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Characteristic for Novel Soft Type Quasinormal Operator Relies on Soft Hilbert Domain and Related Outcomes

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Abstract

The softness of the image Hilbert domain has a momentous role in the analytical characterizations of operators. Therefore, the elaborate study of these classes of soft operators (S-operators) and the comprehensive investigation of the algebraic and analytical traits of their softness Hilbert domains is of great importance. In Soft set Theory (SST), research in this specialty is of great importance. The prime effort of this article is application of novel kinds of general normal S-operator in SST, named as soft $\widehat{\Omega}$ −quasi normal operator. The crucial characteristic of the proposed operator is explained, including its major merits. Furthermore, the necessary stipulations are also discussed to verify the various merits of the proposed operator.

Keywords: FS-operator, FS-normed space, FS-inner product, FS-Hilbert space

خاصية لمؤثر شبه السوي نوع الناعم الجديد المعتمد على مجال فضاء هلبرت الناعم والنتائج ذات الصلة

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الخلاصة

هناك اهمية بالغة للصورة الناعمة في مجال هلبرت الناعم في الوصف التحليلي للمؤثرات، لذلك فان الدراسة التفصيلية لهذه الفئات من المؤثرات الناعمة (SF) والتحقق الشامل من صفاتها الجبرية والتحليلة في مجال فضاءات هلبرت الناعمة الخاصة بها له اهمية كبيرة. في نظرية المجموعات الناعمة (SST) ،يعد البحث في هذا التخصص له اهمية كبيرة لذلك فان العمل الرئيسي في هذا المقال هوة تطبيق نوع مكتشف من المؤثرات الناعمة على (SST) يسمى المؤثر الناعم شبه الاعتيادي− $\widehat{\Omega}$, حيث تم شرح الخاصية الاساسية لهذا المؤثر بما في ذلك مزاياه الرئيسية، علاوة على ذلك تمت مناقشة الشرط الضرورية لتحقق الصفات المختلفة لهذا المؤثر المقترح.

1. Introduction

In mathematical analysis, the leading area is the Functional Analysis (FA) which dates to the 18th century. The historical origins of FA lie in the realization functions spaces and relevant

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traits of function transformations, such as Fourier transform as transformations determining, for instance, unitary operators permit its acting to between function spaces. In Operator Theory (OT), this feature is crucial. Numerous research efforts have been conducted in the FA, utilize the operators, [1-10]. Moreover, operators often emerge while solving problems in a variety of themes, involving physics, complex analysis, differential equations, see [11-20]. On the other hand, Soft Set Theory (SST), originally offered by Molodtsov [21] in 1999, is a novel and interesting generalization of the mathematical approach to overcome uncertain and not plainly realized problems that cannot be simulated by classical techniques in numerous disciplines such as social science, medical sciences, economics, computer science, physics and engineering. In other words, SST is applied when there is no explicit stated mathematical modelling. This theory has acquired a great interest in the domain of mathematical modelling and decision-making. In 2002, Maji et al., [22] initiated particularly interested in applying the SST to decision-making problems, employing coarse mathematics. Later, in 2003, Maji et al., [23] also investigated and presented diverse operations related to SST, such as equality, subset, super set and complement of a soft set, absolute soft set, null soft set, soft binary operations, union and intersection operations, De Morgan's laws along with several outcomes are attained in SST. Since then, because SST is released from numerous of the difficulties that have disordered the customary theoretical approaches, sundry investigators have contributed significantly to systemically developing this theory, see [24-32]. Particularly, in connection with mathematical analysis, the notions and related outcomes of soft real sets and soft real numbers were first discussed by Das and Samanta [33] in 2012. During the year 2013, the work in terms of softness was again intensified by Das and Samanta [34-37], due to their importance in the domain of functional analysis. In this regard, they discovered a variety of soft spaces that are soft linear spaces (SF-LS), soft vector space (SF-VS), soft metric space (SF-MS), soft normed space (SF-NS), soft inner product space (SF-IPS) and soft Hilbert space (SF-HS). After then Yazar et al., [38] in 2019 studied on SF-INP and SF-HS and a number of related features and examples. In this context, the softness of the image domain has a dynamic role in the characterization of linear operators. Therefore, the comprehensive analytical and algebraic features of such operators are of crucial interest. The soft linear operator (SF-operator) defined on SF-LS was first coined by Das and Samanta [39] in 2013. They also deliberated the inverse of this operator and its prime features. Afterwards, Das and Samanta [40] in 2017 considered a variety of versions of SF-operators, namely self-adjoint SF-operator, normal SF-operator, unitary SF-operator, isometric SF-operator, and square root of positive SF-operator on SF-LS. In 2020, Aboud el al., [41] investigated distinct concepts correlating with invertible SF-operator that includes spectrum, eigenvalue, eigenvector along with eigenspace. Derived several features have been described for this operator sort. After that, in 2021, Osmin et al., [42] offered a novel version of the SF-operator on SF-HS, called a hyponormal SF-operator and inspected several features for this introduced operator. Important recent contributions include numerous general classes of SF-operators, such as [43-55]. In this article, a new SF-operator connected with the soft Hilbert domain is introduced and examined, namely soft $\widehat{\Omega}$ −quasi normal operator. The characterization merit as leading stipulation for the proposed operator is discussed. Moreover, several traits are considered to the proposed operator, embedding algebraic and analytical outcomes.

2. Elementary principles in soft theory

In this part, some important priorities in the realm of soft mathematics are mentioned.

Definition 2.1. [21] Assume that the universal set X should have the parameters make up the set E, such that $A \subseteq E$. A soft set over X is an order pair (J, A), where $J: A \to p(x)$ is a mapping from A into power sets of X.

Definition 2.2. [33] Let $\mathbb{R}$ be the set of real number, $p(\mathbb{R})$ be the set of power of $\mathbb{R}$ and let A be a set of constraints. A mapping $J : A \rightarrow p(\mathbb{R})$ is then referred to as a soft real set. The symbol for it is (J, A).
Definition 2.3. [34] Let A be a set of parameters and $\mathbb{C}$ the set of complex numbers. Let $p(\mathbb{C})$ be the collection of all non-empty bounded subsets of C. A mapping $J : A \rightarrow p(\mathbb{C})$ is then referred to as a soft complex set. The symbol for it is (J, A).
Definition 2.4. [36] Assume that A is a parameter set and that X is a vector space over a field F. Consent a special set of G over X. If $G(\beta)$ is a vector subspace of X for each $\beta \in A$, then G is described as a soft vector space of X over F.

