P_2, \ldots, P_n$ in M such that $P_0 \subseteq P_1 \subseteq \ldots \subseteq P_n$, and there is no similar sequence with n+2 of proper prime submodule [6]. In this case we write dim(M)=n.

It was proved in [6] that if M is a multiplication module, then $\dim(M) \leq 1$.

We prove the following theorem:

Theorem 1.7: Let M be an almost multiplication module, then $\dim(M) \leq 1$.

Proof: Since M is an almost multiplication module, then the R_P-module M_P is a multiplication module for each prime ideal P in R. But R_P is a local ring, therefore every submodule of M_P is cyclic, and $\dim M_P \leq 1$, [6]. If dim(M) > 1, then there exists a proper prime submodules P_1, P_2, P_3 in M such that $P_1 \subseteq P_2 \subseteq P_3$. Since P_3 is a proper prime submodule in M, then $\overline{P}_3=(P_3:M)$ is a prime ideal in R, [3]. Hence, there exists a one -to- one map f from the set of prime submodules of M that contained in P_3 in to the set of proper prime submodules in $M_{\overline{P}_3}$, [6]. Now, we have $f(P_1) \subseteq f(P_2) \subseteq f(P_3)$ proper prime submodules in $M_{\overline{P}_3}$ and hence $\dim(M_{\overline{P}_3})>1$ which is a contradiction. Thus $\dim(M) \leq 1$.

A submodule N of an R-module M is said to have the weak cancellation property if whenever $AN \subseteq BN$ where each of A and B is an ideal in R, then $A \subseteq B+\text{ann}(N)$ and N is said to have the cancellation property if whenever $AN \subseteq BN$, then $A \subseteq B$, [10].

Proposition 1.8: Let M be a faithful, finitely generated and an almost multiplication module, then R is an almost multiplication ring.

Proof: Let P be a prime ideal in R and A,B are ideals in R_P such that $B \subseteq A$, then $BM_P \subseteq AM_P$. Since M is an almost multiplication module, then there exists an ideal C in R_P such that $BM_P=CAM_P$. But M_P is a multiplication module and R_P is a local ring, therefore M_P is cyclic, [6]. Thus M_p has the weak cancellation property and hence $B=CA+\text{ann}(M_P)$, [10]. Since M is finitly generated, than $(\text{ann } M)_P=\text{ann } M_P$. Also M is faithful, therefore $\text{ann } M_P=0$ and hence B=CA. Thus A is a multiplication ideal and R is an almost multiplication ring.

The proof of the following proposition is similar to the proof of (1.3).

Proposition (1.9): Let $f: M \rightarrow \overline{M}$ be an epimorphism. If M is an almost multiplication module, then $\overline{M}$ is an almost multiplication module.

Proposition (1.10): Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be a short exact sequence .If B is an almost multiplication module, then each of A and C is an almost multiplication module.

Proof: Since $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is a short exact sequence, then $0 \longrightarrow A_P \xrightarrow{f_P} B_P \xrightarrow{g_P} C_P \longrightarrow 0$ is a short exact sequence for each prime ideal P in R [8.p.15].Now since B_P is a multiplication module, then A_P and C_P are multiplication modules for each prime ideal P in R .Thus A and C are almost multiplication modules.

Remark 1.11: Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be a short exact sequence. If each of A and C is an almost multiplication modules, then B may not be an almost multiplication module as the following example shows: Consider the following short exact sequence $0 \longrightarrow Z \xrightarrow{i} Z \oplus Z \xrightarrow{\pi} Z \longrightarrow 0$

where i is the inclusion map and Π is the projection map. It is clear that Z is a multiplication module and hence an almost multiplication module. Let P=(0).It is known that $Z_{(0)}=Q$. Now consider the following short exact sequence $0 \longrightarrow Z_{(0)} \xrightarrow{i(0)} (Z \oplus Z)_{(0)} \xrightarrow{\pi(0)} Z_{(0)} \longrightarrow 0$

Q is multiplication as Q- module. But $(Z \oplus Z)_{(0)}= Z_{(0)} \oplus Z_{(0)}=Q \oplus Q$ is not multiplication as Q-module .

Recall that a module M is said to be a weak multiplication module if every prime submodule of M is a multiplication submodule, [6].

It is clear that every multiplication module is a weak multiplication module.

It was proved in [6] that every weak multiplication module is an almost multiplication module and hence we have the following remark.

