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Stability and Bifurcation in Fractional Order Derivatives of a Tri-Trophic Food Chain with Harvesting

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Abstract

This paper primarily focuses on the study of an ecological model in which fractional differential equations (FDEs) are utilized to construct the model rather than using traditional first order differential equations. A model of tri-trophic food chain is explored using the Beddington–DeAngelis (BD) functional response with harvesting in each species. The impact of harvesting on the system's behaviour is observed, that brings about the occurrence of both bifurcations, Transcritical and Hopf bifurcation, which depends not only on fractional order derivative parameters, but also on the harvesting parameters. The existence, uniqueness, non-negativity, and boundedness of the solutions are explored. Next, the stability of all feasible equilibrium points is determined locally. Additionally, sufficient conditions are established to make certain the global asymptotic stability of each point of equilibrium, except the trivial equilibrium point, by selecting an appropriate Lyapunov function. Finally, numerical simulations are given to illustrate the outcomes of our dynamical analysis of the model.

Keywords: Food chain model, Beddington-DeAngelis, Fractional order model, Stability, Bifurcation, Numerical simulation.

الاستقرار والتشعب في المشتقات الكسرية لسلسلة غذائية ثلاثية التغذية مع الحصاد

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القسم الرياضيات, الكلية العلوم, الجامعة السليمانية, المحافظة السليمانية, العراق

الخلاصة

يتم في هذا البحث استكشاف نموذجًا بيئيًا مصاغا باستخدام معادلات تفاضلية كسرية، بدلاً من معادلات تفاضلية الاعتيادية ذات الدرجة الأولى. كما يتم فحص نموذج السلسلة الغذائية ثلاثية الاجناس والتي تشمل دالة استجابة وظيفية من نوع بدنكتن–ديانجلس، مع الأخذ في الاعتبار الحصاد لكل نوع. ان الدراسة هذه تستكشف كيفية تأثير الحصاد على سلوك النظام، وتبين وجود كل من تشعب فوق الحرجة وتشعب هوبف، اعتمادًا على درجة المشتقة الكسرية ومعلمات الحصاد. كما بين البحث وجود الحلول وفرادتها وعدم سلبيتها وحدودها. وعلاوة على ذلك، فإنه يحقق في الاستقرار المحلي لجميع نقاط التوازن الممكنة ويضع شروطاً كافية

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لضمان الاستقرار المقارب الشامل لكل نقطة توازن، باستثناء النقطة الصفرية، من خلال اختيار دالة ليابونوف المناسبة. وأخيرًا، تم استخدام محاكاة عددية لإثبات النتائج النظرية.

1. Introduction

Fractional differential equations (FDEs) are increasingly being adopted in mathematical modeling due to their ability to accurately describe both linear and nonlinear phenomena [1], [2], [3]. Recently, the study of the behaviours of biological models has seen an increase in the use of mathematical models based on FDEs. The impact of FDEs on the prey predator model has been examined by several authors in [4]. Because the prey and predator interactions are very frequently in nature, mathematical ecology has drawn significant interest from researchers in recent years [5], [6], [7]. When predators fight for food, interference between them occurs frequently, which is missing from the most popular Holling-II functional response. To address this issue, the BD functional response, proposed by Beddington and DeAngelis, is much more similar to the Holling type II response but includes an extra component in the denominator to account for interference among predators [8], [9]. Numerous studies have also been conducted on multi-species food chains, the literatures [10-14] show that these models exhibit chaotic behaviour when using Holling-type or BD nonlinear functional responses [5], [15], [16]. The dynamics of a three-species prey- predator system were the subject of research by Hastings and Powell (1991) [17]. The effect of the fractional-order derivative on a prey- predator model with predator harvesting has been studied by Javidi and Nyamoradi [18]. According to the above-mentioned literature review, fractional calculus is well suited for describing the hereditary characteristics and memory of diverse systems, which are not taken into account by traditional integer-order models. We consider a three-species prey-predator model to analyse the effect of the fractional-order derivative on prey-predator models. The original Hastings and Powell model's dynamics in the context of population harvesting are another goal of the current work. We investigate how population harvesting affects the tri-trophic food chain model proposed by Hastings and Powell (1991). Population dynamics are typically strongly impacted by harvesting [19]. The severity of this impact depends on the type of harvesting strategy employed, which can range from rapid population decline to complete population preservation. This paper explores the tri-trophic food chain model behaviour with a standard predator-dependent of BD type functional response, both theoretically and numerically. In our proposed model, we also examine the role of harvesting in controlling chaos and its influence on system behaviour, particularly the occurrence of bifurcation.

To incorporate a BD functional response in both consumer species, Y and Z, a set of nonlinear differential equations describing pairwise interactions between three species, X, Y, and Z, has been developed as follows:

$$\begin{aligned}
\frac{dX}{dT} &= rX\left(1-\frac{X}{K}\right)-\frac{a_1XY}{1+b_1X+b_2Y},\\
\frac{dY}{dT} &= \frac{a_2XY}{1+b_1X+b_2Y}-\frac{a_3YZ}{1+b_3Y+b_3Z}-D_1Y,\\
\frac{dZ}{dT} &= \frac{a_4YZ}{1+b_3Y+b_4Z}-D_2Z.
\end{aligned} \tag{1}$$

Here, X represents the number of prey species, Y represents the number of intermediate predators that feed on X, and Z represents the number of super predators that feed on Y. The variable T denotes time. The constant K denotes the carrying capacity of prey species X, while the constant r represents the intrinsic growth rate. Parameters a_1 and a_3 denote attack rates, whereas a_2 and a_4 represent conversion rates. Constants b_1, b_2, b_3 and b_4 are parameters associated with the functional response. While D_1 and D_2 represent constant mortality rates for species Y and Z, respectively. All parameters are positive.

The impact of harvesting on ecological populations is considered one of the most crucial topics in population ecology. Hunting can lead to harvesting, which may cause population decline [20]. Future harvesting strategies will benefit from analyzing harvest quantities in relation to population sizes.

Here, Q_1, Q_2 and Q_3 represent the rate of harvesting of population of the prey, intermediate predator and super predator, correspondingly. Assuming $Q_i \in (0,1)$ for $i = 1,2,3,$ the following system can be used to represent system (1) under these presumptions:

$$\begin{aligned}\frac{dX}{dT} &= rX\left(1-\frac{X}{K}\right)-\frac{a_1XY}{1+b_1X+b_2Y}-Q_1X,\\ \frac{dY}{dT} &= \frac{a_2XY}{1+b_1X+b_2Y}-\frac{a_3YZ}{1+b_3Y+b_4Z}-D_1Y-Q_2Y,\\ \frac{dZ}{dT} &= \frac{a_4YZ}{1+b_3Y+b_4Z}-D_2Z-Q_3Z.\end{aligned} \tag{2}$$

To non-dimensionless the system, we employ the scaling $x=\frac{X}{K}$, $y=\frac{a_1}{r}Y$, $z=\frac{a_3}{r}Z$, $t=rT$. After simplification system (2) takes the form:

$$\begin{aligned}\frac{dx}{dt} &= x\left[1-x-\frac{y}{1+ax+by}-q_1\right],\\ \frac{dy}{dt} &= y\left[\frac{e_1x}{1+ax+by}-\frac{z}{1+my+nz}-d_1-q_2\right],\\ \frac{dz}{dt} &= z\left[\frac{e_2y}{1+my+nz}-d_2-q_3\right],\end{aligned} \tag{3}$$

where $a=b_1K,\ b=\frac{rb_2}{a_1},\ q_1=\frac{Q_1}{r},\ e_1=\frac{a_2K}{r},\ m=\frac{rb_3}{a_1},\ n=\frac{rb_4}{a_3},\ d_1=\frac{D_1}{r},\ q_2=\frac{Q_2}{r},\ e_2=\frac{a_4}{a_1},\ d_2=\frac{D_2}{r}\ and\ q_3=\frac{Q_3}{r}$.

The parameters q_1, q_2, q_3 represent dimensionless harvesting parameters. Since no population may be harvested at a rate exceeding 100%, these parameters cannot exceed the maximum value of 1. For the fractional order three–species prey–predator model, we extended the integer order model (3) as follows:

$$\begin{aligned}{}^{C}D^{\alpha}\,x(t) &= x\left[1-x-\frac{y}{1+ax+by}-q_1\right]=G_1(x,y,z),\\ {}^{C}D^{\alpha}y(t) &= y\left[\frac{e_1x}{1+ax+by}-\frac{z}{1+my+nz}-d_1-q_2\right]=G_2(x,y,z),\\ {}^{C}D^{\alpha}z(t) &= z\left[\frac{e_2y}{1+my+nz}-d_2-q_3\right]=G_3(x,y,z),\end{aligned} \tag{4}$$

with the initial conditions, $x(0)=x_0>0$, $y(0)=y_0>0$ and $z(0)=z_0>0$. When $\alpha \in (0,1)$ and ${}^{C}D^{\alpha}$ is the sense of Caputo fractional derivative.

2. Preliminaries

This section presents several definitions and important lemmas related to Caputo fractional order derivatives, which are used extensively throughout this study.

Definition 2.1. [21] The Caputo derivative represents a normalization of the Riemann Liouville formula. For a function $f(t)$ with $t>t_0=0$, the $\alpha-th$ order of Caputo derivative is defined as:

$$ {}^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} (t-s)^{(n-\alpha)-1} \frac{d^n}{ds^n} f(s) ds , $$

where $n-1 < \alpha < n, n \in \mathbb{N}$, $n = \lceil \alpha \rceil$ and ${}^{C}D^{\alpha}$ denotes the standard Caputo differentiation. Following [22]. If $\alpha \in (0,1)$, then for a function $f(t)$ with $t > t_0 = 0$, the Caputo derivative of order α reduced to

$$ {}^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{0}^{t} (t-s)^{-\alpha} \frac{d}{ds} f(s) ds . $$

Lemma 2.2. [23] Let $\alpha \in (0,1]$ and $\lambda < 0$. Then, $E_{\alpha,\alpha}(\lambda t^{\alpha})$ and $E_{\alpha,\alpha+1}(\lambda t^{\alpha})$ monotonically tend to zero as $t \longrightarrow \infty$.

Lemma 2.3. [23] For any continuous function $x(t)$ on $[0,+\infty)$ that satisfies ${}^{C}D^{\alpha}x(t) \leq \gamma - \lambda x(t)$, $x(t_0) = x_0$, where $\alpha \in (0,1]$, $\lambda, \gamma \in \mathbb{R}$ *and* $\lambda \neq 0$. The following inequality holds:

$$ x(t) \leq \left(x_0 - \frac{\gamma}{\lambda}\right) E_{\alpha}(-\lambda t^{\alpha}) + \frac{\gamma}{\lambda} . $$

Definition 2.4. [24] An autonomous system with the Caputo derivative ${}^{C}D^{\alpha} = Ax(t)$ is said to have an equilibrium point X^*.

Locally stable if $\forall \epsilon > 0$, $\exists\ \delta > 0$ such that $\|X(t) - X^*\| < \epsilon$ holds for $\forall X_0 \in \{Z : \|Z - X^*\| < \delta\}$ and $\forall\, t > 0$;

Locally asymptotically stable if X^* is locally stable and $\lim_{t\to+\infty} X(t) = X^*$.

Lemma 2.5. The equilibrium points for a system ${}^{C}D^{\alpha} = Ax(t)$, are asymptotically stable in accordance with stability given in [25], when all the eigenvalues λ_i, $(i = 1,2,\dots,n)$ of the Jacobian matrix $J(X^*)$, fulfil the condition:

$$ |arg(\lambda(J(X^*))| = |arg(\lambda_i)| > \frac{\alpha\pi}{2} . $$

Lemma 2.6. [26] Let $\beta(t) \in \mathbb{R}^+$ be a differentiable function. Then, for any $t > t_0$, $\alpha \in (0,1]$ and $\beta^* \geq 0$, one has

$$ {}^{C}D^{\alpha}\left[\beta(t) - \beta^* - \beta^* Ln\frac{\beta(t)}{\beta^*}\right] \leq \left(1 - \frac{\beta^*}{\beta(t)}\right) {}^{C}D^{\alpha}\beta(t). $$

3. Analysis

3.1. Existence and Uniqueness

Theorem 3.1.1. The system (4) has a unique solution for any non-negative initial conditions.

