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On the Estimation of $P(Y_1 < X < Y_2)$ in Cased Inverted Kumaraswamy Distribution

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Abstract
This paper deals with the estimation of the stress-strength reliability for a component which has a strength that is independent on opposite lower and upper bound stresses, when the stresses and strength follow Inverse Kumaraswamy Distribution. D estimation approaches were applied, namely the maximum likelihood, moment, and shrinkage methods. Monte Carlo simulation experiments were performed to compare the estimation methods based on the mean squared error criteria.

Keywords: Inverted Kumaraswamy distribution, Stress - Strength reliability, Maximum likelihood estimator, Moment estimator, Shrinkage estimator

في حالة توزيع كومارا سوامي المعكوس $P(Y_1 < X < Y_2)$ حول تقدير

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الخلاصة
يتعلق موضوع البحث بتقدير معولية الاجهاد- المتانة لنظام يحتوي على مركبة واحدة لديها متانة وتتعرض الى اجهاد محدد من الاعلى والاسفل في حالة كل من الاجهاد والمتانة مستقلين و يتبعان توزيع كومارا سوامي المعكوس بأستخدام طرائق تقدير مختلفة منها الامكان الاعظم, مقدر العزوم ومقدرات التقلص. استخدمت طريقة محاكاة مونتي كارلو للمقارنة بين طرائق التقدير المستخدمة بالاعتماد على معيار متوسط مربعات الخطأ.

1. Introduction

The stress-strength model in the reliability research describes the life of a component which has a random strength X and is subjected to a random stress Y. This problem arises in the classical stress–strength reliability where one is interested in estimating the proportion of the times the random strength X of a component exceeds the random stress Y to which the component is subjected [1]. An important case is the estimation of $R = P(Y_1 < X < Y_2)$ which represents the situation where the strength X should not only be greater than stress Y_1 but also be smaller than stress Y_2. Because of that, modern engineering systems may have more than two components [2].
For instance, many electronic components cannot work at very high or very low voltages.

Similarly, person's blood pressure should lie within two limits, i.e., systolic and diastolic. The stress-strength model of $P(Y_1 < X < Y_2)$ was studied in many branches of science, such as psychology, medicine, pedagogy, etc.[3].

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Singh introduced the maximum likelihood, minimum variance unbiased, and empirical estimators of R= P$(Y_1 < X < Y_2)$, where X, Y_1 and Y_2 are mutually independent random variables based on normal distribution [4]. In 1998, Ivshin investigated the Maximum Likelihood (MLE) and the UMVUE of R when X, Y_1 and Y_2 are either uniform or exponential random variables with unknown location parameters [5]. Wang et al., in 2013, made statistical inference for P$(Y_1 < X < Y_2)$ via two methods, which are the nonparametric normal approximation and the jackknife empirical likelihood. Then, classical and real data sets were analyzed using these two proposed methods [6]. In 2013, Hassan et al. focused on the estimation of R=$(Y_1 < X < Y_2)$, where stresses Y_1 and Y_2 and the strength X have Weibull distribution with common known shape and scale parameters in presence of k outliers, where MLE, Moment method (MOM), and mixture estimators are obtained. The results of the simulation study showed that the mixture estimators are better and easier ones [7].

This paper focuses on the estimation of the $R= P(Y_1 < X < Y_2)$, under the assumption that, X, Y_1, and Y_2 are independent and that these stress and strength variables have Inverse Kumaraswamy Distribution. The maximum likelihood estimator, moment estimator, and some shrinkage estimation methods were used. In addition, Monte Carlo simulation was performed for comparing the different methods of estimation.

The rest of the paper is organized as follows: Section 2 clarifies the Inverted Kumaraswamy Distribution. Section 3 describes the stress-strength models. Section 4 deals with the Maximum likelihood Estimation , Moment Method , and Shrinkage Estimation Method of $P(Y_1 < X < Y_2)$. Section 5 presents the simulation study. Section 6 demonstrates the effectiveness of the proposed methods through numerical results. Finally, in Section 7, a conclusion is provided.

2. Inverted Kumaraswamy Distribution

In 1980, Kumaraswamy proposed the Kumaraswamy Distribution (KumD) by fixing some parameters for beta distribution [8]. However, it has a closed-form cumulative distribution function which is invertible and for which the moments do exist. The KumD method was widely applied for testing natural phenomena such as test scores, temperatures and daily hydrological data of rain fall, [9-12].

After that, Abd Al-Fattah [13] derived the inversion of KumD by using the transformation $X = \frac{1-T}{T}$; $T\sim$KumD (α ,β) [8].