Definition 2.5. [37] Considering V to be the purest soft vector space, the mapping is possible $\|.\|: SE(V) \longrightarrow R(A)^*$ if the following feature are requiring,
1- $\|s_e\| \geq \tilde{0}$, for all $s_e \in \tilde{V}$.
2- $\|s_e\| = \tilde{0}$, if and only if $s_e = 0$.
3- $\|\alpha s_e\| = |\tilde{\alpha}|\|s_e\|$, for all $s_e \in \tilde{V}$ and for every soft scalar $\tilde{\alpha}$.
4- $\|s_e + t_e\| \leq \|s_e\| + \|t_e\|$ for all s_e, $t_e \in \tilde{V}$.
Which is referred to as a soft norm on the soft vector space $\tilde{V}$.
Definition 2.6. [40] Let $SE(\tilde{V})$ be the mapping of a soft vector space. $\langle .,. \rangle: SE(\tilde{V}) \times SE(\tilde{V}) \rightarrow R(A)^*$ is called a soft inner product on $SE(\tilde{V})$ and for every soft real number $\tilde{\alpha}$:
1- $\langle s_e, s_e \rangle \geq \tilde{0}$ and $\langle s_e, s_e \rangle = \tilde{0}$ if and only if $s_e = \tilde{0}$.
2- $\langle t_e, s_e \rangle = \langle s_e, t_e \rangle$.
3- $\langle s_e, \tilde{\alpha} t_e \rangle = \tilde{\alpha}\langle s_e, t_e \rangle = \langle \tilde{\alpha} s_e, t_e \rangle$
4- $\langle s_e, w_e \rangle + \langle t_e, w_e \rangle = \langle s_e + t_e, w_e \rangle$, the triple $(SE(\tilde{V}), \langle .,. \rangle)$ is called soft inner product space.
Definition 2.7. [38] A soft sequence $\{s_{en}^{\mathfrak{n}}\}$ in soft normed space $(\tilde{V}, \widetilde{\|.\|})$ is called:
1- Soft convergent, if for ε, there is $\mathfrak{n}_0 \in \mathcal{N}$ such that $\|s_{en}^{\mathfrak{n}} - s_e\| < \varepsilon$ for each $\mathfrak{n} \geq \mathfrak{n}_0$. That is to say that $\|s_{en}^{\mathfrak{n}} - s_e\| \rightarrow 0$ as $\mathfrak{n}, \mathfrak{m} \rightarrow \infty$ and s_e is called soft convergent point.
2- Soft Cauchy, if for ε, there is $\mathfrak{n}_0 \in \mathcal{N}$ such that $\|s_{en}^{\mathfrak{n}} - s_{em}^{\mathfrak{m}}\| < \varepsilon$ for each $\mathfrak{n}$, $\mathfrak{m} \geq \mathfrak{n}_0, \mathfrak{n} > \mathfrak{m}$. That is to say that $\|s_{en}^{\mathfrak{n}} - s_{em}^{\mathfrak{m}}\| \rightarrow 0$ as $\mathfrak{n}, \mathfrak{m} \rightarrow \infty$.
Definition 2.8. [38] The soft complete inner product space $(SE(\tilde{V}), \langle .,. \rangle)$ is also known as a soft Hilbert space (SH-space) and is indicated by $SE(H)$.
Definition 2.9. [37] Let $\widehat{\Omega}: SE(H) \rightarrow SE(H)$ be a soft operator in SH-space H, then $\widehat{\Omega}$ is namely soft linear operator (SL-operator), then $\widehat{\Omega} \in \mathcal{L}(H)$ if for every $s_{e1}, s_{e2} \in H$ attain $\widehat{\Omega}(\alpha s_{e1} + \beta s_{e2}) = \alpha\widehat{\Omega}(s_{e1}) + \beta\widehat{\Omega}(s_{e2})$, and α, β are S −scalars.
Definition 2.10. [40] Let $\widehat{\Omega}: SE(H) \rightarrow SE(H)$ be S-operator, is called a soft bounded operator (SB-operator), if there is $\mathfrak{m} \in R^+(A)$ such that $\|\widehat{\Omega}(s_{e1})\| \leq \mathfrak{m}\|s_{e1}\|$, for every $s_{e1} \in SE(H)$.

3. Proposed generalized soft operator and its traits

This part of the article includes the most important essential characteristics of the operator which are studied through the theories presented.
Definition 3.1. A soft bounded operator linear $\widehat{\Omega}: SE(H) \rightarrow SE(H)$ namely soft $\widehat{\Omega}$ −quasi normal operator if $\widehat{\Omega}^*((\widehat{\Omega}^*)^{\mathfrak{n}}\widehat{\Omega}) = ((\widehat{\Omega}^*)^{\mathfrak{n}}\widehat{\Omega})\widehat{\Omega}^*$, $\mathfrak{n} \in \mathcal{N}$, where $\widehat{\Omega}^*$ is soft djoint of operator.
To illustrate this definition, one can see the operator matrix, $\widehat{\Omega} = [\tilde{0}\ \tilde{0}\ \tilde{1}\ \tilde{0}]$ to be soft $\widehat{\Omega}$ −quasi normal operator where $\mathfrak{n} > 2$, but not satisfy in the case $\mathfrak{n} = 1$. These proofs now provide certain properties of the soft $\widehat{\Omega}$ −quasi normal operator.
Remark 3.2. It is exciting to note that every soft normal operator is soft $\widehat{\Omega}$ −quasi normal operator.
Proposition 3.3. Let $\widehat{\Omega}: SE(H) \rightarrow SE(H)$ is soft $\widehat{\Omega}$ −quasi normal operator, then

1- $\alpha\widehat{\Omega}$ is a soft $\widehat{\Omega}-$ quasi normal operator, where $\alpha \in R(A)$.
2- $\frac{\widehat{\Omega}}{\widehat{\wp}}$ is a soft $\widehat{\Omega}-$ quasi normal operator, where $\widehat{\wp}$ is closed sub-space.
3- $\widehat{\Omega}^*$ is a soft $\widehat{\Omega}-$ quasi normal operator, where $\widehat{\Omega}$ is soft normal operator.