Proof. To ensure the existence and uniqueness of system (4)'s solutions in the domain, we are seeking an adequate and essential condition. $\Psi \times (0,T]$, where $\Psi = \{(x,y,z) \in \mathbb{R}^3 : \max(|x|,|y|,|z|) \leq M\}$, where M is a positive constant.

Following the approach described in [27]. Consider a mapping $G(V) = \big(G_1(V), G_2(V), G_3(V)\big)$ and for any $V = (x,y,z)$, $\overline{V} = (\overline{x}, \overline{y}, \overline{z})$, $V, \overline{V} \in \Psi$, it comes from (4) that:

$$ \|G(V) - G(\overline{V})\| = |G_1(V) - G_1(\overline{V})| + |G_2(V) - G_2(\overline{V})| + |G_3(V) - G_3(\overline{V})| \leq W_1\|V - \overline{V}\|, $$

where:

$$ W_1 = Max\,\{1 + q_1 + 3M + bM^2,\ d_1 + q_2 + 2M + (a+n)M^2,\ d_2 + q_3 + M + mM^2\}. $$

Thus, $G(V)$ fulfil Lipschitz's requirement. This proves that the system (4) has a unique solution. ■

3.2 Non-negativity and Boundedness

Theorem 3.2.1. All solutions of system (4) that start in $\Psi_+ = \{(x,y,z) \in \mathbb{R}^3 : x \geq 0 , y \geq 0 , z \geq 0\}$ remain uniformly bounded and non-negative.

Proof. We show that, all solutions are uniformly bounded for initial conditions in Ψ_+. Define a function $W_2(t) = x(t) + \frac{1}{e_1}y(t) + \frac{1}{e_1e_2}z(t)$, then we have:

$${}^{C}D^{\alpha}W_2(t) + q_1W_2(t) \leq x - x^2 + \frac{1}{e_1}(q_1 - q_2)y + \frac{1}{e_1e_2}(q_1 - q_3)z.$$

Choose $q_1 < min\{q_2, q_3\}$, then ${}^{C}D^{\alpha}W_2(t) + q_1W_2(t) \leq x - x^2 = -\left(x - \frac{1}{2}\right)^2 + \frac{1}{4} \leq \frac{1}{4}$.

By using Lemma 2.2

$$W_2(t) \leq \left(W_2(0) - \frac{1}{4q_1}\right)E_{\alpha}(-q_1t^{\alpha}) + \frac{1}{4q_1}.$$

As $t \to \infty$, by Lemma 2.2 we have $W_2(t) \leq \frac{1}{4q_1}$. Therefore, all solutions of system (4) with initial conditions in Ψ_b, where:

$$\Psi_b = \left\{(x, y, z) \in \mathbb{R}_+^3 : W_2(t) \leq \frac{1}{4q_1} + \epsilon \;, \epsilon > 0\right\}. \tag{5}$$

The non-negativity of solutions of system (4) can be shown as follows. From the first equation of system (4):

$${}^{C}D^{\alpha}x(t) = x\left(1 - x - \frac{y}{1 + ax + by} - q_1\right). \tag{6}$$

From (5), we have this observation:

$$x + \frac{1}{e_1}y + \frac{1}{e_1e_2}z \leq \frac{1}{4q_1}. \tag{7}$$

Depending on (6) and (7) we obtain:

$${}^{C}D^{\alpha}x(t) = x\left(1 - x - \frac{y}{1 + ax + by} - q_1\right) \geq \left(1 - \frac{1 + e_1}{4q_1} - q_1\right)x = s_1x,$$

where $s_1 = 1 - \frac{1+e_1}{4q_1} - q_1$.

Considering Lemma 2.3

$$x(t) \geq x_0E_{\alpha}(s_1t^{\alpha}).$$

Since $x_0 \geq 0$ and by Mittag-Leffler function positivity, then $E_{\alpha}(s_1t^{\alpha}) > 0$. Hence for any $\alpha \in (0,1)$, then $x(t) \geq 0$.

From second equation of system (4)

$${}^{C}D^{\alpha}y(t) = y\left(\frac{e_1x}{1 + ax + by} - \frac{z}{1 + my + nz} - d_1 - q_2\right) \geq \left(-\frac{e_1e_2}{4q_1} - d_1 - q_2\right)y = -s_2y,$$

where $s_2 = \frac{e_1e_2}{4q_1} + d_1 + q_2$. Therefore, $y(t) \geq y_0E_{\alpha}(-s_2t^{\alpha})$, thus $y(t) \geq 0$.

From the third equation in system (4) we obtain:

$${}^{C}D^{\alpha}z(t) = z\left(\frac{y}{1 + my + nz} - d_2 - q_3\right) \geq (-d_2 - q_3)z = -s_3z,$$

where $s_3 = d_2 + q_3$. So, $z(t) \geq z_0E_{\alpha}(-s_3t^{\alpha})$, consequently $z(t) \geq 0$ and Theorem 3.2.1 is entirely proven.
■

3.3 Equilibria and Local Stability

The equilibrium points of system (4) are obtained by solving the simultaneous equations: ${}^{C}D^{\alpha}x(t) = 0$, ${}^{C}D^{\alpha}y(t) = 0$ and ${}^{C}D^{\alpha}z(t) = 0$.

Possible equilibria are:

i. The trivial equilibrium point $E_0 = (0,0,0)$ is always exist.

ii. The axial equilibrium point $E_1 = (1 - q_1, 0,0)$ is always exist since $(q_1 < 1)$.

iii. Super predator free equilibrium point $E_2 = (x_2, y_2, 0)$ exists in Int R_+^3 if the following equations has a positive solution

$$1 - x_2 - \frac{y_2}{1 + ax_2 + by_2} - q_1 = 0, \tag{8}$$
$$\frac{e_1 x_2}{1 + ax_2 + by_2} - (d_1 + q_2) = 0. \tag{9}$$
From equation (9), we have
$$y_2 = \frac{(e_1 - (d_1 + q_2)a)x_2 - d_1 - q_2}{bd_1 + bq_2}. \tag{10}$$
From equation (8) and (10), we obtain
$$P(x_2) = Ax_2{}^2 + Bx_2 + C = 0, \tag{11}$$
where:

$A = be_1 > 0$, $B = be_1q_1 - ad_1 - aq_2 - be_1 + e_1$ and $C = -(d_1 + q_2) < 0$.

According to Descarte's rule of sign [28], $P(x_2) = 0$ has a positive x_2 . And $y_2 > 0$ provided:
$$0 < \frac{d_1 + q_2}{e_1 - (d_1 + q_2)a} < x_2 < 1 - q_1. \tag{12}$$
iv. The co-existence equilibrium point $E_3 = (x_3, y_3, z_3)$, exists in Int R^3_+ if the following equations has a positive solution
$$1 - x_3 - \frac{y_3}{1 + ax_3 + by_3} - q_1 = 0, \tag{13}$$
$$\frac{e_1 x_3}{1 + ax_3 + by_3} - \frac{z_3}{1 + my_3 + nz_3} - (d_1 - q_2) = 0, \tag{14}$$
$$\frac{e_2 y_3}{1 + my_3 + nz_3} - (d_2 - q_3) = 0. \tag{15}$$
From (13) we obtain
$$y_3 = \frac{(1 - x_3 - q_1)(1 + ax_3)}{(1 - b + bq_1) + bx_3}. \tag{16}$$
By putting (16) in (15) we have
$$x_3 = \frac{1}{n}\left(\left(\frac{e_2}{d_2 + q_3} - m\right)\left(\frac{(1 - x_3 - q_1)(1 + ax_3)}{(1 - b + bq_1) + bx_3}\right) - 1\right). \tag{17}$$
After substituting (16) and (17) in (14) we obtain
$$A_3x_3^3 + A_2x_3^2 + A_1x_3 + A_0 = 0, \tag{18}$$
where:

$A_3 = bne_1e_2 > 0,$

$A_2 = 2bne_1e_2q_1 + am(d_2 + q_3) + ne_1e_2 - [(d_1 + q_2)an + (bne_1 + a)]e_2,$

$A_1 = m(d_2 + q_3)(1 + aq_1) + e_2\big[a + n\big(be_1q_1^2 + a(d_1 + q_2) + e_1(b + q_1)\big)\big]$
$\quad - [\,ne_2q_1(ad_1 + aq_1 + 2be_1) + e_2(aq_1 + 1) + (d_2 + q_3)(am + b)$
$\quad + ne_2(e_1 + d_1 + q_2)],$

$A_0 = \big((d_2 + q_3)(mq_1 + b) + e_2(d_1 + q_2) + 1\big)$
$\quad - [e_2q_1(n(d_1 + q_2) + 1) + (d_2 + q_3)(bq_1 + m + 1)].$

Now, Equation (18) has a unique positive real root x_3, according to Descarte's rule of sign [28], if one of the following holds:

i. $A_2 > 0$ and $A_0 < 0$.

ii. $A_1 < 0$ and $A_0 < 0$.

Then we have
$$y_3 = \frac{(1 + ax_3)(1 - x_3 - q_1)}{1 - b(1 - x_3 - q_1)}, \tag{19}$$
$$z_3 = \frac{1}{n}\left(\left(\frac{e_2}{d_2 + q_3} - m\right)y_3 - 1\right). \tag{20}$$

Thus, y_3 and z_3 are positive under the conditions

$$0 < 1 - q_1 - \frac{1}{b} < x_3 < 1 - q_1, \tag{21}$$

$$0 < d_2 + q_3 < \frac{e_2 y_3}{m y_3 + 1}, \tag{22}$$

respectively.

To analyze the stability of each equilibrium point, we employ the Jacobian matrix. For the fractional order system (4), the Jacobian matrix is given by:

$$\begin{aligned}
{}^{C}D^{\alpha}x(t) &= x\left[1 - x - \frac{y}{1 + ax + by} - q_1\right] = x f_1(x,y,z),\\
{}^{C}D^{\alpha}y(t) &= y\left[\frac{e_1 x}{1 + ax + by} - \frac{z}{1 + my + nz} - d_1 - q_2\right] = y f_2(x,y,z),\\
{}^{C}D^{\alpha}z(t) &= z\left[\frac{e_2 y}{1 + my + nz} - d_2 - q_3\right] = z f_3(x,y,z),
\end{aligned} \tag{23}$$

where:

$$\begin{aligned}
f_1(x,y,z) &= 1 - x - \frac{y}{1 + ax + by} - q_1,\\
f_2(x,y,z) &= \frac{e_1 x}{1 + ax + by} - \frac{z}{1 + my + nz} - d_1 - q_2,\\
f_3(x,y,z) &= \frac{e_2 y}{1 + my + nz} - d_2 - q_3.
\end{aligned}$$

The Jacobian matrix of (23), is derived as follow:

$$J(x,y,z) = \begin{bmatrix} x\frac{\partial f_1}{\partial x} + f_1 & x\frac{\partial f_1}{\partial y} & x\frac{\partial f_1}{\partial z} \\ y\frac{\partial f_2}{\partial x} & y\frac{\partial f_2}{\partial y} + f_2 & y\frac{\partial f_2}{\partial z} \\ z\frac{\partial f_3}{\partial x} & z\frac{\partial f_3}{\partial y} & z\frac{\partial f_3}{\partial z} + f_3 \end{bmatrix}, \tag{24}$$

where:

$$\frac{\partial f_1}{\partial x} = -1 + \frac{ay}{(1 + ax + by)^2}, \quad \frac{\partial f_1}{\partial y} = -\frac{1 + ax}{(1 + ax + by)^2}, \quad \frac{\partial f_1}{\partial z} = 0,$$

$$\frac{\partial f_2}{\partial x} = \frac{e_1(1 + by)}{(1 + ax + by)^2}, \quad \frac{\partial f_2}{\partial y} = -\frac{b e_1 x}{(1 + ax + by)^2} + \frac{mz}{(1 + my + nz)^2},$$

$$\frac{\partial f_2}{\partial z} = -\frac{(1 + my)}{(1 + my + nz)^2},$$

$$\frac{\partial f_3}{\partial x} = 0, \quad \frac{\partial f_3}{\partial y} = \frac{e_2(1 + nz)}{(1 + my + nz)^2} \text{ and } \frac{\partial f_3}{\partial z} = -\frac{e_2 n y}{(1 + my + nz)^2}.$$

Theorem 3.3.1. The equilibrium point $E_0 = (0,0,0)$ of system (4) is a saddle point.