Remark 1.12: Let M be a weak multiplication module, then $\text{dom}(M) \leq 1$.

Proposition 1.13: Let R be a semi-local ring and let M be a weak multiplication R-module, then M is a multiplication R-module.

Proof: Since M is a weak multiplication module, then every prime submodule of M is a multiplication submodule. Since R is a semi-local ring, then every prime submodule of M is cyclic, [6], and hence every submodule of M is a multiplication submodule, [6]. Thus M is a multiplication module.

The proof of the following proposition is similar to the proof of (1.3).

Proposition 1.14: Let $f:M\rightarrow\overline{M}$ be an epimorphism. If M is a weak multiplication module, then $\overline{M}$ is a weak multiplication module.

An R-module M is called a semi–multiplication module if every proper submodule of M is a multiplication submodule, [6].

It is clear that every multiplication module is a semi–multiplication module.

It is known that a semi–multiplication module may not have a maximal submodule, for example $Z_{p\infty}$ as Z-module is a weak multiplication module and $Z_{p\infty}$ has no maximal submodule.

Recall that a module M is called apc-module, if $AM\subseteq M$ for each proper ideal A of R ,[6].

Proposition1.14: Let M be a semi–multiplication R-module. If M is apc-module, then M has a maximal submodule.

Proof: Case1: If M is a multiplication module, then M has a maximal submodule.

Case2: suppose that M is not a multiplication submodule of M and assume M has no maximal submodule. Let K be a proper submodule of M, then there exists a proper submodule N of M such that $K\subseteq N\subseteq M$. Thus there exists a proper ideal I of R such that $K=IN\subseteq IM\subseteq M$. Then IM is a multiplication submodule of M and hence there exist an ideal J in R s.t K=JIM. Thus M is a multiplication submodule of M which is a contradiction. Hence M has a maximal submodule.

It is known that if R is a notherian ring and M is apc-module and semi–multiplication module, then M is noetherian, [6].

We prove that the converse is true when M has an element x such that ann(x)=0.

Proposition 1.16: Let M be apc-module which is noetherian and semi-multiplication. If there exists an element $x\in M$ such that ann(x)=0, then R is noetherian.

Proof: If M=Rx, then M is finitely generated, faithful, and multiplication and hence R is noetherian, [9]. Assume $M\neq Rx$, then Rx is a multiplication, faithful and finitely generated module and hence R is a noetherian ring, [9].

Compare the following theorem with theorem (1.6).

Theorem 1.17: Let M be indecoposable R-module, then M is semi- multiplication if and only if every proper essential submodule of M is a multiplication submodule.

Proof: Let $0\neq N$ be a proper submodule of M, then there exists a submodule K of M such that $N\oplus K$ is essential in M, [4, p.75]. Since M indecomposable and $N\neq 0$, then $N\oplus K\neq M$ and hence $N\oplus K$ is a multiplication submodule of M. But $N\approx(N\oplus K)/K$ and $(N\oplus K)/K$ is a multiplication submodule of M/K, by (1.3), therefore N is a multiplication submodule and M is a semi–multiplication module.

Using an argument similar to that used in the proof of proposition (1.3), (1.4). One can get the following:

Proposition 1.18: Let $f:M\rightarrow\overline{M}$ be an epimorphism. If M is a semi–multiplication R-module, then $\overline{M}$ is a semi–multiplication R-module.

Proposition 1.19: Let $0\rightarrow A\xrightarrow{f}B\xrightarrow{g}C\rightarrow 0$ be a short exact sequence. If B is a semi-multiplication R-module, then each of A and C is a semi-multiplication R-module.

Remark 1.20: Let $0\rightarrow A\xrightarrow{f}B\xrightarrow{g}C\rightarrow 0$ be a short exact sequence. If each of A and C is a semi–multiplication R-module, then B may not be semi-multiplication R-module as the following example shows consider the following short exact sequence $0\rightarrow Z_{p\infty}\xrightarrow{i}Z_{p\infty}\oplus Z_{p\infty}\xrightarrow{\pi}Z_{p\infty}\rightarrow 0$

where i is the inclusion map and π is the projection map. It is clear that $Z_{p\infty}$ as Z–module is a semi–multiplication module. But $Z_{p\infty}\oplus Z_{p\infty}$ is not semi–multiplication because $Z_{p\infty}$ is a proper submodule of $Z_{p\infty}\oplus Z_{p\infty}$ which is not a multiplication submodule.