Proof. 1- $(\alpha\,\widehat{\Omega})^*\left((\alpha\,\widehat{\Omega})^*\right)^{\mathfrak{n}}(\alpha\,\widehat{\Omega}) = \alpha\,\widehat{\Omega}^*\,\left(\alpha^{\mathfrak{n}}(\widehat{\Omega}^*)\right)^{\mathfrak{n}}\,(\alpha\,\widehat{\Omega})$

$$= \alpha\alpha^{\mathfrak{n}}\alpha\,\widehat{\Omega}^*\,(\widehat{\Omega}^*)^{\mathfrak{n}}\,\widehat{\Omega}$$
$$= \alpha^{\mathfrak{n}}\alpha\alpha\,(\widehat{\Omega}^*)^{\mathfrak{n}}\,\widehat{\Omega}\,\widehat{\Omega}^*$$
$$= \left(\alpha^{\mathfrak{n}}(\widehat{\Omega}^*)\right)^{\mathfrak{n}}\,(\alpha\,\widehat{\Omega})\,\alpha\widehat{\Omega}^*$$
$$= \left((\alpha\,\widehat{\Omega})^*\right)^{\mathfrak{n}}(\alpha\,\widehat{\Omega})\,(\alpha\,\widehat{\Omega})^*.$$

Therefore, $\alpha\,\widehat{\Omega}$ is soft $\widehat{\Omega}-$quasi normal operator.

2- $\left(\frac{\widehat{\Omega}}{\widehat{\wp}}\right)^*\left(\left(\frac{\widehat{\Omega}}{\widehat{\wp}}\right)^{\mathfrak{n}}\left(\frac{\widehat{\Omega}}{\widehat{\wp}}\right)\right) = \left(\frac{\widehat{\Omega}^*}{\widehat{\wp}}\right)\left(\left(\frac{(\widehat{\Omega}^*)^{\mathfrak{n}}}{\widehat{\wp}}\right)\left(\frac{\widehat{\Omega}^*}{\widehat{\wp}}\right)\right).$

Using some basic steps lead to

$$= \left(\frac{\widehat{\Omega}^*}{\widehat{\wp}}\right)\left(\frac{(\widehat{\Omega}^*)^{\mathfrak{n}}\,\widehat{\Omega}}{\widehat{\wp}}\right) = \left(\frac{\widehat{\Omega}^*\,(\widehat{\Omega}^*)^{\mathfrak{n}}\,\widehat{\Omega}}{\widehat{\wp}}\right)$$
$$= \left(\frac{(\widehat{\Omega}^*)^{\mathfrak{n}}\,\widehat{\Omega}\,\widehat{\Omega}^*}{\widehat{\wp}}\right)$$
$$= \left(\frac{(\widehat{\Omega}^*)^{\mathfrak{n}}\,\widehat{\Omega}}{\widehat{\wp}}\right)\left(\frac{\widehat{\Omega}^*}{\widehat{\wp}}\right)$$
$$= \left(\left(\frac{(\widehat{\Omega}^*)^{\mathfrak{n}}}{\widehat{\wp}}\right)\left(\frac{\widehat{\Omega}}{\widehat{\wp}}\right)\right)\left(\frac{\widehat{\Omega}^*}{\widehat{\wp}}\right).$$

So, $\frac{\widehat{\Omega}}{\widehat{\wp}}$ is a soft operator of the type soft $\widehat{\Omega}-$quasi normal operator. This proposition is weak hereditary property only satisfies on closed subspace.

3- $(\widehat{\Omega}^*)^*\left((\widehat{\Omega}^*)^*\right)^{\mathfrak{n}}(\widehat{\Omega}^*) = \left((\widehat{\Omega}^*)^*\right)^{\mathfrak{n}}\,(\widehat{\Omega}^*)^*\,(\widehat{\Omega}^*)$

$$= \left((\widehat{\Omega}^*)^*\right)^{\mathfrak{n}}(\widehat{\Omega}\,\widehat{\Omega}^*)^*$$
$$= \left((\widehat{\Omega}^*)^*\right)^{\mathfrak{n}}\,(\widehat{\Omega}^*\,\widehat{\Omega}\,)^*$$
$$= \left(\left((\widehat{\Omega}^*)^*\right)^{\mathfrak{n}}\widehat{\Omega}^*\right)(\widehat{\Omega}^*)^*.$$

Therefore, $\widehat{\Omega}^*$ is soft $\widehat{\Omega}-$quasi normal operator.

Proposition 3.4. Let $\widehat{\Omega}: SE(H) \to SE(H)$ is soft $\widehat{\Omega}-$quasi normal operator, then

$$\left[\widehat{\Omega}^*\left((\widehat{\Omega}^*)^{\mathfrak{n}}\,\widehat{\Omega}\right)\right]^{\mathfrak{q}} = \left[\left((\widehat{\Omega}^*)^{\mathfrak{n}}\,\widehat{\Omega}\right)\,\widehat{\Omega}^*\right]^{\mathfrak{q}}, \qquad \mathfrak{q},\mathfrak{n} \in \mathcal{N}.$$

Proof. Now, this proposition using the role of mathematical induction for proving its since soft $\widehat{\Omega}-$quasi normal operator the result is true for $\mathfrak{q} = 1$, so

$$\widehat{\Omega}^*\left((\widehat{\Omega}^*)^{\mathfrak{n}}\,\widehat{\Omega}\right) = \left((\widehat{\Omega}^*)^{\mathfrak{n}}\,\widehat{\Omega}\right)\,\widehat{\Omega}^*\,. \tag{1}$$

Assumption this law satisfies as $\mathfrak{q} = \mathfrak{p}$,

$$\left[\widehat{\Omega}^*\left((\widehat{\Omega}^*)^{\mathfrak{n}}\,\widehat{\Omega}\right)\right]^{\mathfrak{p}} = \left[\left((\widehat{\Omega}^*)^{\mathfrak{n}}\,\widehat{\Omega}\right)\,\widehat{\Omega}^*\right]^{\mathfrak{p}}. \tag{2}$$