Proof. If any of the eigenvalues λ_i of the $J(E_0)$ fulfil $|arg(\lambda_i)| < \frac{\alpha\pi}{2}$, then the equilibrium point $E_0 = (0,0,0)$ is an unstable saddle point. In accordance with Matignon's condition [25]. At $E_0 = (0,0,0)$, the $J(E_0)$ of system (4) is:

$$J(E_0) = \begin{bmatrix} 1 - q_1 & 0 & 0 \\ 0 & -(d_1 + q_2) & 0 \\ 0 & 0 & -(d_2 + q_3) \end{bmatrix}.$$

The corresponding eigenvalues are $\lambda_{01} = 1 - q_1$, $\lambda_{02} = -(d_1 + q_2)$ and $\lambda_{03} = -(d_2 + q_3)$. Clearly, we can see that $|arg(\lambda_{01})| = 0 < \frac{\alpha\pi}{2}$ for all $\alpha \in (0,1)$. Thus, $E_0 = (0,0,0)$ is saddle point.

■

Theorem 3.3.2 The equilibrium point $E_1 = (1-q_1, 0,0)$ of system (4) is locally asymptotically stable, when

$$\frac{e_1(1-q_1)}{1+a(1-q_1)} < d_1 + q_2. \tag{25}$$

Proof. We calculated $J(E_1)$ and acquired:

$$J(E_1) = \begin{bmatrix} -(1-q_1) & -\frac{(1-q_1)}{1+a(1-q_1)} & 0 \\ 0 & \frac{e_1(1-q_1)}{1+a(1-q_1)} - (d_1+q_2) & 0 \\ 0 & 0 & -(d_2+q_3) \end{bmatrix}.$$

Then, the corresponding eigenvalues of $J(E_1)$ are $\lambda_{11} = -(1-q_1)$, $\lambda_{12} = \frac{e_1(1-q_1)}{1+a(1-q_1)} - (d_1+q_2)$ and $\lambda_{13} = -(d_2+q_3)$. Note that $|arg(\lambda_{11})| = |arg(\lambda_{13})| = \pi > \frac{\alpha\pi}{2}$ for all $\alpha \in (0,1)$, but $|arg(\lambda_{12})| = \pi > \frac{\alpha\pi}{2}$ for all $\alpha \in (0,1)$ when $\frac{e_1(1-q_1)}{1+a(1-q_1)} < (d_1+q_2)$. Consequently, E_1 becomes locally asymptotically stable under the condition (25). ■

Observation 3.3.3. Regarding Theorem 4 the stability of E_0 is not depend on the fractional derivative order α. However, the mortality rate and intermediate predator harvesting rate (d_1 and q_2 respectively), control the stability behavior of E_1.

Now stability of $E_2 = (x_2, y_2, 0)$ has been investigated, $J(E_2)$ is also assessed:

$$J(E_2) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

where $a_{11} = x_2\left(-1 + \frac{ay_2}{L_0^2}\right), a_{12} = -x_2\left(\frac{1+ax_2}{L_0^2}\right) < 0, a_{13} = 0, a_{21} = y_2\left(\frac{e_1(1+by_2)}{L_0^2}\right) > 0,$ $a_{22} = -y_2\left(\frac{be_1x_2}{L_0^2}\right) < 0, a_{23} = -\frac{y_2}{1+my_2} < 0, a_{31} = 0$, $a_{32} = 0, a_{33} = \frac{e_2y_2}{1+my_2} - (d_2+q_3)$ and $L_0 = 1 + ax_2 + by_2$.

The eigenvalues correlating with $E_2 = (x_2, y_2, 0)$ are the following characteristic equation's roots:

$$(\lambda - a_{33})(\lambda^2 + A\lambda + B) = 0, \tag{26}$$

where

$$A = -(a_{11} + a_{22}) = \frac{x_2}{L_0^2}\left(L_0^2 + y_2(be_1 - a)\right), \tag{27}$$

$$B = \frac{e_1x_2y_2}{L_0^2}\left(bx_2 + \frac{1}{L_0}\right) > 0. \tag{28}$$

Since the discriminant of the quadratic part of (26) is defined by $\Delta = A^2 - 4B$, then by examining the system's associated characteristic equation at $E_2 = (x_2, y_2, 0)$, we acquired the stability condition that described in the next theorem.

Theorem 3.3.4 The equilibrium point $E_2 = (x_2, y_2, 0)$ is

i. Locally asymptotically stable, when

$$0 < \frac{e_2y_2}{1+my_2} < d_2 + q_3 \tag{29}$$

and

$$0 < a < be_1 + \frac{L_0^2}{y_2}. \tag{30}$$

ii. Locally asymptotically stable, when (29) holds and

$$0 < \frac{L_0^2}{y_2} + be_1 < a, \tag{31}$$

$$y_2\left(\frac{be_1 - a}{L_0}\right)^2 < a + e_1\left(3b + \frac{4}{x_2L_0}\right), \tag{32}$$

$$tan^{-1}\left(\frac{-\sqrt{-\Delta}}{A}\right) > \frac{\alpha\pi}{2}. \tag{33}$$

Proof: (i) From Equation (26), it has the following eigenvalues $\lambda_{21} = a_{33} = \frac{e_2y_2}{1+my_2} - (d_2 + q_3)$ and $\lambda_{22,23} = \frac{-A\mp\sqrt{\Delta}}{2}$. If $0 < \frac{e_2y_2}{1+my_2} < d_2 + q_3$, then $|arg(\lambda_{21})| = \pi > \frac{\alpha\pi}{2}, \forall\alpha \in (0,1)$. If $0 < a < be_1 + \frac{L_0^2}{y_2}$ then $A > 0$ which implies that λ_{22} and λ_{23} have two negative real parts. Thus, according to second point of Routh-Hurwitz criterion in FDEs, $E_2 = (x_2, y_2, 0)$ is locally asymptotically stable $\forall\ \alpha \in (0,1)$.
(ii) The characteristic Equation (26) possesses $\lambda_{21} = \frac{e_2y_2}{1+my_2} - (d_2 + q_3)$ and a pairs of complex conjugate eigenvalues with positive real parts, when $0 < \frac{L_0^2}{y_2} + be_1 < a$ and $y_2\left(\frac{be_1-a}{L_0}\right)^2 < a + e_1\left(3b + \frac{4}{x_2L_0}\right)$ as follows: $\lambda_{22,23} = \frac{-A\mp i\sqrt{-\Delta}}{2}$. If $0 < \frac{e_2y_2}{1+my_2} < d_2 + q_3$, then $|\arg(\lambda_{21})| = \pi > \frac{\alpha\pi}{2}$. If $0 < \frac{L_0^2}{y_2} + be_1 < a$ and $y_2\left(\frac{be_1-a}{L_0}\right)^2 < a + e_1\left(3b + \frac{4}{x_2L_0}\right)$, then $A < 0$ and $\Delta < 0$ which implies that $\left|arg(\lambda_{22,23})\right| = tan^{-1}\left(\frac{-\sqrt{-\Delta}}{A}\right)$. Hence, $\left|arg(\lambda_{22,23})\right| > \frac{\alpha\pi}{2}$ when $tan^{-1}\left(\frac{-\sqrt{-\Delta}}{A}\right) > \frac{\alpha\pi}{2}$.
As a result, following second point of Routh-Hurwitz criterion in FDEs, we obtain $E_2 = (x_2, y_2, 0)$ is locally asymptotically stable $\forall\ \alpha \in (0, \alpha_1)$, where $\alpha_1 = \frac{2}{\pi}tan^{-1}\left(\frac{-\sqrt{-\Delta}}{A}\right)$. ■

Theorem 3.3.5 The equilibrium point $E_3 = (x_3, y_3, z_3)$ is locally asymptotically stable, when anyone of the below cases holds:
i. $D(P) > 0$, $ay_3 < L_1^2$ and $mz_3L_1^2 < e_1bx_3L_2^2$ for all $\alpha \in (0,1]$.
ii. $D(P) < 0$, $ay_3 \le L_1^2$ and $mz_3L_1^2 \le e_1bx_3L_2^2$ for all $\alpha \in \left(0, \frac{2}{3}\right)$.
iii. $D(P) < 0$, $ay_3 = L_1^2$ and $mz_3L_1^2 = e_1bx_3L_2^2$ for all $\alpha \in [0,1)$.

Proof. Initially $J(E_3)$ of system (4) is provided by

$$J(E_3) = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix},$$

where $b_{11} = \frac{x_3}{L_1^2}(ay_3 - L_1^2)$, $b_{12} = -\frac{(1+ax_3)x_3}{L_1^2} < 0$, $b_{13} = 0$, $b_{21} = \frac{e_1(1+by_3)y_3}{L_1^2} > 0$, $b_{22} = \frac{y_3}{L_1^2L_2^2}(mz_3L_1^2 - e_1bx_3L_2^2)$, $b_{23} = -\frac{(1+my_3)y_3}{L_2^2}$, $b_{31} = 0$, $b_{32} = \frac{e_2(1+nz_3)z_3}{L_2^2} > 0$ and $b_{33} = -\frac{ne_2y_3z_3}{L_2^2} < 0$ with $L_1 = 1 + ax_3 + by_3$ and $L_2 = 1 + my_3 + nz_3$.

Now, the characteristic equation of $J(E_3)$ is provided by

$$\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3 = 0, \tag{34}$$

where:

$$B_1 = -(b_{11} + b_{22} + b_{33}), \qquad B_2 = b_{11}b_{22} + b_{11}b_{33} + b_{22}b_{33} - b_{23}b_{32} - b_{12}b_{21},$$
$$B_3 = b_{12}b_{21}b_{32} + b_{11}b_{23}b_{32} - b_{11}b_{22}b_{33}.$$

To determine the local stability of $E_3 = (x_3, y_3, z_3)$, we evaluate the discriminant of (34) using the definition of the discriminant, we obtain:

$$D(P) = 18B_1B_2B_3 + (B_1B_2)^2 - 4B_3(B_1)^3 - 4(B_2)^3 - 27(B_3)^2.$$

Regarding condition (i) if it exists, then $D(P) > 0$, $B_1 > 0$, $B_3 > 0$ and $B_1B_2 - B_3 > 0$, which makes $E_3 = (x_3, y_3, z_3)$ locally asymptotically stable $\forall\ \alpha \in (0, 1]$ as demonstrated in third point of Routh-Hurwitz criterion in FDEs [29].

If condition (ii) exists, then $D(P) < 0$, $B_1 \geq 0$, $B_2 \geq 0$ and $B_3 > 0$, which makes $E_3 = (x_3, y_3, z_3)$ locally asymptotically stable $\forall\ \alpha \in \left(0, \frac{2}{3}\right)$, as demonstrated in fourth point of Routh-Hurwitz criterion in FDEs [29].

If condition (iii) exists, then $D(P) < 0$, $B_1 > 0$, $B_2 > 0$ and $B_1B_2 - B_3 = 0$, which makes $E_3 = (x_3, y_3, z_3)$ locally asymptotically stable $\forall\ \alpha \in [0, 1)$, as demonstrated in fifth point of Routh-Hurwitz criterion in FDEs [29]. ■

3.4 Bifurcation Analysis

In this section, we prove the existence of a Transcritical bifurcation around $E_1 = (1 - q_1, 0, 0)$ using Sotomayor's theorem. Additionally, we investigate the existence of Hopf bifurcation in system (4) around $E_2 = (x_2, y_2, 0)$ and $E_3 = (x_3, y_3, z_3)$.

Theorem 3.4.1. The system (4) undergoes a Transcritical bifurcation considering the bifurcation parameter q_2 around $E_1 = (1 - q_1, 0, 0)$ if

$$q_2 = \frac{e_1(1 - q_1)}{1 + a(1 - q_1)} - d_1. \tag{35}$$

Proof. The Jacobian matrix of system (4) at $E_1 = (1 - q_1, 0, 0)$ with $q_2 = q_2^* = \frac{e_1(1-q_1)}{1+a(1-q_1)} - d_1$, possess a zero eigenvalue, in the form of:

$$J(E_1)|_{q_2=q_2^*} = \begin{bmatrix} -(1 - q_1) & -\dfrac{1 - q_1}{1 + a(1 - q_1)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -(d_2 + q_3) \end{bmatrix}.$$

Here $\lambda_{11} = -(1 - q_1) < 0$, $\lambda_{12} = 0$ and $\lambda_{13} = -(d_2 + q_3) < 0$.

Let $V_1 = (v_1, v_2, v_3)^T = \left(v_1, -\left(1 + a(1 - q_1)\right)v_1, 0\right)^T$ and $V_2 = (w_1, w_2, w_3)^T = (0, w_2, 0)^T$ be the two eigenvectors correlating with the zero eigenvalue ($\lambda_{12} = 0$) of the $J(E_1)$ and $\left(J(E_1)\right)^T$, respectively. Where v_1 and w_2 are any non-zero real numbers.