The probability density function (pdf) of a r.v.X which follows the inverted Kumaraswamy distribution, $X\sim$ IKum (α ,β), can be written as:

$$f(x,\alpha,\beta) = \alpha\beta(1+x)^{-(\alpha+1)}(1-(1+x)^{-\alpha})^{\beta-1} \;, x > 0\,;\; \alpha,\beta > 0 \tag{1}$$

where α and β are shape parameters.

Also, the cumulative distribution function (CDF) of X has the form below:

$$F(x;\alpha,\beta) = (1-(1+x)^{-\alpha})^{\beta} \;, x > 0\,, \alpha,\beta > 0 \tag{2}$$

3. Stress-Strength models $P(Y_1 < X < Y_2)$

The reliability of a component (or system) under such a situation may be defined as $R = P(Y_1 < X < Y_2)$, where Y_1, Y_2 are two independent random stress variables, such that $Y_1 \sim IKumD(\alpha,\beta_2)$, $Y_2 \sim IKumD(\alpha,\beta_3)$ and X be an independent random strength variable, such that $X\sim IKumD(\alpha,\beta_1)$, with a known parameter α. Therefore, the (S-S) reliability is defined as below:

$$R= P(Y_1 < X < Y_2)$$
$$= \int_0^\infty P(Y_1 < X\,, X < Y_2) f(x)\,dx$$
$$= \int_0^\infty F_{y_1}(x)\,\bar{F}_{y_2}(x)\,f(x)dx = \int_0^\infty Fy_1(x)[1-Fy_2(x)]f(x)\,dx$$
$$= \int_0^\infty Fy_1(x)f(x)\,dx - \int_0^\infty Fy_1(x)\,Fy_2(x)f(x)dx$$
$$= \frac{\beta_1}{\beta_1+\beta_2} - \int_0^\infty (1-(1+x)^{-\alpha})^{\beta_2}(1-(1+x)^{-\alpha})^{\beta_3}\,.\,\alpha\beta_1(1+x)^{-(\alpha+1)}.\,(1-(1+x)^{-\alpha})^{\beta_1-1}\,dx$$
$$= \frac{\beta_1}{\beta_1+\beta_2} - \frac{\beta_1}{\beta_1+\beta_2+\beta_3}\int_0^\infty \alpha(\beta_1+\beta_2+\beta_3)(1+x)^{-(\alpha+1)}\,(1-(1+x)^{-\alpha})^{(\beta_1+\beta_2+\beta_3)-1}\,dx$$

Then we get

$$R(x) = \frac{\beta_1}{\beta_1+\beta_2} - \frac{\beta_1}{\beta_1+\beta_2+\beta_3}$$

$$R(x) = \frac{\beta_1\beta_3}{(\beta_1+\beta_2)(\beta_1+\beta_2+\beta_3)} \tag{3}$$

4. Estimation methods of $R= P(Y_1 < X < Y_2)$

4.1 The Maximum Likelihood Estimation (MLE)

The maximum likelihood has been an important and commonly used method, since it contains the properties required for good estimation. The likelihood function is given by:
$l = L(x_1, x_2, \dots, x_n; \beta_1, \alpha)$
$= \prod_{i=1}^{n} f(x_i)$
When $x_1, x_2, \dots . x_n$ is a strength random sample of X from IKumD (α, β_1), where β_1 is an unknown parameter and α is known
$= \prod_{i=1}^{n} \alpha\beta_1(1 + x_i)^{-(\alpha+1)} (1 - (1 + x_i)^{-\alpha})^{\beta_1 - 1}$

$$= \alpha^n \beta_1^n \prod_{i=1}^{n}(1 + x_i)^{-(\alpha+1)} . \prod_{i=1}^{n}(1 - (1 + x_i)^{-\alpha})^{\beta_1 - 1} \quad (4)$$

By taking ln to both sides, we have
$ln\, l = n\, ln\, \alpha + n\, ln\, \beta_1 - (\alpha + 1) \sum_{i=1}^{n} ln(1 + x_i) + (\beta_1 - 1) \sum_{i=1}^{n} ln(1 - (1 + x_i)^{-\alpha})$
The partial derivative of ln l with respect to β_1 which is equal to zero is given by:
$\frac{\partial \ln l}{\partial \beta_1} = \frac{n}{\beta_1} + \sum_{i=1}^{n} \ln(1 - (1 + x_i)^{-\alpha}) = 0$
Hence, the MLE estimator for the unknown shape parameter β_1 will be:

$$\hat{\beta}_{1_{MLE}} = \frac{-n}{\sum_{i=1}^{n} \ln(1-(1+x_i)^{-\alpha})} \quad (5)$$