Must show this law as $\mathfrak{q} = \mathfrak{p}+1$ up on steps (1) and (2) from this proposition in order to prove this result

$$\left[\widehat{\Omega}^*\left((\widehat{\Omega}^*)^{\mathfrak{n}}\,\widehat{\Omega}\right)\right]^{\mathfrak{p}+1} = \left[\widehat{\Omega}^*\left((\widehat{\Omega}^*)^{\mathfrak{n}}\,\widehat{\Omega}\right)\right]^{\mathfrak{p}}\left[\widehat{\Omega}^*\left((\widehat{\Omega}^*)^{\mathfrak{n}}\,\widehat{\Omega}\right)\right].$$

From Equations (1) and (2), leads to

$$\left[\widehat{\Omega}^*\left((\widehat{\Omega}^*)^{\mathfrak{n}}\,\widehat{\Omega}\right)\right]^{\mathfrak{p}+1} = \left[\left((\widehat{\Omega}^*)^{\mathfrak{n}}\,\widehat{\Omega}\right)\,\widehat{\Omega}^*\right]^{\mathfrak{p}}\left[\left((\widehat{\Omega}^*)^{\mathfrak{n}}\,\widehat{\Omega}\right)\,\widehat{\Omega}^*\right] = \left[\left((\widehat{\Omega}^*)^{\mathfrak{n}}\,\widehat{\Omega}\right)\,\widehat{\Omega}^*\right]^{\mathfrak{p}+1}.$$

So, we show the law $\mathfrak{q} = \mathfrak{p}+1$, so one can have

$$\left[\widehat{\Omega}^*\left((\widehat{\Omega}^*)^{\mathfrak{n}}\,\widehat{\Omega}\right)\right]^{\mathfrak{q}} = \left[\left((\widehat{\Omega}^*)^{\mathfrak{n}}\,\widehat{\Omega}\right)\,\widehat{\Omega}^*\right]^{\mathfrak{q}}.$$

Proposition 3.5. Every soft self-adjoint operator is a soft $\widehat{\Omega}$ −quasi normal operator.

Proof. Notice that $\widehat{\Omega}^*\left(\left(\widehat{\Omega}^*\right)^{\pi}\widehat{\Omega}\right) = \widehat{\Omega}\left(\widehat{\Omega}^{\pi}\,\widehat{\Omega}\right) = \widehat{\Omega}^{\pi+2}$ and $\left(\left(\widehat{\Omega}^*\right)^{\pi}\widehat{\Omega}\right)\widehat{\Omega}^* = \left(\widehat{\Omega}^{\pi}\right)\widehat{\Omega}\,\widehat{\Omega} = \widehat{\Omega}^{\pi+2}$. Therefore, it is a soft $\widehat{\Omega}$ −quasi normal operator.

Corollary 3.6. Given any soft operator $\widehat{\Omega}$ on $SE(H)$, then $\left(\widehat{\Omega} + \widehat{\Omega}^*\right)$, $\widehat{\Omega}\,\widehat{\Omega}^*$, $\widehat{\Omega}^*\widehat{\Omega}$, $I + \widehat{\Omega}^*\widehat{\Omega}$ and $I + \widehat{\Omega}\,\widehat{\Omega}^*$ are soft $\widehat{\Omega}$ −quasi normal operator.

Remark 3.7. Given two soft $\widehat{\Omega}$ −quasi normal operators, $\widehat{\Omega}_1$ and $\widehat{\Omega}_2$ it can be said that $\widehat{\Omega}_1 + \widehat{\Omega}_2$ is not always a soft $\widehat{\Omega}$ −quasi normal operator. Take the following instance to demonstrate that.

Example 3.8. If $\widehat{\Omega}_1 = \left[\hat{0}\ \widehat{-1}\ \hat{1}\ \hat{0}\right]$ and $\widehat{\Omega}_2 = \left[\hat{1}\ \hat{0}\ \hat{0}\ \hat{0}\right]$ are two soft $\widehat{\Omega}$ −quasi normal operators. If $\pi = 2$, then

$$\left(\widehat{\Omega}_1 + \widehat{\Omega}_2\right)^*\left(\left(\left(\widehat{\Omega}_1 + \widehat{\Omega}_2\right)^*\right)^2\left(\widehat{\Omega}_1 + \widehat{\Omega}_2\right)\right) = \left[\hat{1}\ \hat{1}\ \hat{0}\ \hat{0}\right]\left(\left[\hat{1}\ \hat{1}\ \hat{0}\ \hat{0}\right]\ \left[\hat{1}\ \hat{0}\ \hat{1}\ \hat{0}\right]\right)$$
$$= \left[\hat{1}\ \hat{0}\ \hat{0}\ \hat{0}\right], \quad (1)$$

and

$$\left(\left(\left(\widehat{\Omega}_1 + \widehat{\Omega}_2\right)^*\right)^2\left(\widehat{\Omega}_1 + \widehat{\Omega}_2\right)\right)\left(\widehat{\Omega}_1 + \widehat{\Omega}_2\right)^* = \left(\left[\hat{1}\ \hat{1}\ \hat{0}\ \hat{0}\right]\ \left[\hat{1}\ \hat{0}\ \hat{1}\ \hat{0}\right]\right)\left[\hat{1}\ \hat{1}\ \hat{0}\ \hat{0}\right]$$
$$= \left[\hat{1}\ \hat{1}\ \hat{0}\ \hat{0}\right]. \quad (2)$$

Thus, from (1) and (2), we yield that $\widehat{\Omega}_1 + \widehat{\Omega}_2$ is not soft $\widehat{\Omega}$ −quasi normal operator.

Theorem 3.9. Assumption $\widehat{\Omega}_1, \widehat{\Omega}_2 : SE(H) \to SE(H)$ be two soft $\widehat{\Omega}$ −quasi normal operators which is known on in a soft Hilbert space with conditions satisfies $\widehat{\Omega}_1 + \widehat{\Omega}_2$ soft normal operator.