§2: The tensor product of multiplication modules and their generalizations.

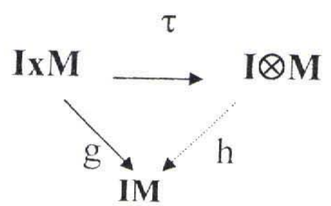
In this section we study the properties of the tensor product of multiplication modules and their generalizations. The following proposition was proved in [1].

Proposition 2.1: Let each of M and N be R-modules. If M is a multiplication submodule of M and N is a multiplication submodule of N, then $M \otimes N$ is a multiplication submodule of $M \otimes N$. The following proposition was proved in [2].

Proposition 2.2: Let each of M and N be R-modules. If N is a multiplication submodule of N and M is a multiplication module, then $M \otimes N$ is a multiplication module. We prove the following.

Proposition 2.3: If M is a multiplication submodule of M and I is a multiplication ideal, then IM is a multiplication module.

Proof : Consider the following diagram



where τ is the tensor map and g is a map defined by g(r,m)=rm. $\forall r \in I$, $\forall m \in M$, its clear that g is a bilinear map and hence there is a homomorphism h from $I \otimes M$ into IM such that $h \circ t = g$. It can be easily checked that h is an epimorphism and hence IM is a multiplication module by (1.3).

Proposition 2.4: Let each of M and N be R-modules. If M is an almost multiplication module and N is a multiplication submodule of N, then $M \otimes N$ is an almost multiplication module.

Proof: Let p be a prime ideal, we want to show that $(M \otimes N)_P$ is a multiplication R_P–module. We have $(M \otimes N)_P \approx M_P \otimes N_P$, [8, P.75]. Since M is an almost multiplication module, then M_P is a multiplication R_P-module. But N is a multiplication submodule of N, then N_P is a multiplication submodule of N_P. Thus $M_P \otimes N_P$ is a multiplication module by (2.2) and hence $M \otimes N$ is an almost multiplication module.

Corollary 2.5: If each of M and N is an almost multiplication module, then $M \otimes N$ is an almost multiplication module.

Corollary 2.6: Let M be an R-module. If M is a multiplication submodule of M and I is an almost multiplication ideal, then IM is an almost multiplication module.

When the module M is weak multiplication, we prove the following:

Proposition 2.7: Let each of M and N be R-modules. If M is a weak multiplication module and N is multiplication submodule of N, then $M \otimes N$ is a weak multiplication module.

Proof: Let K be a prime submodule of $M \otimes N$. Since M is a weak multiplication module, then M is a multiplication submodule of M and hence $M \otimes N$ is a multiplication submodule of $M \otimes N$, by (2.1). Thus

$$K = (K : M \otimes N)(M \otimes N)$$
$$= (K : M \otimes N) M \otimes N.$$

Since K is a prime submodule of $M \otimes N$, then $(K : M \otimes N)$ is a prime ideal in R, and hence $(K : M \otimes N)M$ is a prime submodule of M, [18, prop. (1.27)]. Thus $(K : M \otimes N)M$ is a multiplication submodule of M. Therefore

$K = (K : M \otimes N)\ M \otimes N$ is a multiplication submodule of $M \otimes N$.

Corollary 2.8: If each of M and N is a weak multiplication module, then $M \otimes N$ is a weak multiplication module.

Using an argument similar to that used in the proof of prop. (2.3), we prove the following:

Corollary 2.9: Let M be an R-module. If M is a multiplication submodule of M and I is a weak multiplication ideal, then IM is a weak multiplication module.

§ 3: The module of homomorphisms of multiplication modules and their generalizations.

In this section we study the properties of the module Hom(M, N) when M or N is a multiplication or generalized multiplication module.

The following proposition was proved in [1].

Proposition 3.1: Let each of M and N be an R-module. If M is a finitely generated multiplication submodule of M and N is a multiplication submodule of N such that ann $M \subseteq$ ann N, then Hom(M, N) is a multiplication submodule of Hom(M, N).

Our first result in this section is the following

Proposition 3.2: Let each of M and N be R-modules. If M is a finitely generated multiplication R-module and N is a multiplication submodule of N such that ann $M \subseteq$ ann N, then Hom(M, N) is a multiplication module.