In the same way, let $y_{1_1}, y_{1_2}, \dots, y_{1_{m1}}$ and $y_{2_1}, y_{2_2}, \dots, y_{2_{m2}}$ be stress random variable from IKumD (α, β_2) and IKumD (α, β_3), respectively, and the MLE for the unknown shape parameters β_2 , β_3 will be

$$\hat{\beta}_{2_{MLE}} = \frac{-m_1}{\sum_{j=1}^{m_1} \ln\left(1-(1+y_{1j})^{-\alpha}\right)} \quad (6)$$

$$\hat{\beta}_{3_{MLE}} = \frac{-m_2}{\sum_{r=1}^{m_2} \ln(1-(1+y_{2r})^{-\alpha})} \quad (7)$$

By substitutingequ ation (5), (6) and (7) in equation (3), we get

$$\hat{R}_{MLE} = \frac{\hat{\beta}_{1_{MLE}} \hat{\beta}_{3_{MLE}}}{(\hat{\beta}_{1_{MLE}} + \hat{\beta}_{2_{MLE}})(\hat{\beta}_{1_{MLE}} + \hat{\beta}_{2_{MLE}} + \hat{\beta}_{3_{MLE}})} \quad (8)$$

4.2 The Moment Method (MOM)

In this subsection, the moment estimation method will be used to estimate the parameter β for IKumD, when the parameter α is known based on equality of the sample and population means.
Assume that $X \sim IKumD(\alpha, \beta)$, then the non –central moment of X will be:
$\mathrm{E}(X^r) = \beta \sum_{j=0}^{r} \binom{r}{j} -1^{r-j} B\left(1 - \frac{j}{\alpha}, \beta\right) r= 1,2,\dots.$
Hence, population means of X, Y_1 and Y_2 when $\alpha > 1$ are respectively given by
$\mathrm{E}(X) = \beta_1 B\left(1 - \frac{1}{\alpha}, \beta_1\right) - 1$
$\mathrm{E}(Y_1) = \beta_2 B\left(1 - \frac{1}{\alpha}, \beta_2\right) - 1$
$\mathrm{E}(Y_2) = \beta_3 B\left(1 - \frac{1}{\alpha}, \beta_3\right) - 1$
where $B(.,.)$ refer to Beta distribution.
And when equalizing the sample mean with the corresponding population mean we get

$$\bar{X} = \frac{\sum_{i=1}^{n} x_i}{n} = \beta_1 B\left(1 - \frac{1}{\alpha}, \beta_1\right) - 1$$

$\bar{Y_1} = \frac{\sum_{j=1}^{m_1} y_j}{m_1} = \beta_2 B\left(1 - \frac{1}{\alpha}, \beta_2\right) - 1$
$\bar{Y_2} = \frac{\sum_{j=1}^{m_2} y_j}{m_2} = \beta_2 B\left(1 - \frac{1}{\alpha}, \beta_3\right) - 1$

Consequently, $$\hat{\beta}_{1_{MOM}} = \frac{1 + \sum_{i=1}^{n} \frac{x_i}{n}}{B\left(1 - \frac{1}{\alpha}, \beta_{1_0}\right)} \quad (9)$$

$$\hat{\beta}_{2_{MOM}} = \frac{1 + \sum_{j=1}^{m_1} \frac{y_{1j}}{m_1}}{B\left(1 - \frac{1}{\alpha}, \beta_{2_0}\right)} \quad (10)$$

and $$\hat{\beta}_{3_{MOM}} = \frac{1 + \sum_{r=1}^{m_2} \frac{y_{2r}}{m_2}}{B\left(1 - \frac{1}{\alpha}, \beta_{3_0}\right)} \quad (11)$$

where β_{i_0} is a prior value of β_i, i= 1,2,3

By substituting the equations (9),(10) and(11) in the equation (3), we get the approximate estimator of R as below:

$$\hat{R}_{MOM} = \frac{\hat{\beta}_{1_{MOM}}\hat{\beta}_{3_{MOM}}}{\left(\hat{\beta}_{1_{MOM}}+\hat{\beta}_{2_{MOM}}\right)\left(\hat{\beta}_{1_{MOM}}+\hat{\beta}_{2_{MOM}}+\hat{\beta}_{3_{MOM}}\right)} \quad (12)$$

4.3 The Shrinkage Estimation Method (Sh)

In 1968, Thompson proposed the shrink usual estimator $\hat{\beta}$ (ex. MLE or Unbiased estimator) of the parameter β to prior information β_0, using shrinkage weight factor $\psi(\hat{\beta})$, such that $0\le \psi(\hat{\beta}) \le 1$. Thompson stated that "We are estimating β where we believe that β_0 is close to the true value of β and something bad happens if $\beta_0 \approx \beta$, and we do not use β_0". Thus, Thompson gave the form of shrinkage estimator of β say $\hat{\beta}_{Sh}$ as bellow:

$$\hat{\beta}_{Sh} = \psi(\hat{\beta})\hat{\beta}_{ub} + (1-\psi(\hat{\beta}))\beta_0 \quad (13)$$

where the unbiased estimator $\hat{\beta}_{ub}$ was applied as a usual estimator of β, β_0 is a very close value to β as a prior information (initial estimate), and $\psi(\hat{\beta})$ denotes the shrinkage weight factor as mentioned above, such that $0\le \psi(\hat{\beta}) \le 1$, which may be a function of $\hat{\beta}_{ub}$, a function of sample size, a constant, or may be found through minimizing the mean square error of $\hat{\beta}_{sh}$ (ad hoc basis) [14- 17].