Proof. By using the definition of is soft $\widehat{\Omega}$ −quasi normal operator

$$\left(\widehat{\Omega}_1 + \widehat{\Omega}_2\right)^*\left(\left(\left(\widehat{\Omega}_1 + \widehat{\Omega}_2\right)^*\right)^{\pi}\left(\widehat{\Omega}_1 + \widehat{\Omega}_2\right)\right) = \left(\widehat{\Omega}_1^* + \widehat{\Omega}_2^*\right)\left(\left(\left(\widehat{\Omega}_1 + \widehat{\Omega}_2\right)^{\pi}\right)^*\left(\widehat{\Omega}_1 + \widehat{\Omega}_2\right)\right)$$
$$= \left(\widehat{\Omega}_1^* + \widehat{\Omega}_2^*\right)\left(\left(\sum_{\kappa=0}^{\pi}(\pi\ \ \kappa)\ \widehat{\Omega}_1^{\pi-\kappa}\ \widehat{\Omega}_2^{\kappa}\right)^*\left(\widehat{\Omega}_1 + \widehat{\Omega}_2\right)\right),$$

we gain

$$= \left(\widehat{\Omega}_1^* + \widehat{\Omega}_2^*\right)\left(\left(\widehat{\Omega}_1^{\pi} + \sum_{\kappa=0}^{\pi}(\pi\ \ \kappa)\ \widehat{\Omega}_1^{\pi-\kappa}\ \widehat{\Omega}_2^{\kappa} + \widehat{\Omega}_2^{\pi}\right)^*\left(\widehat{\Omega}_1 + \widehat{\Omega}_2\right)\right)$$
$$= \widehat{\Omega}_1^*\left(\left(\widehat{\Omega}_1^{\pi} + \sum_{\kappa=0}^{\pi}(\pi\ \ \kappa)\ \widehat{\Omega}_1^{\pi-\kappa}\ \widehat{\Omega}_2^{\kappa} + \widehat{\Omega}_2^{\pi}\right)^*\left(\widehat{\Omega}_1 + \widehat{\Omega}_2\right)\right)$$
$$+\widehat{\Omega}_2^*\left(\left(\widehat{\Omega}_1^{\pi} + \sum_{\kappa=0}^{\pi}(\pi\ \ \kappa)\ \widehat{\Omega}_1^{\pi-\kappa}\ \widehat{\Omega}_2^{\kappa} + \widehat{\Omega}_2^{\pi}\right)^*\left(\widehat{\Omega}_1 + \widehat{\Omega}_2\right)\right)$$
$$= \left(\widehat{\Omega}_1^*\left(\widehat{\Omega}_1^{\pi} + \sum_{\kappa=0}^{\pi}(\pi\ \ \kappa)\ \widehat{\Omega}_1^{\pi-\kappa}\ \widehat{\Omega}_2^{\kappa} + \widehat{\Omega}_2^{\pi}\right)^*\right)\left(\widehat{\Omega}_1 + \widehat{\Omega}_2\right)$$
$$+\left(\widehat{\Omega}_1^*\left(\widehat{\Omega}_1^{\pi} + \sum_{\kappa=0}^{\pi}(\pi\ \ \kappa)\ \widehat{\Omega}_1^{\pi-\kappa}\ \widehat{\Omega}_2^{\kappa} + \widehat{\Omega}_2^{\pi}\right)^*\right)\left(\widehat{\Omega}_1 + \widehat{\Omega}_2\right)$$
$$= \left(\widehat{\Omega}_1^*\left(\widehat{\Omega}_1^{\pi} + \sum_{\kappa=0}^{\pi}(\pi\ \ \kappa)\ \widehat{\Omega}_1^{\pi-\kappa}\ \widehat{\Omega}_2^{\kappa} + \widehat{\Omega}_2^{\pi}\right)\right)^*\left(\widehat{\Omega}_1 + \widehat{\Omega}_2\right)$$

$$+\left(\widehat{\Omega}_2^*\left(\widehat{\Omega}_1^{\mathfrak{n}}+\sum_{\kappa=0}^{\mathfrak{n}}(\mathfrak{n}\ \ \kappa)\ \widehat{\Omega}_1^{\mathfrak{n}-\kappa}\ \widehat{\Omega}_2^{\kappa}+\widehat{\Omega}_2^{\mathfrak{n}}\right)^*\right)\left(\widehat{\Omega}_1+\widehat{\Omega}_2\right)$$

$$=\left(\left(\widehat{\Omega}_1^{\mathfrak{n}}+\sum_{\kappa=0}^{\mathfrak{n}}(\mathfrak{n}\ \ \kappa)\ \widehat{\Omega}_1^{\mathfrak{n}-\kappa}\ \widehat{\Omega}_2^{\kappa}+\widehat{\Omega}_2^{\mathfrak{n}}\right)\widehat{\Omega}_1^*\right)^*\left(\widehat{\Omega}_1+\widehat{\Omega}_2\right)$$

$$+\left(\left(\widehat{\Omega}_1^{\mathfrak{n}}+\sum_{\kappa=0}^{\mathfrak{n}}(\mathfrak{n}\ \ \kappa)\ \widehat{\Omega}_1^{\mathfrak{n}-\kappa}\ \widehat{\Omega}_2^{\kappa}+\widehat{\Omega}_2^{\mathfrak{n}}\right)\widehat{\Omega}_2^*\right)^*\left(\widehat{\Omega}_1+\widehat{\Omega}_2\right)$$

$$=\left(\widehat{\Omega}_1^{\mathfrak{n}}+\sum_{\kappa=0}^{\mathfrak{n}}(\mathfrak{n}\ \ \kappa)\ \widehat{\Omega}_1^{\mathfrak{n}-\kappa}\ \widehat{\Omega}_2^{\kappa}+\widehat{\Omega}_2^{\mathfrak{n}}\right)^*\widehat{\Omega}_1^*\left(\widehat{\Omega}_1+\widehat{\Omega}_2\right)$$

$$+\left(\widehat{\Omega}_1^{\mathfrak{n}}+\sum_{\kappa=0}^{\mathfrak{n}}(\mathfrak{n}\ \ \kappa)\ \widehat{\Omega}_1^{\mathfrak{n}-\kappa}\ \widehat{\Omega}_2^{\kappa}+\widehat{\Omega}_2^{\mathfrak{n}}\right)^*\widehat{\Omega}_2^*\left(\widehat{\Omega}_1+\widehat{\Omega}_2\right)$$

$$=\left(\widehat{\Omega}_1^{\mathfrak{n}}+\sum_{\kappa=0}^{\mathfrak{n}}(\mathfrak{n}\ \ \kappa)\ \widehat{\Omega}_1^{\mathfrak{n}-\kappa}\ \widehat{\Omega}_2^{\kappa}+\widehat{\Omega}_2^{\mathfrak{n}}\right)^*\left(\widehat{\Omega}_1+\widehat{\Omega}_2\right)\widehat{\Omega}_1^*$$