Consider $\frac{\partial G}{\partial q_2} = \begin{pmatrix} 0 \\ -y \\ 0 \end{pmatrix} = F_{q_2}(X, \tau)$. Thus ${V_2}^T\left(F_{q_2}(E_1, q_2^*)\right) = (0, w_2, 0)\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0$. Therefore, in accordance with Sotomayor's theorem [30] for local bifurcation, the system (4) does not exhibit a saddle-node bifurcation near E_1 at $q_2^* = \frac{e_1(1-q_1)}{1+a(1-q_1)} - d_1$.

Now, $DF_{q_2}(E_1, q_2^*) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then, ${V_2}^T\left(DF_{q_2}(E_1, q_2^*)V_1\right) = -\left(1 + a(1 - q_1)\right)v_1w_2 \neq 0$, and ${V_2}^T\left(D^2G(E_1, q_2^*)(V_1, V_1)\right) = \left(\frac{e_1(v_1 - b(1-q_1)v_2)}{\left(1+a(1-q_1)\right)^2} - 1\right)(2v_2w_2) \neq 0$.

By Sotomayor's theorem [30], system (4) has a Transcritical bifurcation around $E_1 = (1 - q_1, 0, 0)$, when $q_2 = q_2^* = \frac{e_1(1-q_1)}{1+a(1-q_2)} - d_1$ as q_2 passes through the value of q_2^*, while no Saddle-node bifurcation can occur. ■

Observation 3.4.2. The existence of Transcritical bifurcation at $E_1 = (1 - q_1, 0, 0)$ depends on harvesting rate of intermediate predator population (q_2).

Theorem 3.4.3. In system (4), when the parameter α crosses the critical value $\alpha_1 = \frac{2}{\pi} tan^{-1}\left(\frac{-\sqrt{-\Delta}}{A}\right)$, a Hopf bifurcation occurs at $E_2 = (x_2, y_2, 0)$ when the conditions (31) and (32) hold.

Proof. Considering the conditions (31) and (32) and the critical value $\alpha_1 = \frac{2}{\pi} tan^{-1}\left(\frac{-\sqrt{-\Delta}}{A}\right)$, where $\Delta = A^2 - 4B$, as A and B were defined in (27), (28). We define

$$\mu = -\frac{1}{2}A = -\frac{1}{2}\left(\frac{x_2}{L_0^2}\left(L_0^2 + y_2(be_1 - a)\right)\right), \tag{36}$$

$$\text{and } \theta = \frac{1}{2}\sqrt{-\Delta} = \frac{1}{2}\sqrt{\frac{4e_1x_2y_2}{L_0^2}\left(bx_2 + \frac{1}{L_0}\right) - \left(\frac{x_2}{L_0^2}\left(L_0^2 + y_2(be_1 - a)\right)\right)^2}\ . \tag{37}$$

Regarding (31) and (36), we obtain $\mu = -\frac{1}{2}\left(\frac{x_2}{L_0^2}\left(L_0^2 + y_2(be_1 - a)\right)\right) > 0$. Furthermore, from conditions (31), (32) and (37), we acquire a pair of complex conjugate eigenvalues $\lambda_{22,23} = \mu \mp \theta$ with $\mu > 0$. Therefore, condition (i) of the Hopf bifurcation theorem [31] holds.

From (29), (36) and (37), and based on the definition of $H(\alpha)$ in [31], we have $H(\alpha_1) = \frac{\alpha_1\pi}{2} - min\{|arg(\lambda_{21})|, |arg(\lambda_{22})|, |arg(\lambda_{23})|\} = 0$. Thus, condition (ii) of the Hopf bifurcation theorem [31] holds.

Additionally, from the definition of $H(\alpha)$, we obtain $\left.\frac{d\,H(\alpha)}{d\alpha}\right|_{\alpha=\alpha_1} = \frac{\pi}{2} \neq 0$, which implies that condition (iii) of the Hopf bifurcation theorem [31] is also holds.

Consequently, from Theorem 8. in [31], when parameter α crosses the critical value α_1, system (4) undergoes a Hopf bifurcation at $E_2 = (x_2, y_2, 0)$.

■

The existence of a Hopf bifurcation in system (4) at $E_3 = (x_3, y_3, z_3)$ is established, with α as the bifurcation parameter. When a stable focus equilibrium point changes to an unstable one, this bifurcation occurs and a limit cycle appears simultaneously as the bifurcation parameter changes.

Theorem 3.4.4. Suppose that $S_1, S_2 \in \mathbb{R}$ and $3(S_1 + S_2) < B_1 < -\frac{3}{2}(S_1 + S_2)$. A Hopf bifurcation occurs at $E_3 = (x_3, y_3, z_3)$, when α crosses $\alpha_2 = \frac{2}{\pi}\left|arg(\lambda_{32,33})\right|$.

Proof. From $J(E_3)$, we obtain the characteristic polynomial $\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3 = 0$. Using Cardano's formula, the eigenvalues are given by $\lambda_{31} = S_1 + S_2 - \frac{B_1}{3}$ and $\lambda_{32,33} = w_1 \mp w_2 i$, where $w_1 = -\frac{S_1+S_2}{2} - \frac{B_1}{3}$ and $w_2 = \frac{\sqrt{3}}{2}(S_1 - S_2)$, where:

$$S_1 = \sqrt[3]{P_1 + \sqrt{P_2^3 + P_1^2}}\,, \qquad S_2 = \sqrt[3]{P_1 - \sqrt{P_2^3 + P_1^2}},$$

$$P_1 = \frac{9B_1B_2 - 27B_3 - 2B_1}{54} \text{ and } P_2 = \frac{3B_2 - B_1^2}{9}.$$

By assumption $S_1, S_2 \in \mathbb{R}$, and if $3(S_1 + S_2) < B_1 < -\frac{3}{2}(S_1 + S_2)$, then $\lambda_{31} = S_1 + S_2 - \frac{B_1}{3} < 0$ and $w_1 = -\frac{S_1+S_2}{2} - \frac{B_1}{3} > 0$. Hence, the Jacobian matrix $J(E_3)$ has one negative real eigenvalue and a pair of complex conjugate eigenvalues $\lambda_{32,33} = w_1 \mp w_2 i$ with positive real parts. Based on the definition of $H(\alpha)$ in [31], It's simple to verify that

$H(\alpha_2) = \frac{\alpha_2 \pi}{2} - min\{|arg(\lambda_{31})|, |arg(\lambda_{32})|, |arg(\lambda_{33})|\} = 0$, and $\frac{d\,H(\alpha)}{d\alpha}\Big|_{\alpha=\alpha_2} = \frac{\pi}{2} \neq 0$.
Observe that $\alpha_2 = \frac{2}{\pi}\left|arg(\lambda_{32,33})\right| = \frac{2}{\pi} tan^{-1}\left(\left|\frac{w_2}{w_1}\right|\right)$. Based on the Hopf bifurcation theorem [31], the sign of $H(\alpha)$ may change when the parameter α crosses the critical value α_2. That is. $E_3 = (x_3, y_3, z_3)$ is asymptotically stable for $\alpha \in (0, \alpha_2)$ and is unstable for $\alpha \in (\alpha_2, 1)$. Hence, one may claim that Hopf bifurcation in system (4) takes place near $E_3 = (x_3, y_3, z_3)$ at $\alpha = \alpha_2$. ■

3.5 Global Stability
The stability of $E_1 = (1 - q_1, 0, 0)$, $E_2 = (x_2, y_2, 0)$ and $E_3 = (x_3, y_3, z_3)$ of the system (4) are investigated globally asymptotically in this section.

Theorem 3.5.1. If $e_1(1 - q_1) \leq d_1 + q_2$ holds, then the equilibrium $E_1 = (1 - q_1, 0, 0)$ of system (4) is globally asymptotically stable.
Proof. Consider the positive definite Lyapunov function $V_*(x, y, z)$, defined as:
$$V_*(x, y, z) = \left(x(t) - (1 - q_1) - (1 - q_1)Ln\left(\frac{x(t)}{1 - q_1}\right)\right) + \frac{1}{e_1} y(t) + \frac{1}{e_1 e_2} z(t).$$
Here $V_*(x, y, z)$ is a continuous function such that $V_*(x, y, z) > 0$ for all values of $\big(x(t), y(t), z(t)\big) \neq (1 - q_1, 0, 0)$ and $V_*(x, y, z) = 0$ only at $(1 - q_1, 0, 0)$.
Computing the α order derivative of $V_*(x, y, z)$ along the solution of system (4), it follows from Lemma 2.6, that:
$$\begin{aligned}
{}^{C}D^{\alpha}\, v_*(x, y, z) &\leq \left(\frac{x(t) - (1 - q_1)}{x(t)}\right) {}^{C}D^{\alpha} x(t) + \frac{1}{e_1}\, {}^{C}D^{\alpha} y(t) + \frac{1}{e_1 e_2}\, {}^{C}D^{\alpha} z(t) \\
&= \big(x - (1 - q_1)\big)\left(1 - x - \frac{y}{1 + ax + by} - q_1\right) \\
&\quad + \frac{1}{e_1}\left(\frac{e_1 xy}{1 + ax + by} - \frac{yz}{1 + my + nz} - (d_1 + q_2)y\right) \\
&\quad + \frac{1}{e_1 e_2}\left(\frac{e_2 yz}{1 + my + nz} - (d_2 + q_3)z\right) \\
&= -\big(x - (1 - q_1)\big)^2 - \frac{1}{e_1 e_2}(d_2 + q_3)z + \left(\frac{1 - q_1}{1 + ax + by} - \frac{1}{e_1}(d_1 + q_2)\right)y \\
&\leq -\big(x - (1 - q_1)\big)^2 - \frac{1}{e_1 e_2}(d_2 + q_3)z + \left((1 - q_1) - \frac{1}{e_1}(d_1 + q_2)\right)y
\end{aligned}$$
In consequence, ${}^{C}D^{\alpha} V_*(x, y, z) \leq 0$ where $e_1(1 - q_1) < d_1 + q_2$.
Regarding proposition 2. in [26] we obtain that, if $e_1(1 - q_1) \leq d_1 + q_2$ then $E_1 = (1 - q_1, 0, 0)$ is globally asymptotically stable. ■

Theorem 3.5.2. The equilibrium point $E_2 = (x_2, y_2, 0)$ of system (4) is globally asymptotically stable in the subregion $\phi_1 = \left\{ (x, y, z) \in \mathbb{R}_+^3 :\ 0 < y < \frac{m^*}{e_2 - m^*}\ where\ m^* = \frac{1 + ax_2}{e_1(1 + by_2)} \right\}$, under the condition
$$ay_2 \leq 1. \tag{38}$$

Proof. Consider the positive definite Lyapunov function $V_*(x, y, z)$, defined as:

$$V_*(x,y,z) = \left(x(t) - x_2 - x_2 \operatorname{Ln}\left(\frac{x(t)}{x_2}\right)\right) + \frac{1+ax_2}{e_1(1+by_2)}\left(y(t) - y_2 - y_2 \operatorname{Ln}\left(\frac{y(t)}{y_2}\right)\right) + z(t).$$

Here $V_*(x,y,z)$ is continuous function such that $V_*(x,y,z) > 0$ for all values of $\big(x(t),y(t),z(t)\big) \neq (x_2,y_2,0)$ and $V_*(x,y,z) = 0$ only at $(x_2,y_2,0)$.