There is no doubt if our assumption in this work is to take $\beta_{i_0} = [\beta_i - 0.001]$ as a prior information of β_i for i =1, 2, 3.

Note that $\beta_{i_{MLE}}$ is a biased estimator, since $E(\hat{\beta}_{i_{MLE}}) = \frac{\lambda}{\lambda-1}\beta_i \neq \beta_i$, λ refers to n, m_1 or m_2.

Thus, $\hat{\beta}_{i_{ub}} = \frac{\lambda-1}{\lambda}\hat{\beta}_{i_{MLE}}$ becomes an unbiased estimator of β_i, whereas $E(\hat{\beta}_{i_{ub}}) = \beta_i$.

Likewise,

$\mathrm{var}(\hat{\beta}_{i_{ub}}) = \frac{(\beta_i)^2}{(\lambda-2)}$ i.e.,

$$\hat{\beta}_{1_{ub}} = \frac{n-1}{-\sum_{i=1}^{n}\ln(1-(1+x_i)^{-\alpha})} \quad (14)$$

$$\hat{\beta}_{2_{ub}} = \frac{m_1-1}{-\sum_{j=1}^{m_1}\ln\left(1-\left(1+y_{1j}\right)^{-\alpha}\right)} \quad (15)$$

and $$\hat{\beta}_{3_{ub}} = \frac{m_2-1}{-\sum_{r=1}^{m_2}\ln(1-(1+y_{2r})^{-\alpha})} \quad (16)$$

4.3.1 The Constant shrinkage factor (Sh1)

In this subsection, the constant shrinkage weight factor will be assumed as $\Psi(\hat{\beta}_1) = \Psi(\hat{\beta}_2) = \Psi(\hat{\beta}_3) = 0.01$, and this implies to the following shrinkage estimators

$$\hat{\beta}_{1_{Sh1}} = \Psi(\hat{\beta}_1)\hat{\beta}_{1_{ub}} + \left(1-\Psi(\hat{\beta}_1)\right)\beta_{1_0} \quad (17)$$

$$\hat{\beta}_{2_{Sh1}} = \Psi(\hat{\beta}_2)\hat{\beta}_{2_{ub}} + \left(1-\Psi(\hat{\beta}_2)\right)\beta_{2_0} \quad (18)$$

$$\hat{\beta}_{3_{Sh1}} = \Psi(\hat{\beta}_3)\hat{\beta}_{3_{ub}} + \left(1-\Psi(\hat{\beta}_3)\right)\beta_{3_0} \quad (19)$$

The substitution of the equations (17), (18) and (19) in equation (3) leads to the estimation of (S-S) reliability using the constant shrinkage estimator $\hat{R}_{Sh1}$ as below:

$$\hat{R}_{Sh1} = \frac{\hat{\beta}_{1_{Sh1}}\hat{\beta}_{3_{Sh1}}}{\left(\hat{\beta}_{1_{Sh1}}+\hat{\beta}_{2_{Sh1}}\right)\left(\hat{\beta}_{1_{Sh1}}+\hat{\beta}_{2_{Sh1}}+\hat{\beta}_{3_{Sh1}}\right)} \quad (20)$$

4.3.2 The modified Thompson type shrinkage weight factor (Th)

The shrinkage weight factor, which was considered by Thompson in 1968, will be modified in this subsection as follows:

$$\varphi(\hat{\beta}_i) = \frac{\left(\hat{\beta}_{i_{ub}}-\hat{\beta}_{i_0}\right)^2}{\left(\hat{\beta}_{i_{ub}}-\hat{\beta}_{i_0}\right)^2+var\left(\hat{\beta}_{i_{ub}}\right)}(0.001) \quad \text{for i=1, 2, 3} \quad (21)$$

where, $var(\hat{\beta}_{i_{ub}})$ is as defined in section (4.3).

