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On ST-Essential (Complement) Submodules

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Abstract

Let R be an associative ring with identity and let D be a left R-module. As a generalization of T-essential submodules, we introduce the concept of the small T-essential submodule. Let T be a proper submodule of a module D. A submodule N such that $N \nleq T$ is small T-essential (ST-essential) and denoted by $N \leq_{STe} D$, if for each submodule L of a module D, such that $N \cap L \leq T$, implies that $L \leq T$. We also define ST-complement submodules and show the relationships between ST-essential and S-closed, ST-essential and S-singular, and ST-complement and ST-essential submodules. Some properties and theories about these concepts are also provided.

Keywords: S-Essential T-Essential ST-Essential T-Complement ST-Complement.

المقاسات الجزئيه الصغيرة الكبيره(المكملة) من النمطT

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الخلاصه

لتكن R حلقه تجميعيه ذات عنصر محايد ولتكن D مقاسا احاديا ايسر معرف عليها. كتعميم لمفهوم المقاسات الجزئيه الكبيره من النمط T . قدمنا تعريف المقاسات الجزئيه الصغيره الكبيره من النمط T . ليكن T مقاس جزئي فعلي من D . المقاس الجزئي N بحيث $N \nleq T$ يدعى مقاسا صغيرً كبيرً من النمط T, اذا كان لكل مقاس جزئي L من D , بحيث $N \cap L \leq T$ فأن $L \leq T$ وسنوضح العلاقه بين المقاسات الجزئيه الصغيره الكبيره من النمط T والمقاسات الصغيره المغلقه. كذلك عرفنا المقاس الجزئي المكمل من النمط T , ودرسنا خواصه.

1.Introduction

Let R be an associative ring with unitary and let D be unitary left R-module. A submodule N of D is essential submodule of D ($N \leq_e D$), if for each submodule L of D, $N \cap L \neq 0$ [1]. A submodule N of D is called small submodule of D ($N \ll D$), if for each L submodule of a module D, such that N+L=D,implies that L=D [2]. A submodule N of a module D is said to be small essential submodule (s-essential) and denoted by $N \leq_{s.e} D$, if N+L=D, for each L small submodule of D, implies that L=D [3, 4]. Let R be a ring and T be a proper submodule of a right R-module D. A submodule N of a

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module D is called T-essential submodule of D (and denoted by $N \leq_{T.e} D$), such that N$\nleq$T and, for each submodule L of D, $N \cap L \leq T$, implies that $L \leq T$. In this paper , as a generalization of T-essential submodule, we introduce the concept of small T-essential submodule. Let T be a proper submodule of a module D. A submodule N, such that N$\nleq$T, is small T- essential (ST-essential) and denoted by $N \leq_{ST.e} D$, if for each small submodule L of a module D, such that $N \cap L \leq T$, implies that $L \leq T$. We also provide some basic properties of this concept. In section two, we introduce the definition of small T- complement submodule .Let N,T be submodules of D. A small submodule L is called ST- complement for N of D, if N is maximal with respect to the property that $N \cap L \leq T$. We give some basic properties and various characterizations of this concept.

2. ST-Essential Submodules

In this section, we introduce the definition of ST-essential submodules, as a generalization of T-essential submodule, and we study some basic properties of this type of submodules.

Definition (2.1): Let T be a proper submodule of a module D. A submodule N, such that N$\nleq$T, is called small T-essential of D (ST − essential) and denoted by $N \leq_{ST.e} D$, if for each small submodule L of D with $N \cap L \leq T$, then $L \leq T$.

Remarks and examples (2.2):

1- It is clear that every T.essential is ST.essential. But the converse is not true. For example: The module $\mathbb{Z}_{24}$ as $\mathbb{Z}$-module. Let T=$6\mathbb{Z}_{24}$, N=$2\mathbb{Z}_{24}$. The small submodules of $\mathbb{Z}_{24}$ are $\{\bar{0}\}$, $6\mathbb{Z}_{24}$,$12\mathbb{Z}_{24}$. If L =$\{\bar{0}\}$, then $N \cap L = \{\bar{0}\} \leq T$, and $L \leq T$. If L=$6\mathbb{Z}_{24}$, then $N \cap L = 12\mathbb{Z}_{24} \leq T$, and $L \leq T$. IfL=$12\mathbb{Z}_{24}$,then$N \cap L = 6\mathbb{Z}_{24} \leq T$,and$L \leq T$.Then$N \leq_{ST.e} \mathbb{Z}$.

But if L=$3\mathbb{Z}_{24}$ is submodule of $\mathbb{Z}_{24}$ of $\mathbb{Z}$, $N \cap L = \{\bar{0}, \bar{6}, \overline{12}, \overline{18}\} \leq T$, then N is not T-essential.

2- The module $\mathbb{Z}_6$ as $\mathbb{Z}$-module. Let T=$\{\bar{0},\bar{3}\}$, N=$\{\bar{0}, \bar{2}, \bar{4}\}$. The only small submodule of $\mathbb{Z}_6$ is $\{\bar{0}\}$. Clearly,$N \cap L = \{\bar{0}\} \leq T$, implies that $L \leq T$.Then $N \leq_{ST.e} \mathbb{Z}_6$.