Computing the α order derivative of $V_*(x,y,z)$ along the solution of system (4), it follows from Lemma 2.6 that:

$$\begin{aligned}
{}^{C}D^{\alpha}V_*(x,y,z) &\leq \left(\frac{x(t)-x_2}{x(t)}\right) {}^{C}D^{\alpha}x(t) + \frac{1+ax_2}{e_1(1+by_2)}\left(\frac{y(t)-y_2}{y(t)}\right) {}^{C}D^{\alpha}y(t) + {}^{C}D^{\alpha}\, z(t) \\
&= (x-x_2)\left(1 - x - \frac{y}{1+ax+by} - q_1\right) \\
&\quad + \frac{1+ax_2}{e_1(1+by_2)}(y-y_2)\left(\frac{e_1 x}{1+ax+by} - \frac{z}{1+my+nz} - (d_1+q_2)\right) \\
&\quad + \frac{e_2 yz}{1+my+nz} - (d_2+q_3)z \\
&\leq -(1-ay_2)(x-x_2)^2 - \frac{be_1x_2(1+ax_2)}{L_0Le_1(1+by_2)}(y-y_2)^2 \\
&\quad + \left(\left(e_2 - \frac{1+ax_2}{e_1(1+by_2)}\right)y + \frac{1+ax_2}{e_1(1+by_2)}y_2\right)\frac{z}{1+my+nz}.
\end{aligned}$$

In consequence, ${}^{C}D^{\alpha}V_*(x,y,z) \leq 0$ in the subregion ϕ_1, under the condition $ay_2 < 1$.
Regarding proposition 2. in [26] we obtain that, if $ay_2 \leq 1$ then $E_2 = (x_2,y_2,0)$ of system (4) is globally asymptotically stable in the subregion ϕ_1. ■

Theorem 3.5.3. The equilibrium point $E_3 = (x_3,y_3,z_3)$ of system (4) is globally asymptotically stable in the subregion $\phi_2 = \left\{(x,y,z) \in \mathbb{R}_+^3 \,:\, 0 < L < \frac{e_1 b x_3}{m z_3 L_1}\right\}$, under the condition

$$ay_3 \leq 1\,. \tag{39}$$

Proof. Consider the positive definite Lyapunov function $V_*(x,y,z)$, defined as:

$$\begin{aligned}
V_*(x,y,z) &= \left(x(t) - x_3 - x_3 \operatorname{Ln}\left(\frac{x(t)}{x_3}\right)\right) + \frac{1+ax_3}{e_1(1+by_3)}\left(y(t) - y_3 - y_3 \operatorname{Ln}\left(\frac{y(t)}{y_3}\right)\right) \\
&\quad + \frac{(1+ax_3)(1+my_3)}{e_1e_2(1+by_3)(1+nz_3)}\left(z(t) - z_3 - z_3 \operatorname{Ln}\left(\frac{z(t)}{z_3}\right)\right).
\end{aligned}$$

Here $V_*(x,y,z)$ is continuous function such that $V_*(x,y,z) > 0$ for all values of $\big(x(t),y(t),z(t)\big) \neq (x_3,y_3,z_3)$ and $V_*(x,y,z) = 0$ only at (x_3,y_3,z_3).

Computing the α order derivative of $V_*(x,y,z)$, along the solution of system (4), it follows from Lemma 2.6 that:

$$\begin{aligned}
{}^{C}D^{\alpha}V_*(x,y,z) &\leq \left(\frac{x(t)-x_3}{x(t)}\right) {}^{C}D^{\alpha}x(t) + \frac{1+ax_3}{e_1(1+by_3)}\left(\frac{y(t)-y_3}{y(t)}\right) {}^{C}D^{\alpha}y(x) \\
&\quad + \frac{(1+ax_3)(1+my_3)}{e_1e_2(1+by_3)(1+nz_3)}\left(\frac{z(t)-z_3}{z(t)}\right) {}^{C}D^{\alpha}z(t)
\end{aligned}$$

$$
\begin{aligned}
&= (x - x_3)\left(1 - x - \frac{y}{1 + ax + by} - q_1\right) \\
&\quad + \frac{1 + ax_3}{e_1(1 + by_3)}(y - y_3)\left(\frac{e_1 x}{1 + ax + by} - \frac{z}{1 + my + nz} - (d_1 + q_2)\right) \\
&\quad + \frac{(1 + ax_3)(1 + my_3)}{e_1 e_2 (1 + by_3)(1 + nz_3)}(z - z_3)\left(\frac{e_2 y}{1 + my + nz} - (d_2 + q_3)\right) \\
&\leq -(1 - ay_3)(x - x_3)^2 - \frac{1 + ax_3}{e_1(1 + by_3)}\left(\frac{be_1 x_3}{(1 + ax_3 + by_3)(1 + ax + by)} - mz_3\right)(y - y_3)^2 \\
&\quad - \frac{ny_3(1 + ax_3)(1 + my_3)}{e_1(1 + my_3 + nz_3)(1 + my + nz)(1 + by_3)(1 + nz_3)}(z - z_3)^2 .
\end{aligned}
$$

In consequence, ${}^{C}D^{\alpha}V_*(x, y, z) \leq 0$ in the subregion ϕ_2 under the condition $ay_3 < 1$. Regarding proposition 2 in [26], we obtain that, if $ay_3 \leq 1$ then $E_3 = (x_3, y_3, z_3)$ of system (4) is globally asymptotically stable in the subregion ϕ_2. ∎

4 Numerical simulation

Approximation and numerical approaches must be utilized since the majority of FDEs lack exact analytic solutions. For FDEs, several numerical techniques have been proposed. In this section, the dynamical behaviour of each equilibrium point has been investigated numerically, with this hypothetical set of parameter values

$$
\begin{gathered}
a = 1,\ \ b = 0.1,\ \ q_1 = 0.2,\ \ e_1 = 1.2,\ \ m = 1,\ \ n = 0.1, \\
d_1 = 0.04,\ \ q_2 = 0.55,\ \ e_2 = 1,\ \ d_2 = 0.1,\ \ q_3 = 0.2.
\end{gathered} \tag{40}
$$

The equilibrium point $E_1 = (0.8, 0, 0)$ is locally and globally asymptotically stable if $q_2 \in (0.49333, 1)$, but it is unstable when $q_2 \in (0, 0.49333)$, which indicates that a Transcritical bifurcation occurs at $q_2^* = q_2 = 0.49333$. For $q_2 = 0.55$ the eigenvalues of the Jacobian matrix at E_1 are $\lambda_1 = -0.8$ $\lambda_1 = -0.05666$ $\lambda_1 = -0.3$. Then, $E_1 = (0.8, 0, 0)$ is locally and globally asymptotically stable for all $\alpha \in (0, 1)$, and the α orders are (0.85 and 1) as demonstrated in Figure (1 and 2) sequentially.

In addition, Figure 3 illustrates a Transcritical bifurcation of E_1 and shows the impact of harvesting rate of intermediate predator population (q_2) on stability of E_1.

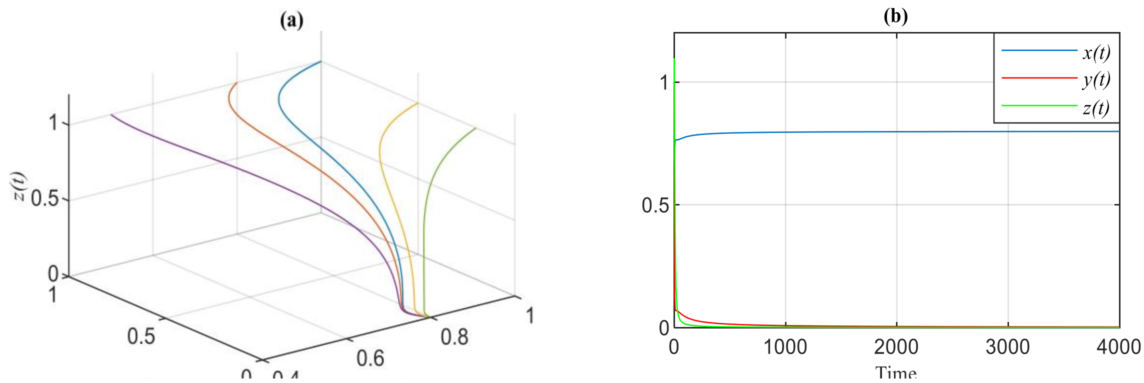


Figure 1: System (4) with α = 0.85 and $q_2 = 0.55$. (a) 3D view of convergence of $E_1 = (0.8, 0, 0)$, (b) The time series starts at $(x_0, y_0, z_0) = (1, 1, 1)$.

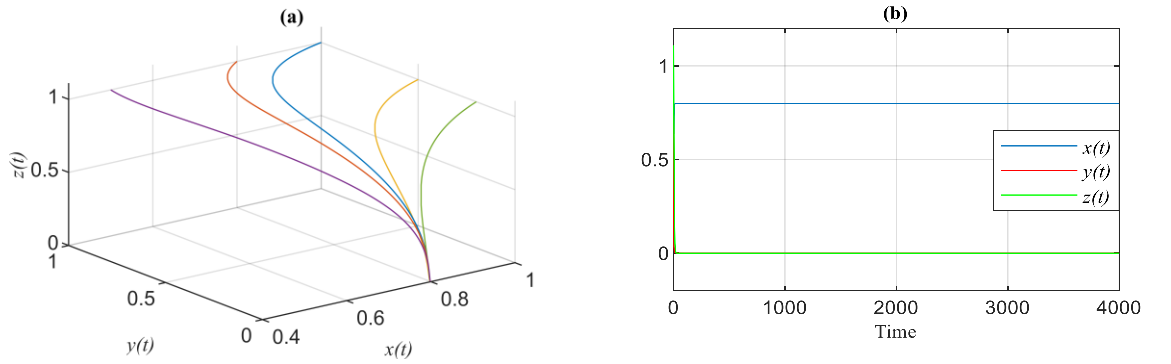


Figure 2: System (4) with $\alpha = 1$ and $q_2 = 0.55$. (a) 3D view of convergence of $E_1 = (0.8, 0, 0)$, (b) The time series starts at $(x_0, y_0, z_0) = (1, 1, 1)$.

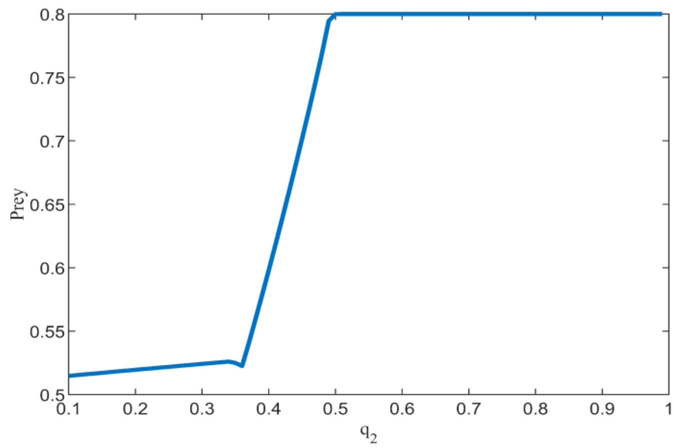
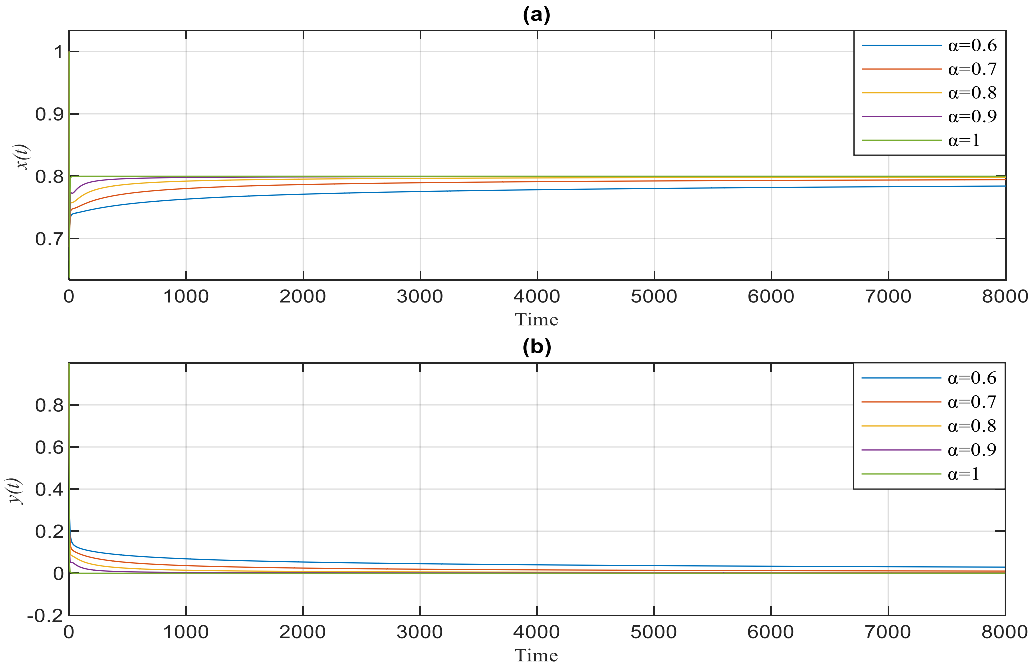


Figure 3: Transcritical bifurcation diagram of $E_1 = (0.8, 0, 0)$.