In consequence, the modified Thompson type shrinkage estimator will be:

$$\hat{\beta}_{i_{Th}} = \varphi(\hat{\beta}_i)\hat{\beta}_{i_{ub}} + \left(1-\varphi(\hat{\beta}_i)\right)\beta_{i_0} \quad \text{, for i = 1, 2, 3} \quad (22)$$

By substituting the equation (22) in the equation (3), we get the modified Thompson type shrinkage estimation of the S-S reliability as below

$$\hat{R}_{Th} = \frac{\hat{\beta}_{1_{Th}}\hat{\beta}_{3_{Th}}}{\left(\hat{\beta}_{1_{Th}}+\hat{\beta}_{2_{Th}}\right)\left(\hat{\beta}_{1_{Th}}+\hat{\beta}_{2_{Th}}+\hat{\beta}_{3_{Th}}\right)} \quad (23)$$

4.3.4 The squared shrinkage weight function (Sq)

This subsection is concerned with the shrinkage estimator based on squared shrinkage weight function which is defined as below:

$$\gamma(\hat{\beta}_i) = \left(\frac{\hat{\beta}_{i_{ub}} - \mathrm{E}\left(\hat{\beta}_{i_{ub}}/\beta_{i_0}\right)}{\sqrt{\boldsymbol{var}(\hat{\beta}_{i_{ub}}/\beta_{i_0})}}\right)^{\mathbf{2}} .\epsilon \,,\ \epsilon = 0.001, \text{ for i = 1, 2, 3} \quad (24)$$

The shrinkage estimator should be:

$$\hat{\beta}_{i_{Sq}} = \gamma(\hat{\beta}_{i_{ub}})\hat{\beta}_{i_{ub}} + \left(1 - \gamma(\hat{\beta}_{i_{ub}})\right)\beta_{i_0} \text{ , i = 1, 2, 3} \quad (25)$$

When substituting the equation (25) in the equation (3), the squared shrinkage estimation of the (S-S) reliability becomes:

$$\hat{R}_{Sq} = \frac{\hat{\beta}_{1_{Sq}}\hat{\beta}_{3_{Sq}}}{\left(\hat{\beta}_{1_{Sq}}+\hat{\beta}_{2_{Sq}}\right)\left(\hat{\beta}_{1_{Sq}}+\hat{\beta}_{2_{Sq}}+\hat{\beta}_{3_{Sq}}\right)} \quad (26)$$

5. Simulation study

The Monte Carlo Simulation (MCS) is an artificial sampling method which may be used for solving complicated problems in analytic formulations and for simulating purely statistical problems [18]. Therefore, MCS is used in this section to investigate the effectiveness of the different estimators of reliability. Different sample sizes (20, 50 and 100) were used based on MSE criteria, with 1000 replicates. The steps of simulation using Mote Carlo approach are as follows:

Step1: Generating random sample sizes of 20, 50 and 100 from the uniform distribution which is defined on the interval (0,1) as $u_1, u_2, \dots, u_n$; $v_1, v_2, \dots, v_{m_1}$ and $w_1, w_2, \dots, w_{m_2}$, respectively.

Step2: Appling the cumulative distribution function to convert the uniform random samples to random samples following the IKumD, as shown below:

$$\mathrm{F}(x) = (1 - (1 + x_i)^{-\alpha})^{\beta_1}$$

$$u_i = (1 - (1 + x_i)^{-\alpha})^{\beta_1}$$

$$x_i = \left[1 - (u_i)^{\frac{1}{\beta_1}}\right]^{-\frac{1}{\alpha}} - 1$$

After that, we get y_{1j} and y_{2r} by using the same method

$$y_{1j} = \left[1 - (v_j)^{\frac{1}{\beta_2}}\right]^{-\frac{1}{\alpha}} - 1$$

$$y_{2r} = \left[1 - (w_r)^{\frac{1}{\beta_3}}\right]^{-\frac{1}{\alpha}} - 1$$

Step3: Computing the MLE for reliability using equation (8).

Step4: Finding the moment estimate for reliability using equation (12).

Step5: Computing the shrinkage estimators of reliability using equations (20), (23) and (26).

Step6: Based on L=1000 trials, MSE is calculated as follows:

$$\mathrm{MSE} = \frac{1}{L}\sum_{i=1}^{L}\left(\hat{R}_i - R\right)^2$$

6. Numerical Results

In this section, simulation results are introduced based on three parameters (β_1, β_2 , β_3) and three sample problem sizes (20, 50,100) that implemented 1000 duplicates. In addition, Tables- 1 to 8 explain the results of the proposed estimation methods. Tables- 2, 4, 6, and 8 present the simulation results for MSE of all the proposed estimation methods. Based on the simulation data, the shrinkage estimator ($\hat{R}_{Th}$), using the modified Thompson type shrinkage weight factor shown in these tables, showed the best results and had less MSE for the $R= P(Y_1 < X < Y_2)$ of the Inverse Kumaraswamy distribution. While the estimator ($\hat{R}_{Sq}$) was in the second rank, followed by Sh1, MOM and MLE.