$$+\left(\widehat{\Omega}_1^{\mathfrak{n}}+\sum_{\kappa=0}^{\mathfrak{n}}(\mathfrak{n}\ \ \kappa)\ \widehat{\Omega}_1^{\mathfrak{n}-\kappa}\ \widehat{\Omega}_2^{\kappa}+\widehat{\Omega}_2^{\mathfrak{n}}\right)^*\left(\widehat{\Omega}_1+\widehat{\Omega}_2\right)\widehat{\Omega}_2^*$$

$$=\left[\left(\widehat{\Omega}_1^{\mathfrak{n}}+\sum_{\kappa=0}^{\mathfrak{n}}(\mathfrak{n}\ \ \kappa)\ \widehat{\Omega}_1^{\mathfrak{n}-\kappa}\ \widehat{\Omega}_2^{\kappa}+\widehat{\Omega}_2^{\mathfrak{n}}\right)^*\left(\widehat{\Omega}_1+\widehat{\Omega}_2\right)\right]\left(\widehat{\Omega}_1^*+\widehat{\Omega}_2^*\right)$$

$$=\left[\left(\widehat{\Omega}_1^{\mathfrak{n}}+\sum_{\kappa=0}^{\mathfrak{n}}(\mathfrak{n}\ \ \kappa)\ \widehat{\Omega}_1^{\mathfrak{n}-\kappa}\ \widehat{\Omega}_2^{\kappa}+\widehat{\Omega}_2^{\mathfrak{n}}\right)^*\left(\widehat{\Omega}_1+\widehat{\Omega}_2\right)\right]\left(\widehat{\Omega}_1+\widehat{\Omega}_2\right)^*.$$

Hence, $\widehat{\Omega}_1+\widehat{\Omega}_2$ is soft $\widehat{\Omega}$ −quasi normal operator.

Remark 3.10. Examine what follows as an example to show that even if $\widehat{\Omega}_1$ and $\widehat{\Omega}_2$ are two soft $\widehat{\Omega}$ −quasi normal operators, $\widehat{\Omega}_1\,\widehat{\Omega}_2$ are not necessarily soft $\widehat{\Omega}$ −quasi normal operators.

Example 3.11. $\widehat{\Omega}_1=[\hat{0}\ \widehat{-1}\ \hat{1}\ \hat{0}]$ and $\widehat{\Omega}_2=[\hat{1}\ \hat{1}\ \hat{1}\ \hat{1}]$ are two soft $\widehat{\Omega}$ −quasi normal operators. If $\mathfrak{n}=2$, then

$$\left(\widehat{\Omega}_1\,\widehat{\Omega}_2\right)^*\left(\left(\left(\widehat{\Omega}_1\,\widehat{\Omega}_2\right)^*\right)^2\left(\widehat{\Omega}_1\,\widehat{\Omega}_2\right)\right)=[\hat{0}\ \hat{1}\ \hat{0}\ \hat{1}]\,([\hat{0}\ \hat{1}\ \hat{0}\ \hat{1}]\ [\hat{0}\ \hat{0}\ \hat{1}\ \hat{1}])=[\hat{1}\ \hat{1}\ \hat{1}\ \hat{1}], \quad (1)$$

and

$$\left(\left(\left(\widehat{\Omega}_1\,\widehat{\Omega}_2\right)^*\right)^2\left(\widehat{\Omega}_1\,\widehat{\Omega}_2\right)\right)\left(\widehat{\Omega}_1\,\widehat{\Omega}_2\right)^*=([\hat{0}\ \hat{1}\ \hat{0}\ \hat{1}]\ [\hat{0}\ \hat{0}\ \hat{1}\ \hat{1}])\,[\hat{0}\ \hat{1}\ \hat{0}\ \hat{1}]=[\hat{0}\ \hat{1}\ \hat{0}\ \hat{1}]. \quad (2)$$

Thus, from (1) and (2), we yield that $\widehat{\Omega}_1\,\widehat{\Omega}_2$ is not soft $\widehat{\Omega}$ −quasi normal operator.

Theorem 3.12. Let $\widehat{\Omega}_1,\widehat{\Omega}_2 : SE(H)\to SE(H)$ be two soft $\widehat{\Omega}$ −quasi normal operators such that $\widehat{\Omega}_1\,\widehat{\Omega}_2=\widehat{\Omega}_2\,\widehat{\Omega}_1$ and $\widehat{\Omega}_2^*\,\widehat{\Omega}_1=\widehat{\Omega}_1\widehat{\Omega}_2^*$ $\widehat{\Omega}_1\,\widehat{\Omega}_2$ soft normal operator.

Proof.

By using the definition of is a soft $\widehat{\Omega}$ −quasi normal operator and hypothesis of this theorem

$$\left(\widehat{\Omega}_1\,\widehat{\Omega}_2\right)^*\left(\left(\left(\widehat{\Omega}_1\,\widehat{\Omega}_2\right)^*\right)^{\mathfrak{n}}\left(\widehat{\Omega}_1\,\widehat{\Omega}_2\right)\right)=\left(\widehat{\Omega}_2^*\,\widehat{\Omega}_1^*\right)\left(\left(\left(\widehat{\Omega}_1\,\widehat{\Omega}_2\right)^*\right)^{\mathfrak{n}}\left(\widehat{\Omega}_1\,\widehat{\Omega}_2\right)\right).$$