Figure 4 illustrates the impact of the fractional order α on convergence speed of solutions, for different α values. All solutions converge to $E_1 = (0.8, 0, 0)$; the solution of system (4) has a greater convergence speed and requires less time when α is larger compared to smaller α value. As the order α approaches 1, the system's solution proceeds toward the integer-order system (3).



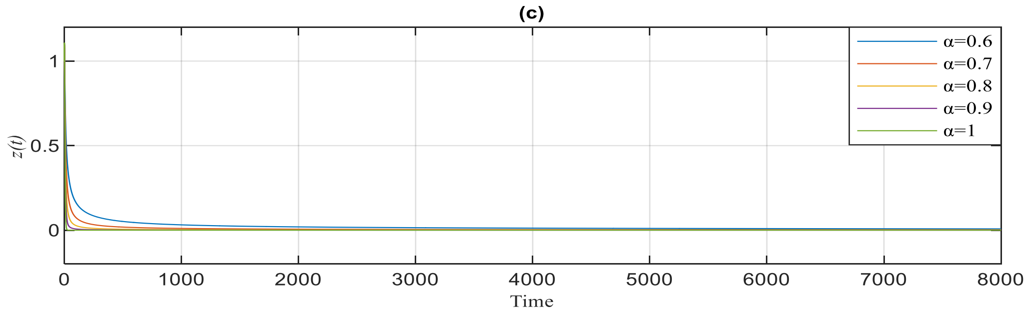


Figure 4: The solution curves of system (4) of equilibrium point $E_1 = (0.8, 0, 0)$ with different values of α, showing impact of α on convergence speed of (a) prey (b) intermediate predator (c) super predator.

Regarding $E_2 = (x_2, y_2, 0)$, which exists when $q_2 = 0.01$ and is locally asymptotically stable when $q_3 \in (0.3712, 1)$ while unstable when $q_3 \in (0, 0.3712)$, with the remaining parameters as given in (40) are same.

By choosing $q_3 = 0.4$, we obtain $E_2 = (0.04718485, 0.8525155, 0)$, which is locally and globally asymptotically stable for all $\alpha \in (0, 1)$, with the α orders of 0.85 and 1, as illustrated in Figure 5 and 6 respectively.

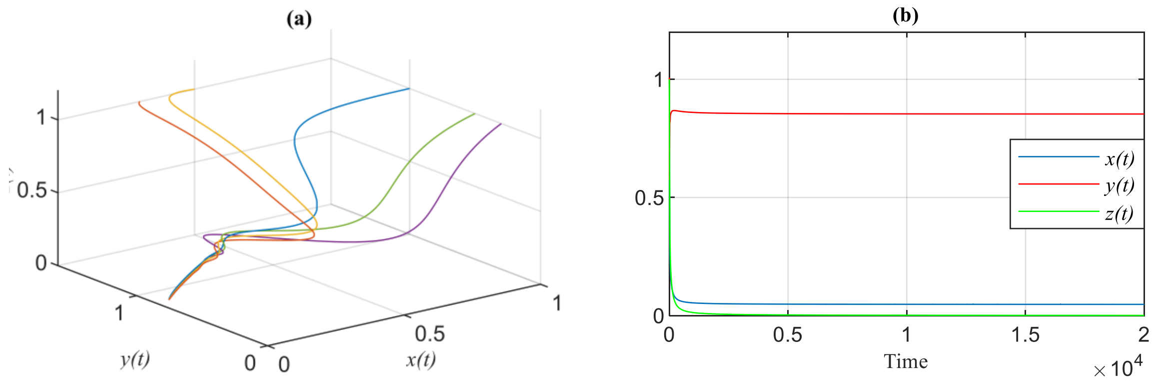


Figure 5: System (4) with $\alpha = 0.85$ and $a = 1$. (a) 3D view of convergence of $E_2 = (0.04718485, 0.8525155, 0)$, (b) The time series starts at $(x_0, y_0, z_0) = (1, 1, 1)$.

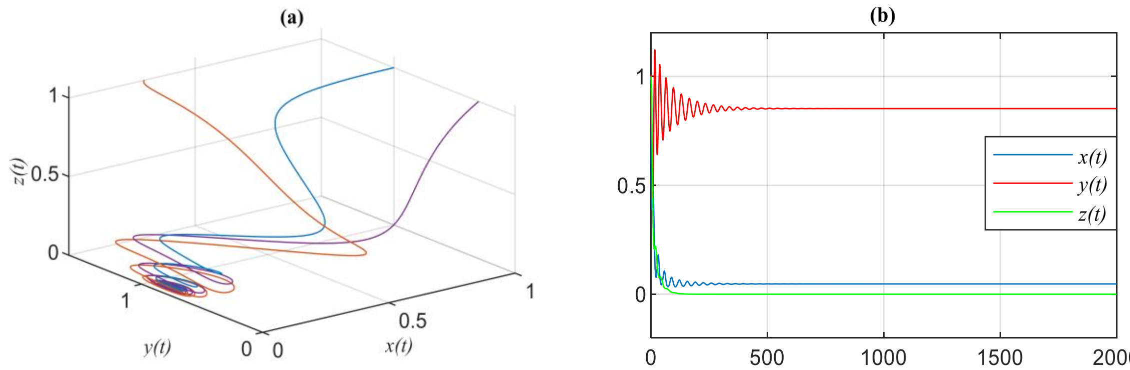


Figure 6: System (4) with $\alpha = 1$ and $a = 1$. (a) 3D view of convergence of $E_2 = (0.04718485, 0.8525155, 0)$, (b) The time series starts at $(x_0, y_0, z_0) = (1, 1, 1)$.

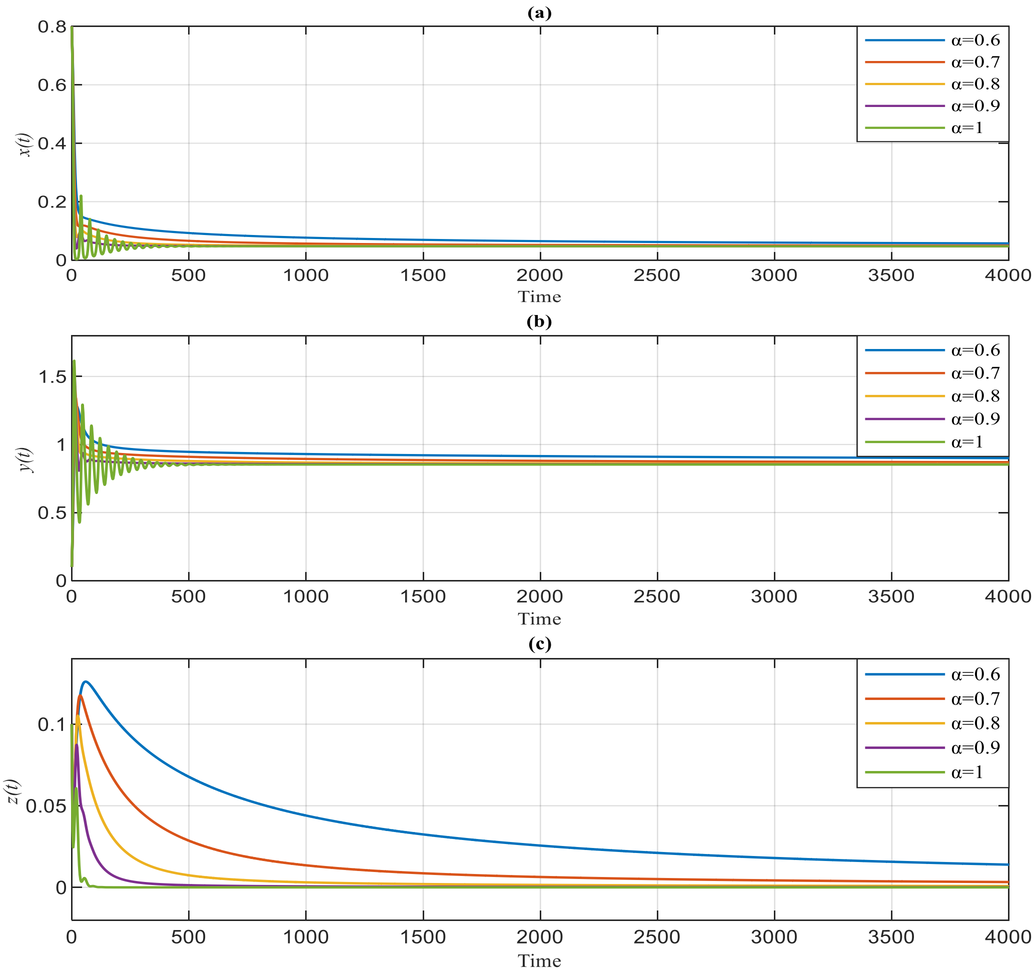


Figure 7: The solution curves of system (4) of equilibrium point E_2 with different values of α, showing impact of α on convergence speed of (a) prey (b) intermediate predator (c) super predator.

Regarding Hopf bifurcation, the parameter value plays a significant role in controlling Hopf bifurcation. For example, for the parameters in (40), if $0 < a < 1.657$, Hopf bifurcation does not occur around E_2 as investigated above. However, when $a > 1.657$, Hopf bifurcation does occur around E_2 as explored in Theorem 8. Therefore, among the parameters in (40), we take $a = 2$ in our example to guarantee the occurrence of Hopf bifurcation around $E_2 = (0.049508, 0.891726, 0)$ with the bifurcation value $\alpha_2 = \frac{2}{\pi} tan^{-1} \frac{0.178172972324115}{0.0046403534085} = 0.9834235$.

Hence, if $\alpha \in (0, 0.9834235)$, E_2 is locally asymptotically stable, as shown in Figure 8. When $\alpha \in (0.9834235, 1)$, E_2 is unstable and experiences a limit cycle around E_2, as demonstrated in Figure 9.

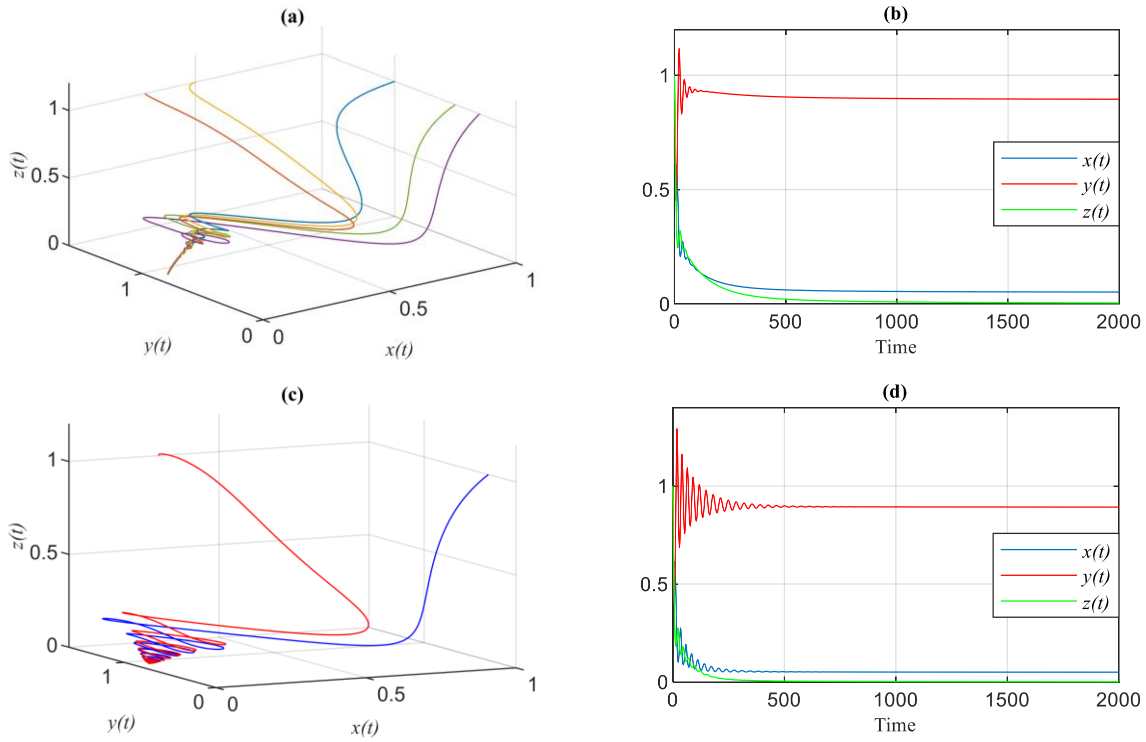


Figure 8: (a) and (c) are 3D views of convergence of $E_2 = (0.049508, 0.891726, 0)$ with $\alpha = 0.9$ and $\alpha = 0.96$, respectively. (b) and (d) are their time series which start at $(x_0, y_0, z_0) = (1, 1, 1)$.