3- The module $\mathbb{Z}_{12}$ as Z-module. Let T=$6\mathbb{Z}_{12}$, N=$2\mathbb{Z}_{12}$. The small submodule of $\mathbb{Z}_{12}$ are $\{\bar{0}\}$, $\{\bar{0}, \bar{6}\}$. If L=$\{\bar{0}\}$, then $N \cap L = \{\bar{0}\} \leq T$ and $L \leq T$. Also, if L=$\{\bar{0}, \bar{6}\}$, then $N \cap L = \{\bar{0}, \bar{6}\} \leq T$, $L \leq T$. Then $N \leq_{ST.e} \mathbb{Z}_{12}$.

4- The module $\mathbb{Z}_{p^\infty}$ as Z-module. By example (2,1,4-3) [5], $\mathbb{Z}_{p^\infty} \leq_{T.e} \mathbb{Z}$. Since every T. essential is ST.essential,then$\mathbb{Z}_{p^\infty} \leq_{ST.e} D$.

5- It is clear that T and N are submodules of a module D, and T=0. Then $N \leq_{ST.e} D$ if and only if $N \leq_{S.e} D$.

Remark (2.3): Let T and N be submodules of a module D, such that N$\nleq$T. Then $N \leq_{ST.e} D$ if and only if for every small submodule L of D, $L \nleq T$, implies that $N \cap L \nleq T$.

Proof: ⇒) Let $N \leq_{ST.e} D$ and let L be small submodule of D, such that $L \nleq T$. Assume that $N \cap L \leq T$. As $N \leq_{ST.e} D$, then $L \leq, T$ which is a contradiction. Thus, $N \cap L \nleq T$.

(⇐ Let L be a small submodule of a module D , such that $N \cap L \leq T$. We want to show that $L \leq T$. Assume that $L \nleq T$, implies that $N \cap L \nleq T$, which is a contradiction. Then $N \leq_{ST.e} D$.

Corollary (2.4): Let T, N be submodules of a module D, such that N$\nleq$T. Then $N \leq_{ST.e} D$ if and only if for each $x \in D$, such that Rx small submodule of D . Rx$\nleq$ T , implies that $N \cap Rx \nleq T$.

Proposition (2.5): Let T, N be a submodules of a module D, such that N$\nleq$T. Then $N \leq_{ST.e} D$ if and only if for each $x \in D - T$, such that R_x small submodule in D . Then there exists $r \in R$, such that $rx \in N - T$.

Proof:⇒) Let $N \leq_{ST.e} D$, let $x \in D - T$, and let R_x small submodule of D . Let $L = R_x$. Then $L \nleq T$. Since $N \leq_{ST.e} D$,by (2.4), then $N \cap R_x \nleq T$. Hence, there exist $n \in N \cap R_x$ and $n \notin T$. Let $n = rx$, for some $r \in R$. Thus, $rx \in N - T$.(⇐ Let L be small submodule of D , such that $N \cap L \leq T$. We want to show that $L \leq T$. Assume that $L \nleq T$ .Then there exists $x \in L - T$. By our assumption , there exists $r \in R$, such that $rx \in N - T$. Clearly, $rx \in L$. Then $L \leq T$.

Proposition (2.6): Let T, N be a submodules of a module D, such that $T \leq N$, but $N \nleq T$, then $N \leq_{ST.e} D$ if and only if $\frac{N}{T} \leq_{s.e} \frac{D}{T}$.

Proof: $\Rightarrow$) Assume that $N \leq_{ST.e} D$ and let $\frac{L}{T}$ be small submodule of $\frac{D}{T}$, such that $\frac{N}{T} \cap \frac{L}{T} = 0$. Then $\frac{N \cap L}{T} = 0$, which implies that $N \cap L = T$. Since $L \leq T$, but $T \leq L$, therefore $T = L$. Hence $\frac{L}{T} = 0$. So $\frac{N}{T} \leq_{S.e} \frac{D}{T}$. ($\Leftarrow$ Let $\frac{N}{T} \leq_{s.e} \frac{D}{T}$ and let V be small submodule of a module D, such that $N \cap V \leq T$. We want to show that $V \leq T$. As $\frac{N}{T} \cap \frac{V+T}{T} = \frac{N \cap (V+T)}{T} = \frac{(N \cap V)+T}{T} = \frac{T}{T} = 0$, bythemodular law, then $\frac{V+T}{T} = 0$, and hence $V + T = T$, so $V \leq T$. Then $N \leq_{ST.e}$.D

Recall that a submodule L of an R-module D is called small closed (s-closed) and denoted by $L \leq_{sc} D$, if L has no proper s-essential extention submodule in D, that is whenever $C \leq D$, such that $L \leq_{s.e} C \leq D$, then L=C[6].

Proposition (2.7): Let T, N be submodules of a module D , such that $N \nleq T$. Then T is s-closed in D, and $N \oplus T \leq_{s.e} D$ if and only if $N \oplus T \leq_{ST.e} D$.