So, having

$$=\left(\widehat{\Omega}_1^*\,\widehat{\Omega}_2^*\right)\left(\left(\widehat{\Omega}_2\,\widehat{\Omega}_1\right)^*\right)^{\mathfrak{n}}\left(\widehat{\Omega}_1\,\widehat{\Omega}_2\right)$$

$$
\begin{aligned}
&= \left(\hat{\Omega}_1^{*}\,\hat{\Omega}_2^{*}\right)\hat{\Omega}_1^{\mathfrak{n}}\,\hat{\Omega}_2^{\mathfrak{n}}\left(\hat{\Omega}_1\,\hat{\Omega}_2\right)\\
&= \hat{\Omega}_1^{*}\left(\hat{\Omega}_1^{\mathfrak{n}}\,\hat{\Omega}_2^{*}\right)\hat{\Omega}_2^{\mathfrak{n}}\left(\hat{\Omega}_1\,\hat{\Omega}_2\right)\\
&= \hat{\Omega}_1^{*}\,\hat{\Omega}_1^{\mathfrak{n}}\,\hat{\Omega}_2^{*}\left(\hat{\Omega}_2^{*\mathfrak{n}}\,\hat{\Omega}_1\right)\hat{\Omega}_2\\
&= \hat{\Omega}_1^{*}\,\hat{\Omega}_1^{\mathfrak{n}}\left(\hat{\Omega}_2^{*}\,\hat{\Omega}_1\right)\hat{\Omega}_2^{*\mathfrak{n}}\,\hat{\Omega}_2\\
&= \hat{\Omega}_1^{*}\,\hat{\Omega}_1^{\mathfrak{n}}\,\hat{\Omega}_1\,\hat{\Omega}_2^{*}\,\hat{\Omega}_2^{*\mathfrak{n}}\,\hat{\Omega}_2\\
&= \hat{\Omega}_1^{*}\left(\hat{\Omega}_1^{\mathfrak{n}}\,\hat{\Omega}_1\right)\hat{\Omega}_2^{*}\left(\hat{\Omega}_2^{*\mathfrak{n}}\,\hat{\Omega}_2\right)\\
&= \left(\hat{\Omega}_1^{\mathfrak{n}}\,\hat{\Omega}_1\right)\hat{\Omega}_1^{*}\left(\hat{\Omega}_2^{*\mathfrak{n}}\,\hat{\Omega}_2\right)\hat{\Omega}_2^{*}\\
&= \hat{\Omega}_1^{\mathfrak{n}}\,\hat{\Omega}_1\left(\hat{\Omega}_2^{\mathfrak{n}}\,\hat{\Omega}_1\right)^{*}\hat{\Omega}_2\,\hat{\Omega}_2^{*}\\
&= \hat{\Omega}_1^{\mathfrak{n}}\,\hat{\Omega}_1\left(\hat{\Omega}_1\,\hat{\Omega}_2^{\mathfrak{n}}\right)^{*}\hat{\Omega}_2\,\hat{\Omega}_2^{*}\\
&= \hat{\Omega}_1^{*\mathfrak{n}}\,\hat{\Omega}_1\,\hat{\Omega}_2^{*\mathfrak{n}}\left(\hat{\Omega}_1^{*}\,\hat{\Omega}_2\right)^{*}\hat{\Omega}_2^{*}\\
&= \hat{\Omega}_1^{*\mathfrak{n}}\,\hat{\Omega}_2^{*\mathfrak{n}}\,\hat{\Omega}_1\,\hat{\Omega}_2\,\hat{\Omega}_1^{*}\,\hat{\Omega}_2^{*}\\
&= \left(\hat{\Omega}_1\,\hat{\Omega}_2\right)^{*\mathfrak{n}}\left(\hat{\Omega}_1\,\hat{\Omega}_2\right)\left(\hat{\Omega}_1\,\hat{\Omega}_2\right)^{*}.
\end{aligned}
$$

Therefore, the product $\hat{\Omega}_1\,\hat{\Omega}_2$ is a soft $\hat{\Omega}$ −quasi normal operator.

Theorem 2.13. Let $\hat{\Omega}_1, \hat{\Omega}_2, \dots, \hat{\Omega}_{\mathfrak{n}}: SE(H) \to SE(H)$ be soft $\hat{\Omega}$ −quasi normal operators, then $\hat{\Omega}_1 \oplus \hat{\Omega}_2 \oplus \dots \oplus \hat{\Omega}_{\mathfrak{n}}$ is soft $\hat{\Omega}$ −quasi normal operator, where $\oplus$ is direct sum

Proof.

By using the definition of direct sum

$$
\begin{aligned}
&\left(\hat{\Omega}_1 \oplus \hat{\Omega}_2 \oplus \dots \oplus \hat{\Omega}_{\mathfrak{n}}\right)^{*}\left(\left(\left(\hat{\Omega}_1 \oplus \hat{\Omega}_2 \oplus \dots \oplus \hat{\Omega}_{\mathfrak{n}}\right)^{*}\right)^{\mathfrak{n}}\left(\hat{\Omega}_1 \oplus \hat{\Omega}_2 \oplus \dots \oplus \hat{\Omega}_{\mathfrak{n}}\right)\right)\\
&= \left(\hat{\Omega}_1^{*} \oplus \hat{\Omega}_2^{*} \oplus \dots \oplus \hat{\Omega}_{\mathfrak{n}}^{*}\right)\left(\left(\hat{\Omega}_1^{*}\right)^{\mathfrak{n}} \oplus \left(\hat{\Omega}_2^{*}\right)^{\mathfrak{n}} \oplus \dots \oplus \left(\hat{\Omega}_{\mathfrak{n}}^{*}\right)^{\mathfrak{n}}\right)\left(\hat{\Omega}_1 \oplus \hat{\Omega}_2 \oplus \dots \oplus \hat{\Omega}_{\mathfrak{n}}\right)\\
&= \left(\hat{\Omega}_1^{*}\left(\hat{\Omega}_1^{*}\right)^{\mathfrak{n}}\hat{\Omega}_1 \oplus \hat{\Omega}_2^{*}\left(\hat{\Omega}_2^{*}\right)^{\mathfrak{n}}\hat{\Omega}_2 \oplus \dots \oplus \hat{\Omega}_{\mathfrak{n}}^{*}\left(\hat{\Omega}_{\mathfrak{n}}^{*}\right)^{\mathfrak{n}}\hat{\Omega}_{\mathfrak{n}}\right)\\
&= \left(\left(\hat{\Omega}_1^{*}\right)^{\mathfrak{n}}\hat{\Omega}_1\,\hat{\Omega}_1^{*} \oplus \left(\hat{\Omega}_2^{*}\right)^{\mathfrak{n}}\hat{\Omega}_2\,\hat{\Omega}_2^{*} \oplus \dots \oplus \left(\hat{\Omega}_{\mathfrak{n}}^{*}\right)^{\mathfrak{n}}\hat{\Omega}_{\mathfrak{n}}\,\hat{\Omega}_{\mathfrak{n}}^{*}\right)\\
&= \left(\left(\hat{\Omega}_1^{*}\right)^{\mathfrak{n}} \oplus \left(\hat{\Omega}_2^{*}\right)^{\mathfrak{n}} \oplus \dots \oplus \left(\hat{\Omega}_{\mathfrak{n}}^{*}\right)^{\mathfrak{n}}\right)\left(\hat{\Omega}_1 \oplus \hat{\Omega}_2 \oplus \dots \oplus \hat{\Omega}_{\mathfrak{n}}\right)\left(\hat{\Omega}_1^{*} \oplus \hat{\Omega}_2^{*} \oplus \dots \oplus \hat{\Omega}_{\mathfrak{n}}^{*}\right)\\
&= \left(\left(\hat{\Omega}_1 \oplus \hat{\Omega}_2 \oplus \dots \oplus \hat{\Omega}_{\mathfrak{n}}\right)^{*}\right)^{\mathfrak{n}}\left(\hat{\Omega}_1 \oplus \hat{\Omega}_2 \oplus \dots \oplus \hat{\Omega}_{\mathfrak{n}}\right)\left(\hat{\Omega}_1 \oplus \hat{\Omega}_2 \oplus \dots \oplus \hat{\Omega}_{\mathfrak{n}}\right)^{*}.
\end{aligned}
$$