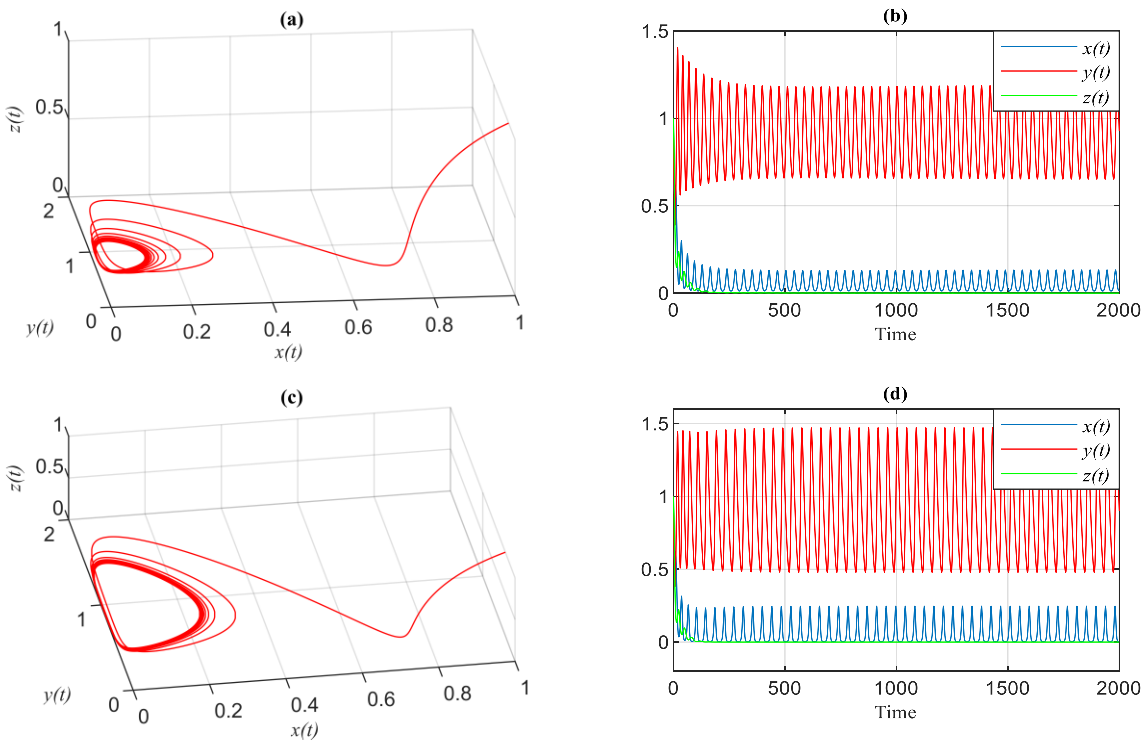


Figure 9: (a) and (c) are 3D views of limit cycle around $E_2 = (0.049508, 0.891726, 0)$ with $\alpha = 0.99$ and $\alpha = 1$, respectively. (b) and (d) are their time series which start at $(x_0, y_0, z_0) = (1, 1, 1)$.

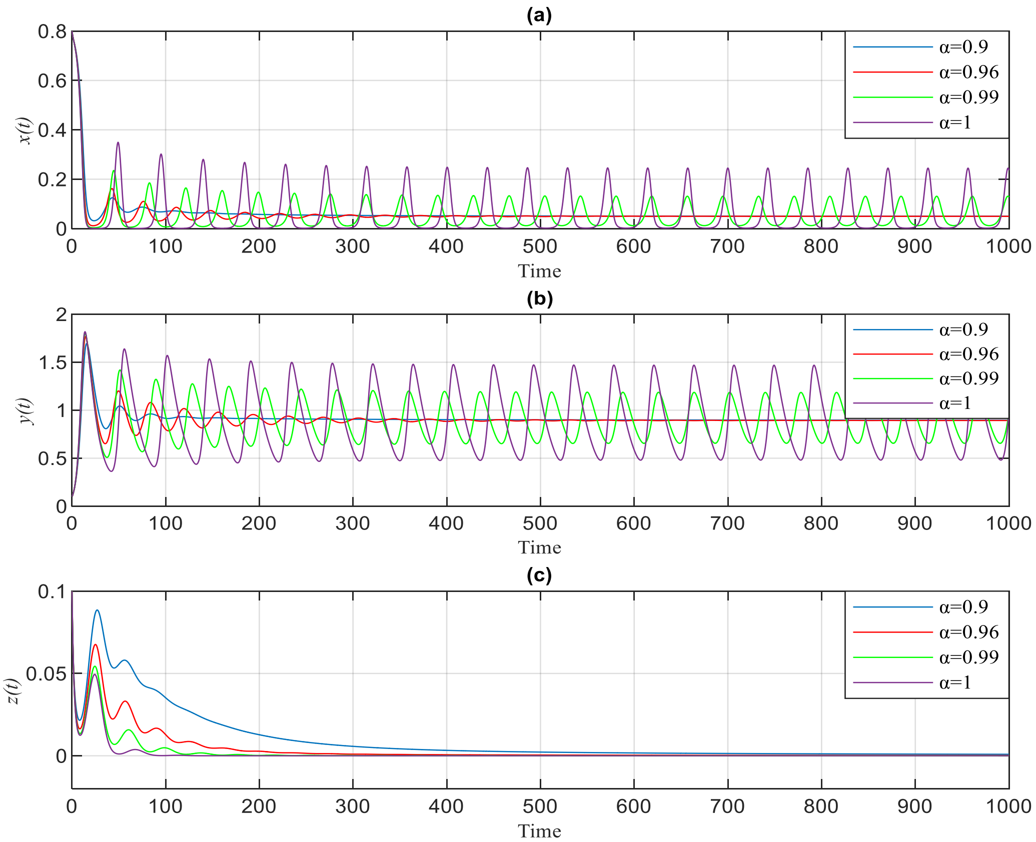


Figure 10: The solution curves of system (4) of equilibrium point E_2 with different values of α, showing impact of α on convergence speed of (a) prey (b) intermediate predator (c) super predator.

Regarding the co-existence equilibrium point E_3, by use of this set of parameter values

$$a = 0.2,\quad b = 0.3,\quad q_1 = 0.01,\quad e_1 = 1,\quad m = 2,\quad n = 0.2,$$
$$d_1 = 0.3,\quad q_2 = 0.2,\quad e_2 = 0.6,\quad d_2 = 0.1,\quad q_3 = 0.01, \tag{41}$$

$E_3 = (0.748535, 0.299294, 0.169623)$ is locally and globally asymptotically stable. As shown in Figure 11 and 12.

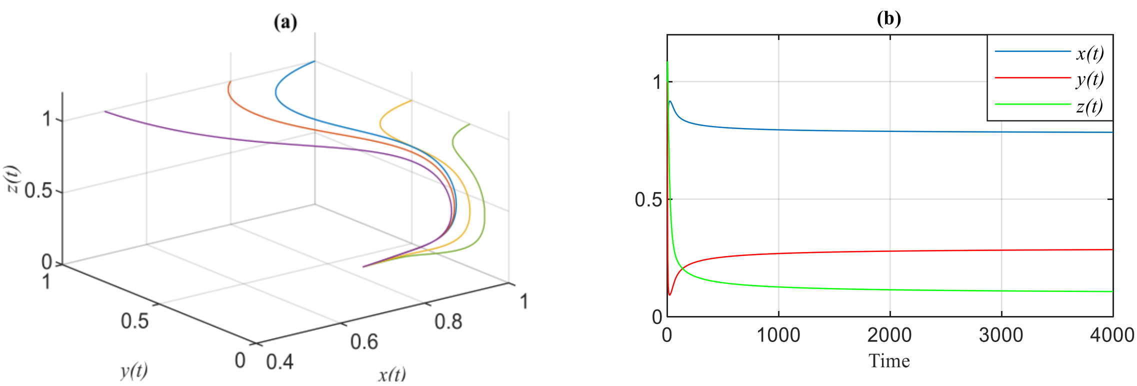


Figure 11: System (4) with $\alpha = 0.8$. (a) 3D view of convergence of $E_3 = (0.748535, 0.299294, 0.169623)$, (b) The time series starts at $(x_0, y_0, z_0) = (1, 1, 1)$.

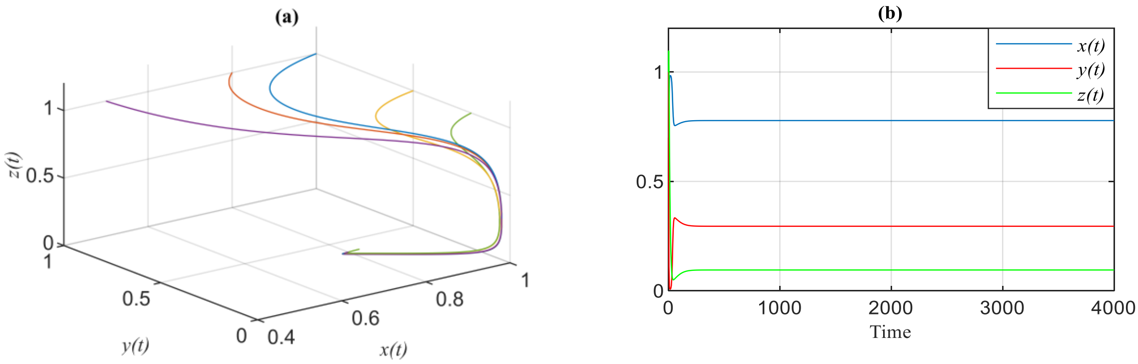


Figure 12: System (4) with α = 1. (a) 3D view of convergence of $E_3 = (0.748535, 0.299294, 0.169623)$, (b) The time series starts at $(x_0, y_0, z_0) = (1, 1, 1)$.

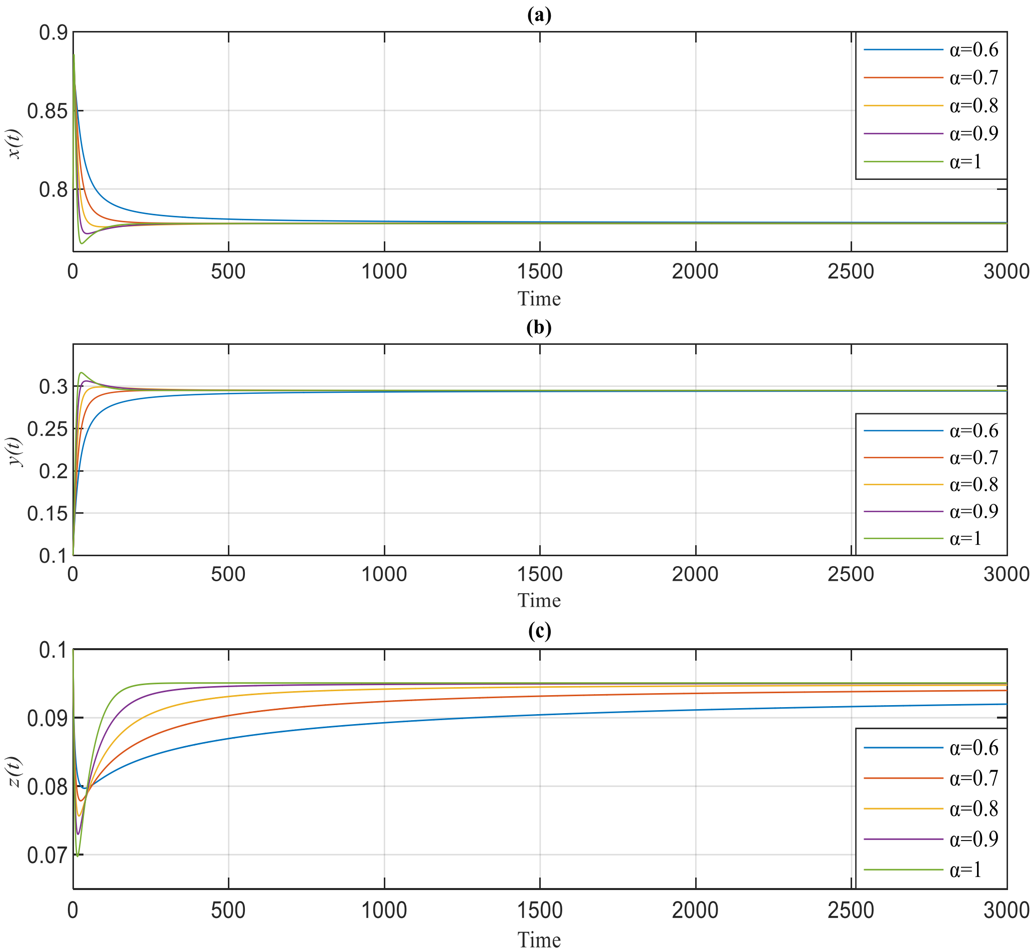


Figure 13: The solution curves of system (4) of equilibrium point E_3 with different values of α, showing impact of α on convergence speed of (a) prey (b) intermediate predator (c) super predator.