Proof: Let each of K and L be submodules of Hom(M, N) such that $L \subseteq K$, then

$(L : \mathrm{Hom}(M,N)) \subseteq (K : \mathrm{Hom}(M,N))$ and hence

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$(L:Hom(M,N))M \subseteq (K:Hom(M,N))M$.But M is a multiplication module, So there is an ideal I in R such that

$(L:Hom(M,N))M=I(K:Hom(M,N))M$.Since M is a finitely generated multiplication module, then M has the weak cancellation property, [10,Th 6.6]. Thus $(L:Hom(M,N))+ann\ M=I(K:Hom(M,N))+ann\ M$. But ann $M \subseteq$ ann Hom(M,N), so

$(L:HomM,N))Hom(M,N)=I(K:Hom(M,N))Hom(M,N)$.Thus L=IK by (3.1) .

Corollary 3.3: Let M be an R–module. If M is a finitely generated and multiplication module, then Hom(M, M) is a multiplication module.

Corollary 3.4: Let M be an R-module. If M is faithful, finitely generated and multiplication, then M^{*} =Hom(M,R) is a multiplication module.

Compare the following proposition with [9, P. 57, Prop.(1.26)]

Proposition 3.5: Let each of M and N be R-modules, such that M is a multiplication submodule of M and ann M =ann Hom(M, N). If Hom(M,N) is a finitely generated and multiplication module, then M is a multiplication module.

Proof: Let each of K and L be submodules of M, such that $L \subseteq K$ and hence

$(L:M) \subseteq (K:M)$ and $(L:M)Hom(M,N) \subseteq (K:M)Hom(M,N)$.

Since Hom(M,N) is a multiplication module, then there exists an ideal I in R such that (L:M)Hom (M,N)=I(K:M)Hom(M,N).

But Hom(M,N) is finitely generated and multiplication module, so Hom(M,N) has the weak cancellation property, [10, Th(6.6)] and hence (L:M)+ann Hom(M,N)=I(K:M)+ann Hom(M,N).

Since ann M=ann Hom(M,N),then (L:M)M=I(K:M)M

But M is a multiplication submodule of M, so L=IK and hence M is a multiplication module.

Corollary 3.6: Let M be a multiplication submodule of M such that ann M =ann Hom(M,M).If Hom(M,M) is finitely generated and multiplication R-module, then M is a multiplication module.

Corollary 3.7: Let M be a multiplication submodule of M such that

ann M= ann M^{*}. If M^{*} is a finitely generated and multiplication module, then M is a multiplication module.

We need the following lemma later

Lemma 3.8: Let each of M and N be R-module and let P be a prime ideal of R. If M is finitely generated, then there exists a monomorphism from the module $(Hom(M,N))_P$ into the module $Hom(M_P,N_P)$.

Proof: Define $\phi:(Hom(M,N))_P \to Hom(M_P,N_P)$ as follows:-

$$\left(\phi\left(\frac{f}{t}\right)\right)\left(\frac{m}{s}\right)=\frac{f(m)}{ts}$$

for each $f \in Hom(M,N)$,each $m \in M$ and for all $t,s \in R-P$.

First we show that ϕ is well define

Let $\frac{f_1}{t_1}=\frac{f_2}{t_2}$ and $\frac{m}{s} \in M_P$, we want to show that $\frac{f_1(m)}{t_1 s}=\frac{f_2(m)}{t_2 s}$ Since $\frac{f_1}{t_1}=\frac{f_2}{t_2}$, then there exists $t_3 \in R-P$ .Such that $t_3t_2f_1=t_3t_1f_2$ and hence $st_3t_2f_1=st_3t_1f_2$. Thus $\frac{f_1(m)}{t_1 s}=\frac{f_2(m)}{t_2 s}$ It is clear that ϕ is a homomorphism, to show that ϕ is a monomorphism. Let $\phi(\frac{f}{t})=\phi(\frac{g}{s})$ and suppose that $M=Rx_1+Rx_2+\ldots+Rx_n$

$$\phi(\frac{f}{t})(\frac{x_i}{1})=\phi(\frac{g}{s})(\frac{x_i}{1})\ \forall\, i ;1 \le i \le n$$

Thus $(\frac{f(x_i)}{t})=(\frac{g(x_i)}{s})$ and hence there is $t_i \in R-P$ such that $t_isf(x_i)=t_itg(x_i)$.