Table 1- Estimation value of R, when $\alpha = 5$, $\beta_1 = 2$, $\beta_2 = 4$ and $\beta_3 = 2.5$.

(n, m_1, m_2)	R	$\hat{R}_{MLE}$	$\hat{R}_{MOM}$	$\hat{R}_{Sh1}$	$\hat{R}_{Th}$	$\hat{R}_{sq}$
(20,20,20)	0.098039	0.099443	0.098186	0.09801018	0.09801784	0.09801554
(50,20,20)	0.098039	0.099618	0.098011	0.09800121	0.09801775	0.09800840
(50,100,20)	0.098039	0.101208	0.098071	0.09805960	0.09801965	0.09803481
(20,100,20)	0.098039	0.100487	0.097933	0.09806347	0.09801898	0.09804146
(100,50,50)	0.098039	0.099100	0.098137	0.09801825	0.09801864	0.09801840
(50,50,50)	0.098039	0.098343	0.097874	0.09801373	0.09801802	0.09801877
(100,100,100)	0.098039	0.098021	0.097881	0.09801357	0.09801786	0.09801695
(50,100,100)	0.098039	0.098665	0.098207	0.09802404	0.09801859	0.09802204
(20,50,100)	0.098039	0.098963	0.098330	0.09802996	0.09801887	0.09802378
(20,100,50)	0.098039	0.099164	0.098030	0.09804206	0.09801877	0.09802901

Table 2- MSE value of R= 0.098039 when $\alpha = 5$, $\beta_1 = 2$, $\beta_2 = 4$ and $\beta_3 = 2.5$.

(n, m_1, m_2)	$\hat{R}_{MLE}$	$\hat{R}_{MOM}$	$\hat{R}_{Sh1}$	$\hat{R}_{Th}$	$\hat{R}_{sq}$	best
(20,20,20)	0.009726	0.000726	1.210E-06	1.256 E-08	7.475 E-07	$\hat{R}_{Th}$
(50,20,20)	0.009010	0.000724	1.109 E-06	1.190 E-08	4.375 E-07	$\hat{R}_{Th}$
(50,100,20)	0.004466	0.000344	4.853E-07	6.962 E-09	1.824 E-07	$\hat{R}_{Th}$
(20,100,20)	0.004338	0.000324	4.908 E-07	7.218 E-09	3.674 E-07	$\hat{R}_{Th}$
(100,50,50)	0.003468	0.000288	3.684 E-07	6.750 E-09	6.165 E-08	$\hat{R}_{Th}$
(50,50,50)	0.003800	0.000316	4.284 E-07	7.475 E-09	1.110 E-07	$\hat{R}_{Th}$
(100,100,100)	0.001860	0.000163	2.0137 E-07	5.944 E-09	3.235 E-08	$\hat{R}_{Th}$
(50,100,100)	0.001975	0.000168	2.077 E-07	5.693 E-09	3.881 E-08	$\hat{R}_{Th}$
(20,50,100)	0.003556	0.000264	4.008 E-07	6.839 E-09	1.064 E-07	$\hat{R}_{Th}$
(20,100,50)	0.002972	0.000203	3.273 E-07	6.323 E-09	7.535 E-08	$\hat{R}_{Th}$

Table 3- Estimation value of R, when $\alpha = 5$, $\beta_1 = 1.5$, $\beta_2 = 2$ and $\beta_3 = 3$.

(n, m_1, m_2)	R	$\hat{R}_{MLE}$	$\hat{R}_{MOM}$	$\hat{R}_{Sh1}$	$\hat{R}_{Th}$	$\hat{R}_{sq}$
(20,20,20)	0.197802	0.196879	0.198410	0.1977912	0.19780724	0.19777577
(50,20,20)	0.197802	0.196774	0.197665	0.19777175	0.19780671	0.19779584
(50,100,20)	0.197802	0.200070	0.198072	0.19785809	0.19780927	0.19782717
(20,100,20)	0.197802	0.199459	0.197572	0.19787192	0.19780912	0.19783801
(100,50,50)	0.197802	0.198355	0.198036	0.19780780	0.19780900	0.19780958
(50,50,50)	0.197802	0.197600	0.197781	0.19780524	0.19780854	0.19780896
(100,100,100)	0.197802	0.197080	0.197757	0.19780080	0.19780805	0.19780703
(50,100,100)	0.197802	0.198354	0.197935	0.19782131	0.19780931	0.19781412
(20,50,100)	0.197802	0.198183	0.198076	0.19782628	0.19780925	0.19780830
(20,100,50)	0.197802	0.198486	0.198029	0.19784449	0.19780929	0.19782415

Table 4- MSE value of R= 0.197802 when = 5 , $\beta_1 = 1.5$, $\beta_2 = 2$ and $\beta_3 = 3$.