In the second case of Theorem 3.3.4, the parameter values are considered as

$$a = 1, \quad b = 3.45, \quad q_1 = 0.6, \quad e_1 = 10.6, \quad m = 10, \quad n = 0.1, \\ d_1 = 0.03, \quad q_2 = 0.04, \quad e_2 = 2, \quad d_2 = 0.03, \quad q_3 = 0.03. \tag{42}$$

This parameter set satisfies the positivity condition for $E_3 = (0.359302, 0.064356, 5.016552)$. It is locally asymptotically stable for all $\alpha \in \left(0, \frac{2}{3}\right)$, as illustrated in Figure.14, and unstable for $\alpha \in \left(\frac{2}{3}, 1\right)$ as shown in Figure.15. In Figure 16, the impact of fractional order is shown on each species.

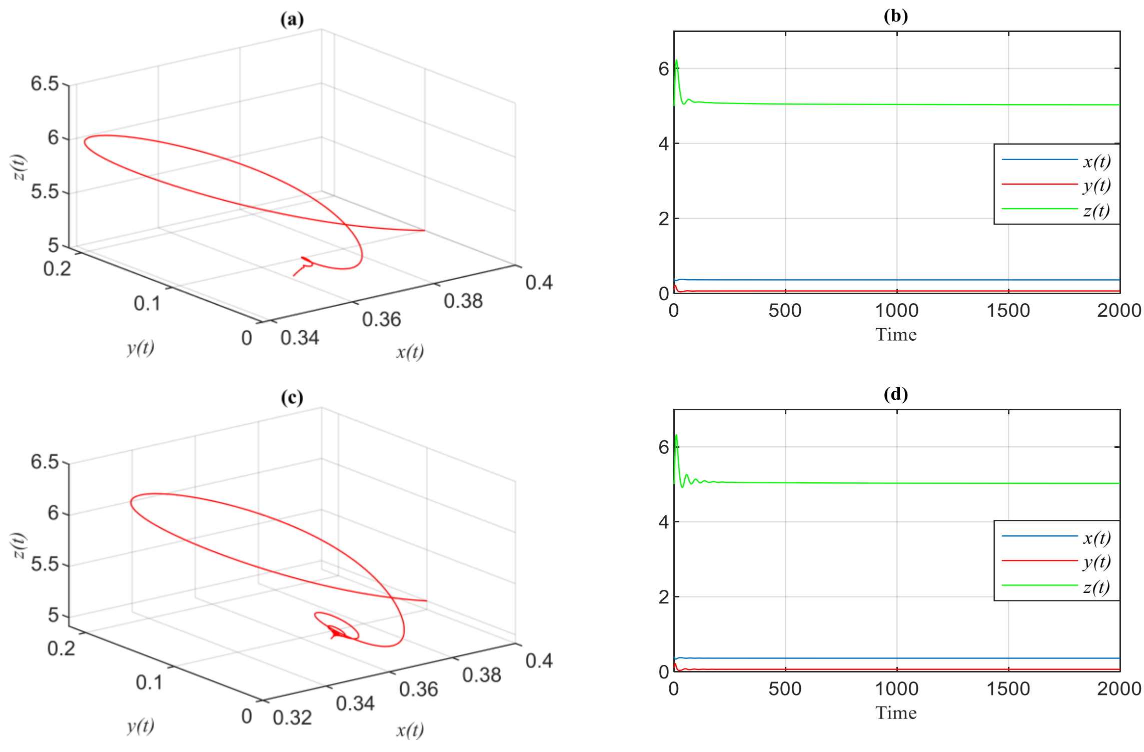


Figure 14: (a) and (c) are 3D views of convergence of $E_3 = (0.359302, 0.064356, 5.016552)$, with $\alpha = 0.6$ and $\alpha = 0.65$, respectively. (b) and (d) are their time series which start at $(x_0, y_0, z_0) = (0.4, 0.1, 5)$.

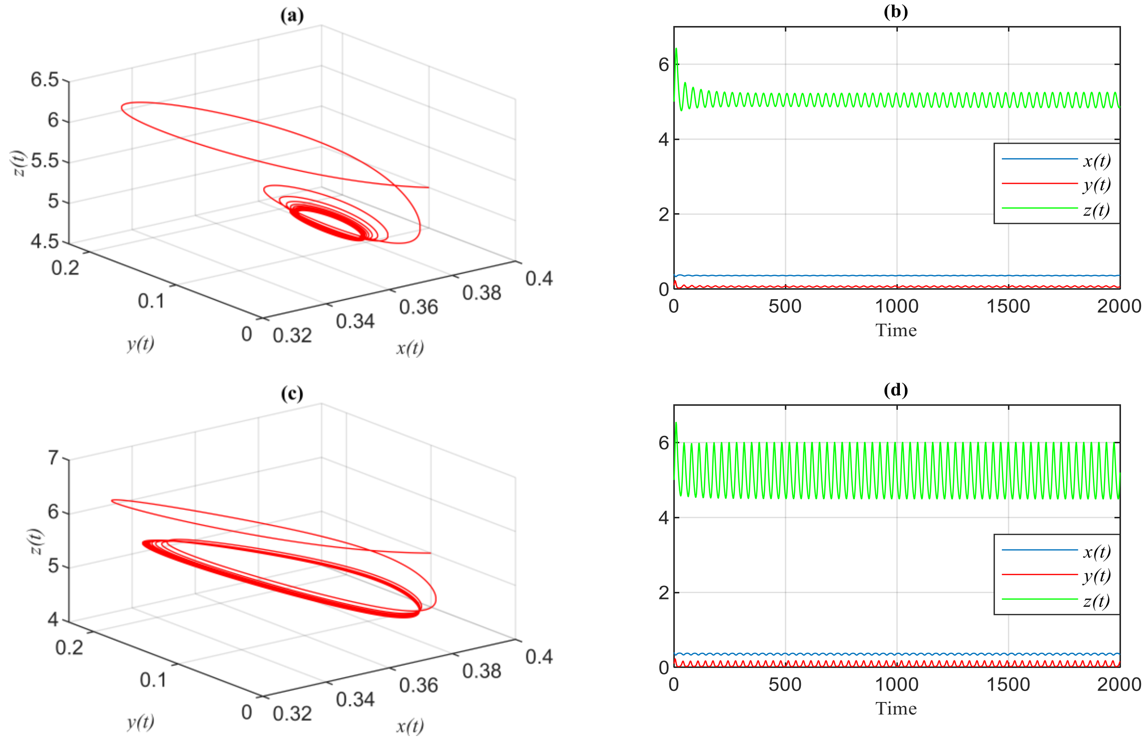


Figure 15: (a) and (c) are 3D views of limit cycle around $E_3 = (0.359302, 0.064356, 5.016552)$, with $\alpha = 0.7$ and $\alpha = 0.75$, respectively. (b) and (d) are their time series which start at $(x_0, y_0, z_0) = (0.4, 0.1, 5)$.

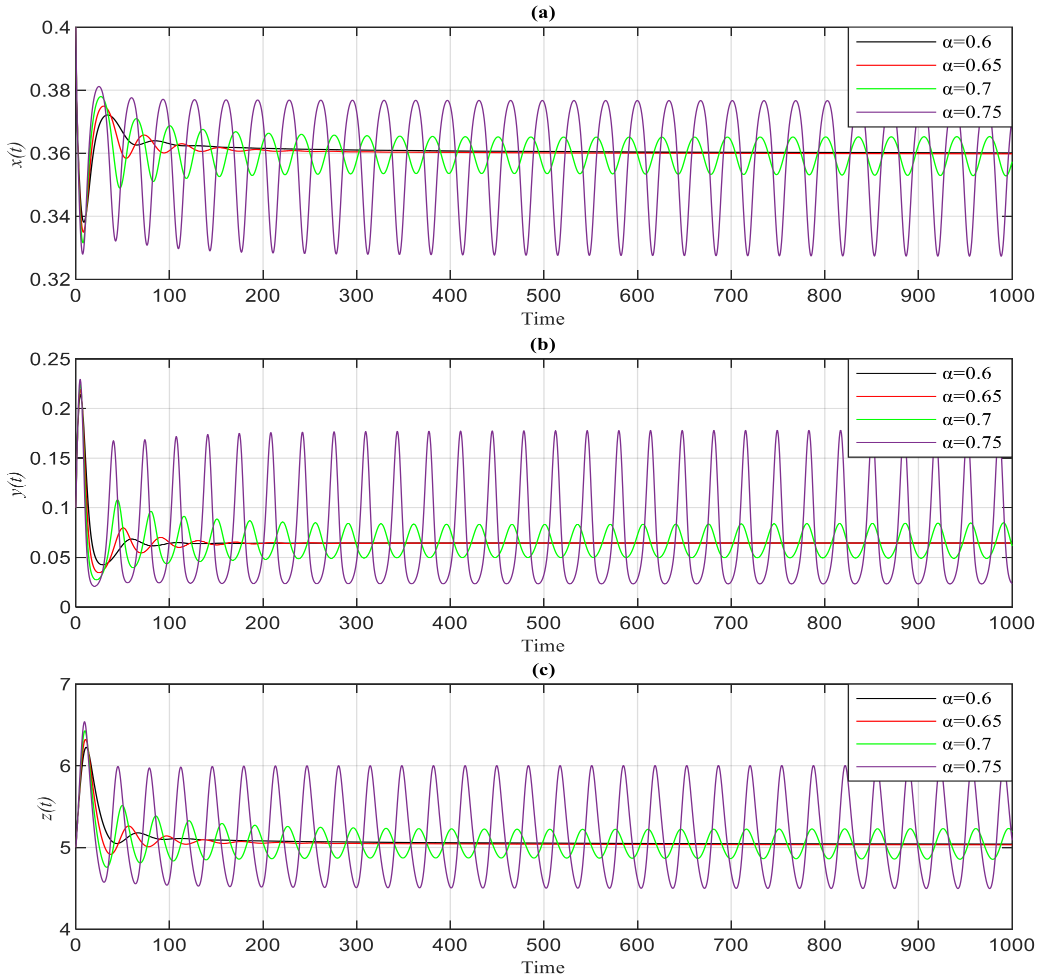


Figure 16: The solution curves of system (4) of equilibrium point E_3 with different values of α, showing impact of α on convergence speed of (a) prey (b) intermediate predator (c) super predator.

5. Conclusions

In this study, we analysed a fractional-order tri-trophic model with harvesting and the BD functional response, demonstrating its superiority over integer-order systems in capturing realistic dynamics. Among the chief benefits of the fractional calculus is that a higher-order system may be converted to a lower-order model via employing fractional-order. First, we created a model of the tri-trophic food chain that included harvesting of each species. Then, we evaluated the conditions for the solutions' non-negativity and uniform boundedness. We have also determined the equilibrium points and their existence. For local stability of various equilibrium points, Routh-Hurwitz criterion is used for FDEs and we established some criterion to verify global stability of E_1, E_2 and E_3, by formulating desirable Lyapunov functions. Our results highlight that the intermediate predator's harvesting parameter controls Transcritical bifurcation at E_1 of system (4) and the derivative order parameter α controls the occurrence of Hopf- bifurcation on both E_2 and E_3 of system (4). Biologically, fractional-order dynamics introduce memory-driven transitions, and controlled harvesting stabilizes populations by mitigating abrupt collapses. These insights, supported by simulations, underscore the interplay of mathematical and ecological principles in trophic systems. Finally, theoretical findings are supported by several numerical simulations.

Conflict of interest: According to the authors, there aren't any conflicting interests.

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