Now, its enough to prove that $t_1t_2\ldots\ldots t_nf=t_1t_2\ldots\ldots t_ng$.

To see this, let $m \in M$, then there is $r_1,\ldots,r_n \in R$ such that $m=r_1x_1+r_2x_2+\ldots\ldots+r_nx_n$,

But $t_1t_2\ldots.t_nsf(m)= t_1t_2\ldots t_nsf(r_1x_1+r_2x_2+\ldots+r_nx_n)$

$=r_1t_1t_2\ldots\ldots t_nsf(x_1)+r_2t_1t_2\ldots\ldots t_nsf(x_2)+\ldots\ldots +r_nt_1t_2\ldots\ldots t_nsf(x_n)$

$=r_1t_1t_2\ldots\ldots t_ntg(x_1)+r_2t_1t_2\ldots\ldots t_ntg(x_2)+\ldots\ldots +r_nt_1t_2\ldots\ldots t_ntg(x_n)$

$=t_1t_2\ldots\ldots t_ntg(m)$

so ϕ is a monomorphism.

When M is a finitely generated and almost multiplication module, we have the following:

Proposition 3.9: Let each of M and N be a finitely generated R-module. If M is an almost multiplication module and N is a multiplication submodule of N such that ann M $\subseteq$ ann N, then Hom(M,N) is an almost multiplication module.

Proof: Let P be a prime ideal in R. It is enough to show that $(Hom(M,N))_P$ is a multiplication module. Since M is an almost multiplication module, then M_P is a multiplication module. But N is a multiplication submodule of N, So N_P is a multiplication submodule of N_P, [6]. But ann M $\subseteq$ ann N, then ann $M_P \subseteq$ ann N_P and hence by (3.2)Hom(M_P,N_P) is a multiplication module. Using lemma(3.8),we can consider $(Hom(M,N))_P$ as a submodule of Hom(M_P,N_P) and hence $(Hom(M,N))_P$ is a multiplication module. Thus Hom(M,N) is an almost multiplication module.

Corollary 3.10: If M is a finitely generated almost multiplivation R-module, then Hom(M,M) is an almost multiplication module.

Corollary 3.11: If M is a faithful, finitely generated almost multiplication module, then M^* is an almost multiplication module.

Proposition 3.12: Let each of M and N be R-modules. If M is finitely generated and a multiplication submodule of M such that

ann M =ann Hom(M,N) and Hom(M,N) is an almost multiplication module, then M is an almost multiplication module.

Proof: Let P be a prime ideal in R. We have to show that M_P is a multiplication module. Let K and L be submodules of M_P such that L $\subseteq$ K and hence $(L:M_P) \subseteq (K:M_P)$. Thus $(L:M_P)(Hom(M,N))_P \subseteq (K:M_P)(Hom(M,N))_P$.

But Hom(M,N) is an almost multiplication module. So $(Hom(M,N))_P$ is a multiplication module and hence there exists an ideal I in R_P such that

$(L:M_P)(Hom(M,N)_P=I(K:M_P)(Hom(M,N))_P$.But $(Hom(M,N))_P$ is cyclic, [6] and hence $(Hom(M,N)_P$ has the weak cancellation property,[10,Th(6.6)]. Thus

$(L:M_P)+ann(Hom(M,N))_P=I(K:M_P)+ann(Hom(M,N))_P$.

Now,each of M and Hom(M,N) is finitely generated, so ann M_P=(ann $M)_P$= $(ann(Hom(M,N))_P$= $ann(Hom(M,N))_P$, [8,p.75] and hence $(L:M_P)M_P= I(K:M_P)M_P$.

Since M is a multiplication submodule, then M_P is a multiplication submodule, [6]. Thus L=IK and hence K is a multiplication submodule of M_P.

Corollary 3.13: Let M be a multiplication submodule of M such that ann(M)=ann(Hom(M,M)). If Hom(M, M) is finitely generated almost multiplication module, then M is an almost multiplication module.

Corollary 3.14: Let M be a multiplication submodule of M such that ann M =annM^*. If M^* is a finitely generated almost multiplication module, then M is an almost multiplication module.

When M is weak multiplication, we have the following:

Proposition 3.15: Let each of M and N be R-modules and let M be a finitely generated weak multiplication module. If N is a multiplication submodule of N such that ann M $\subseteq$ ann N, then Hom(M, N)is a weak multiplication module.