(n, m_1, m_2)	$\hat{R}_{MLE}$	$\hat{R}_{MOM}$	$\hat{R}_{Sh1}$	$\hat{R}_{Th}$	$\hat{R}_{sq}$	best
(20,20,20)	0.002268	0.000163	3.056 E-07	1.894 E-09	2.864 E-07	$\hat{R}_{Th}$
(50,20,20)	0.002113	0.000160	2.704 E-07	1.650 E-09	1.590 E-07	$\hat{R}_{Th}$
(50,100,20)	0.009942	0.000793	1.162 E-06	7.619 E-09	4.429 E-07	$\hat{R}_{Th}$
(20,100,20)	0.001049	0.000072	1.303 E-07	8.088 E-010	5.788 E-08	$\hat{R}_{Th}$
(100,50,50)	0.008370	0.000665	8.979 E-07	5.888 E-09	1.349 E-07	$\hat{R}_{Th}$
(50,50,50)	0.009203	0.000709	1.019 E-06	6.757 E-09	2.239 E-07	$\hat{R}_{Th}$

(100,100,100)	**0.004376**	**0.000363**	**4.593 E-07**	**3.178 E-09**	**6.846 E-08**	$\hat{R}_{Th}$
(50,100,100)	**0.004911**	**0.000395**	**5.285 E-07**	**3.782 E-09**	**1.178 E-07**	$\hat{R}_{Th}$
(20,50,100)	**0.009126**	**0.000659**	**1.058 E-06**	**6.750 E-09**	**4.302 E-07**	$\hat{R}_{Th}$
(20,100,50)	**0.007476**	**0.000540**	**8.760 E-07**	**5.767 E-09**	**2.602 E-07**	$\hat{R}_{Th}$

Table 5- Estimation value of R, when $\alpha = 5$, $\beta_1 = 2.2$, $\beta_2 = 3.3$ and $\beta_3 = 4$.

(n, m_1, m_2)	R	$\hat{R}_{MLE}$	$\hat{R}_{MOM}$	$\hat{R}_{Sh1}$	$\hat{R}_{Th}$	$\hat{R}_{sq}$
(20,20,20)	**0.168421**	**0.170239**	**0.169262**	**0.16842691**	**0.16841795**	**0.16842309**
(50,20,20)	**0.168421**	**0.168540**	**0.168096**	**0.16838549**	**0.16841540**	**0.16840674**
(50,100,20)	**0.168421**	**0.170535**	**0.168252**	**0.16846155**	**0.16841730**	**0.16844194**
(20,100,20)	**0.168421**	**0.171319**	**0.168434**	**0.16848596**	**0.16841839**	**0.16845307**
(100,50,50)	**0.168421**	**0.168980**	**0.168652**	**0.16841051**	**0.16841661**	**0.16840813**
(50,50,50)	**0.168421**	**0.168413**	**0.168608**	**0.16841016**	**0.16841624**	**0.16841465**
(100,100,100)	**0.168421**	**0.167362**	**0.168286**	**0.16840465**	**0.16841580**	**0.16841446**
(50,100,100)	**0.168421**	**0.169267**	**0.168533**	**0.16842939**	**0.16841762**	**0.16842186**
(20,50,100)	**0.168421**	**0.168108**	**0.168991**	**0.16842396**	**0.16841689**	**0.16841798**
(20,100,50)	**0.168421**	**0.169770**	**0.168497**	**0.16845727**	**0.16841785**	**0.16843252**

Table 6- MSE value of R= 0.168421 when = 5 , $\beta_1 = 2.2$, $\beta_2 = 3.3$ and $\beta_3 = 4$.

(n, m_1, m_2)	$\hat{R}_{MLE}$	$\hat{R}_{MOM}$	$\hat{R}_{Sh1}$	$\hat{R}_{Th}$	$\hat{R}_{sq}$	best
(20,20,20)	**0.001893**	**0.000148**	**2.321 E-07**	**1.505 E-09**	**9.107 E-08**	$\hat{R}_{Th}$
(50,20,20)	**0.001907**	**0.000149**	**2.415 E-07**	**1.620 E-09**	**9.028 E-08**	$\hat{R}_{Th}$
(50,100,20)	**0.008164**	**0.000618**	**9.385 E-07**	**6.381 E-09**	**4.548 E-07**	$\hat{R}_{Th}$
(20,100,20)	**0.001006**	**0.000066**	**1.214 E-07**	**7.914 E-010**	**7.148 E-08**	$\hat{R}_{Th}$
(100,50,50)	**0.007738**	**0.000611**	**8.496 E-07**	**6.292 E-09**	**3.536 E-07**	$\hat{R}_{Th}$
(50,50,50)	**0.008263**	**0.000664**	**9.253 E-07**	**6.664 E-09**	**1.927 E-07**	$\hat{R}_{Th}$
(100,100,100)	**0.003893**	**0.000326**	**4.149 E-07**	**3.195 E-09**	**7.241 E-08**	$\hat{R}_{Th}$
(50,100,100)	**0.004251**	**0.000334**	**4.507 E-07**	**3.288 E-09**	**1.950 E-07**	$\hat{R}_{Th}$
(20,50,100)	**0.007750**	**0.000542**	**9.018 E-07**	**6.263 E-09**	**3.731 E-07**	$\hat{R}_{Th}$
(20,100,50)	**0.006328**	**0.000437**	**7.207 E-07**	**4.937 E-09**	**1.427 E-07**	$\hat{R}_{Th}$