So, one can obtain $\hat{\Omega}_1 \oplus \hat{\Omega}_2 \oplus \dots \oplus \hat{\Omega}_{\mathfrak{n}}$ is soft $\hat{\Omega}$ −quasi normal operator.

Theorem 2.14. Let $\hat{\Omega}_1, \hat{\Omega}_2, \dots, \hat{\Omega}_{\mathfrak{n}}: SE(H) \to SE(H)$ be soft $\hat{\Omega}$ −quasi normal operators, then $\hat{\Omega}_1 \otimes \hat{\Omega}_2 \otimes \dots \otimes \hat{\Omega}_{\mathfrak{n}}$ is soft $\hat{\Omega}$ −quasi normal operator, where $\otimes$ tensor product

Proof.

$$
\begin{aligned}
&\left(\hat{\Omega}_1 \otimes \hat{\Omega}_2 \otimes \dots \otimes \hat{\Omega}_{\mathfrak{n}}\right)^{*}\left(\left(\left(\hat{\Omega}_1 \otimes \hat{\Omega}_2 \otimes \dots \otimes \hat{\Omega}_{\mathfrak{n}}\right)^{*}\right)^{\mathfrak{n}}\left(\hat{\Omega}_1 \otimes \hat{\Omega}_2 \otimes \dots \otimes \hat{\Omega}_{\mathfrak{n}}\right)\right)\\
&= \left(\hat{\Omega}_1^{*} \otimes \hat{\Omega}_2^{*} \oplus \dots \otimes \hat{\Omega}_{\mathfrak{n}}^{*}\right)\left(\left(\hat{\Omega}_1^{*}\right)^{\mathfrak{n}} \otimes \left(\hat{\Omega}_2^{*}\right)^{\mathfrak{n}} \otimes \dots \otimes \left(\hat{\Omega}_{\mathfrak{n}}^{*}\right)^{\mathfrak{n}}\right)\left(\hat{\Omega}_1 \otimes \hat{\Omega}_2 \otimes \dots \otimes \hat{\Omega}_{\mathfrak{n}}\right)\\
&= \left(\hat{\Omega}_1^{*}\left(\hat{\Omega}_1^{*}\right)^{\mathfrak{n}}\hat{\Omega}_1 \otimes \hat{\Omega}_2^{*}\left(\hat{\Omega}_2^{*}\right)^{\mathfrak{n}}\hat{\Omega}_2 \otimes \dots \otimes \hat{\Omega}_{\mathfrak{n}}^{*}\left(\hat{\Omega}_{\mathfrak{n}}^{*}\right)^{\mathfrak{n}}\hat{\Omega}_{\mathfrak{n}}\right)\\
&= \left(\left(\hat{\Omega}_1^{*}\right)^{\mathfrak{n}}\hat{\Omega}_1\,\hat{\Omega}_1^{*} \otimes \left(\hat{\Omega}_2^{*}\right)^{\mathfrak{n}}\hat{\Omega}_2\,\hat{\Omega}_2^{*} \otimes \dots \otimes \left(\hat{\Omega}_{\mathfrak{n}}^{*}\right)^{\mathfrak{n}}\hat{\Omega}_{\mathfrak{n}}\,\hat{\Omega}_{\mathfrak{n}}^{*}\right)\\
&= \left(\left(\hat{\Omega}_1^{*}\right)^{\mathfrak{n}} \otimes \left(\hat{\Omega}_2^{*}\right)^{\mathfrak{n}} \otimes \dots \otimes \left(\hat{\Omega}_{\mathfrak{n}}^{*}\right)^{\mathfrak{n}}\right)\left(\hat{\Omega}_1 \otimes \hat{\Omega}_2 \otimes \dots \otimes \hat{\Omega}_{\mathfrak{n}}\right)\left(\hat{\Omega}_1^{*} \otimes \hat{\Omega}_2^{*} \otimes \dots \otimes \hat{\Omega}_{\mathfrak{n}}^{*}\right)\\
&= \left(\left(\hat{\Omega}_1 \otimes \hat{\Omega}_2 \otimes \dots \otimes \hat{\Omega}_{\mathfrak{n}}\right)^{*}\right)^{\mathfrak{n}}\left(\hat{\Omega}_1 \otimes \hat{\Omega}_2 \otimes \dots \otimes \hat{\Omega}_{\mathfrak{n}}\right)\left(\hat{\Omega}_1 \otimes \hat{\Omega}_2 \otimes \dots \otimes \hat{\Omega}_{\mathfrak{n}}\right)^{*}.
\end{aligned}
$$

So, one can obtain $\hat{\Omega}_1 \otimes \hat{\Omega}_2 \otimes \dots \otimes \hat{\Omega}_{\mathfrak{n}}$ is soft $\hat{\Omega}$ −quasi normal operator.

4. Conclusions

During this work, we have discovered another type of soft quasi normal operator, which is an extension of the operator. In order to enhance this research, addition and multiplication operations were introduced for more than one operator through some of the theories that were addressed in this paper. Also, a new structure of the operator of tensor addition and tensor multiplication was discovered.

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