Proof: Let K be a prime submodule of Hom(M, N) and L be a submodule of Hom(M, N) such that L $\subseteq$ K and hence

(L:Hom(M,N)) $\subseteq$ (K:Hom(M,N)). Thus (L:Hom(M,N))M $\subseteq$ (K:Hom(M,N))M. But M is a weak multiplication module and (K:Hom(M,N))M is a prime submodule of M, [2,Prop. (1.27)], so there exists an ideal I in R such that (L:Hom(M,N))M=I(K:Hom(M,N))M. Since M is finitely generated and multiplication submodule, then M has the weak cancellation property, [12, Th(6.6)] and hence

(L:Hom(M,N))+ann M=I(K:Hom(M,N))+ann M.

Thus ((L:Hom(M,N)+ann M)Hom(M,N)= (I(K:Hom(M,N))+ann M)Hom(M,N) since ann M $\subseteq$ ann Hom(M,N), then (L:Hom(M,N))Hom(M,N)= I(K:Hom(M,N))Hom(M,N).

But Hom(M, N)is a multiplication submodule by (3.1), so L=IK. Thus K is a multiplication submodule and Hom (M,N) is a weak multiplication module .

Corollary 3.16: If M is finitely generated and weak multiplication, then Hom(M,M) is weak multiplication .

Corollary 3.17: If M is faithful, finitely generated and weak multiplication module, then M^* is a weak multiplication module.

Proposition 3.18: Let each of M and N be R-modules. If M is a multiplication submodule of M such that ann M=ann Hom(M,N) and Hom(M,N) is finitely generated and weak multiplication, then M is a weak multiplication module.

Proof: Let K be a prime submodule of M and L be a submodule of M such that

L $\subseteq$ K.Then (L:M) $\subseteq$(K:M) and (L:M)Hom(M,N) $\subseteq$(K:M)Hom(M,N). Since K is a prime submodule of M, then (K:M) is a prime ideal in R. But Hom(M,N) is a multiplication submodule, so (K:M)Hom(M,N) is

a prime submodule, [8,Prop.(1.27)]. Since Hom(M,N) is weak multiplication, then there exists an ideal I in R such that (L:M)Hom(M,N)=I(K:M)Hom(M,N).
Now, Hom(M,N) is finitely generated and multiplication submodule, so Hom(M,N) has the weak cancellation property, [12,Th(6.6)].Thus

(L:M)+ann Hom(M,N)=I(K:M)+ann Hom(M,N).

Since ann M=ann Hom(M,N),then

(L:M)M=I(K:M)M.

Also M is a multiplication submodule of M, so L=IK. Thus M is a weak multiplication module.

Corollary 3.19: If M is a multiplication submodule of M such that ann M=ann Hom(M,M) and Hom(M,M) is a finitely generated weak multiplication module, then M is a weak multiplication module.

Corollary 3.20: Let M be a multiplication submodule of M such that ann M=ann M*. If M* is a finitely generated weak multiplication module, then M is a weak multiplication module.

References

1. B. Al-Hashimi and B. AL-Bahrany, *On the tensor product and the module of homomorphisms of multiplication modules*, Iraqi J. sci., vol.35, No.3(**1994**), 799-825.
2. B. Al-Hashimi and A. A. Ahmed, *On submodules of multiplication modules*, M.sc.Thesis, U. Baghdad, (**1992**).
3. B. Al-Hashimi and A. A. Athab, *The prime submodules and semi -prime submodules*, M.sc.Thesis, U. Baghdad, (**1996**).
4. F. W. Anderson and K. R. Fuller, *Rings and categories of modules*, Springer-Verlag, **1991**.
5. Z. A. El-Bast and P. F. Smith, *Multiplication modules* ,comm. in Algebra,16(**1988**), 755-779.
6. R. K. Jain, *Generalized multiplication modules*, Riv.mat. Univ. Parma 7(**1981**), 461-472.
7. F. Kasch, *Modules and rings*, Academic press, London, (NewYork, **1982**).
8. M. D. Larsen and P, F. Smith, *Multiplicative theory of ideals*, Academic press. (NewYork, **1971**).
9. A. G. Naoum and H. R. Hasen, *On multiplication modules and strong multiplication modules*, M.sc, Thesis U. Baghdad, (**1992**).
10. A. G. Naoum and A. S. Mijbass, *Weak cancellation modules*. Kyung. Math . J. 37(**1997**), 73-82.