Table 7- Estimation value of R, when $\alpha = 5$, $\beta_1 = 1.5$, $\beta_2 = 2.5$ and $\beta_3 = 3.5$.

(n, m_1, m_2)	R	$\hat{R}_{MLE}$	$\hat{R}_{MOM}$	$\hat{R}_{Sh1}$	$\hat{R}_{Th}$	$\hat{R}_{sq}$
(20,20,20)	**0.175000**	**0.175100**	**0.175259**	**0.17497916**	**0.17498986**	**0.17498647**
(50,20,20)	**0.175000**	**0.173949**	**0.174784**	**0.17494903**	**0.17498881**	**0.17496093**
(50,100,20)	**0.175000**	**0.178194**	**0.175088**	**0.17504787**	**0.17499261**	**0.17501377**
(20,100,20)	**0.175000**	**0.177755**	**0.175309**	**0.17506322**	**0.17499251**	**0.17502183**
(100,50,50)	**0.175000**	**0.175829**	**0.175268**	**0.17498983**	**0.17499117**	**0.17499070**
(50,50,50)	**0.175000**	**0.175519**	**0.174951**	**0.17499062**	**0.17499093**	**0.17499178**
(100,100,100)	**0.175000**	**0.174322**	**0.174813**	**0.17498140**	**0.17499008**	**0.17498951**
(50,100,100)	**0.175000**	**0.174665**	**0.174830**	**0.17499255**	**0.17499059**	**0.17499440**
(20,50,100)	**0.175000**	**0.175034**	**0.175025**	**0.17500191**	**0.17499081**	**0.17499063**
(20,100,50)	**0.175000**	**0.176298**	**0.175154**	**0.17503136**	**0.17499173**	**0.17500884**

Table 8- MSE value of R= 0.175000 when = 5 , $\beta_1 = 1.5$, $\beta_2 = 2.5$ and $\beta_3 = 3.5$.

(n, m_1, m_2)	$\hat{R}_{MLE}$	$\hat{R}_{MOM}$	$\hat{R}_{Sh1}$	$\hat{R}_{Th}$	$\hat{R}_{sq}$	best
(20,20,20)	0.002098	0.000149	2.682 E-07	1.762 E-09	1.654 E-07	$\hat{R}_{Th}$
(50,20,20)	0.001842	0.000164	2.517 E-07	1.715 E-09	5.704 E-07	$\hat{R}_{Th}$
(50,100,20)	0.008734	0.000673	1.003 E-06	6.963 E-09	3.583 E-07	$\hat{R}_{Th}$
(20,100,20)	0.009556	0.000646	1.143 E-06	7.324 E-09	3.172 E-07	$\hat{R}_{Th}$
(100,50,50)	0.007490	0.000597	7.981 E-07	5.905 E-09	1.248 E-07	$\hat{R}_{Th}$
(50,50,50)	0.008400	0.000624	9.187 E-07	6.700 E-09	1.671 E-07	$\hat{R}_{Th}$
(100,100,100)	0.004116	0.000336	4.354 E-07	3.802 E-09	6.628 E-08	$\hat{R}_{Th}$
(50,100,100)	0.004695	0.000324	4.944 E-07	4.004 E-09	8.798 E-08	$\hat{R}_{Th}$
(20,50,100)	0.008809	0.000572	1.043 E-06	7.230 E-09	4.098 E-07	$\hat{R}_{Th}$
(20,100,50)	0.006834	0.000420	7.887 E-07	5.438 E-09	1.914 E-07	$\hat{R}_{Th}$

7. Conclusions

The estimation of S-S reliability for two parameters using the inverted Kumaraswamy distribution was introduced in this paper using different estimation methods, namely MLE, MOM, and three types of shrinkage methods. Monte Carlo simulation exhibited that the performance of the squared shrinkage estimator ($\hat{R}_{Th}$) had the appropriate behavior, being a more efficient estimator than the others in the sense of MSE.

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