Proof: $\Rightarrow$) Let T be s-closed of D and let $N \oplus T \leq_{s.e} D$ .Then by a previous study [5] (2.1.2-6), $\frac{N \oplus T}{T} \leq_{s.e} \frac{D}{T}$. Then by (2.6), $N \oplus T \leq_{ST.e} D$.($\Leftarrow$ Let $N \oplus T \leq_{ST.e} D$, then $N \cap T = 0$, by (2.6), then $\frac{N \oplus T}{T} \leq_{s.e} \frac{D}{T}$. Now, let $\frac{H}{T}$ be small submodule of $\frac{D}{T}$ and $N \cap T = 0$. Now $\frac{N \oplus T}{T} \cap \frac{H}{T} = \frac{(N \oplus T) \cap H}{T} = \frac{(N \cap H) \oplus T}{T} = \frac{T}{T} = 0$, by the modular law. But $\frac{N \oplus T}{T} \leq_{s.e} \frac{D}{T}$, therefore $\frac{H}{T} = 0$ and hence $H = T$. Thus, T is s-closed of D.

Proposition (2.8): Let T, N be a submodules of a module D, such that $N \nleq T$. If $N \leq_{s.e} D$ and T is s-closed for N of D, then $N + T \leq_{ST.e} D$.

Proof: Let $N \leq_{s.e} D$, $N \leq N + T$, then the same study above [5] (1.1.17) $N + T \leq_{s.e} D$. And since T is s-closed for N of D , by another study [6] (2,1,2-6), we have $\frac{N+T}{T} \leq_{s.e} \frac{D}{T}$, and by (2,6), $N + T \leq_{ST.e} D$.

Proposition (2.9): Let T, N be submodules of a module D, such that $N \nleq T$. And let L be a small submodule of D with $L \leq T$. If $N + T \leq_{ST.e} D$, then $\frac{N+T}{L} \leq_{s(\frac{T}{L}).e} \frac{D}{L}$.

Proof: Let $\frac{C}{L}$ be a small submodule of a module $\frac{D}{L}$, such that $\frac{N+T}{L} \cap \frac{C}{L} \leq \frac{T}{L}$. Then $\frac{(N+T) \cap C}{L} \leq \frac{T}{L}$. Hence, $(N + T) \cap C \leq T$. Since $\frac{C}{L}$ is small submodule of D, and L is small submodule of D. then, by an earlier work [1], C is small submodule of D. As $N + T \leq_{ST.e} D$, therefore, $C \leq T$. Thus, $\frac{C}{L} \leq \frac{T}{L}$.

Proposition (2.10): Let N, N_1 and L, L_1 be submodules of a module D, such that $N \leq_{ST.e} N_1$, $L \leq_{ST.e} L_1$. Then $N \cap L \leq_{ST.e} N_1 \cap L_1$.

Proof: Let K be a small submodule of $N_1 \cap L_1$, such that $K \cap (N \cap L) \leq T$. We want to show that $K \leq T$. Since K is small submodule of $N_1 \cap L_1$, then $K \cap N$ is small submodule of $N_1 \cap L_1 \leq L_1$. Then $K \cap N$ is small submodule of L_1. Then $(K \cap N) \cap L \leq T$, since $L \leq_{ST.e} L_1$. Then $K \cap N \leq T$, since K is small submodule of $N_1 \cap L_1$ .Then K is small submodule of N_1 $N \leq_{ST.e} N_1$. Then $K \leq T$.

Corollary(2.11): Let T, N be submodules of a module D, such that $N \nleq T$.

And let L be small submodule of a module D. If N and L are ST-essential submodules of D , then $N \cap L \leq_{ST.e} D$.

Proposition (2.12): Let $g: D_1 \to D_2$ be an epimorphism and let T,L be submodules of a module D_2. If $L \leq_{ST.e} D_2$, such that $L \nleq T$, then $g^{-1}(L) \leq_{s(g^{-1}(T)).e} D_1$.

Proof: Let K be a small submodule of D_1, such that $g^{-1}(L) \cap K \leq g^{-1}(T)$. Since g is an epimorphism, then $L \cap g(K) \leq T$. And since K be a small submodule of D_1, then $g(K)$ is a small submodule of D_2, $L \leq_{ST.e} D_2$. Then $g(K) \leq T$ and g is an epimorphism, then $K \leq g^{-1}(T)$.

Let $g: D_1 \to D_2$ be a homomorphism and let T,N be a submodules of D_1. If $N \leq_{ST.e} D_1$ then it is not necessary that $g(N) \leq_{sg(T).e} D_2$. For example:

Let $g: \mathbb{Z}_6 \to \mathbb{Z}_6$ be a map defined by $g(x) = 3x, \forall x \in \mathbb{Z}$. Let T=$\{\bar{0}, \bar{3}\}$ and N=$\{\bar{0}, \bar{2}, \bar{4}\}$. Then $g(T) = \{\bar{0}, 3\}$ and $g(N) = \{\bar{0}\}$. By (1,2-1), $N \leq_{ST.e} \mathbb{Z}_6$. But $\{\bar{0}\} \cap \{\bar{0}, \bar{2}, \bar{4}\} = \{\bar{0}\} \leq g(T)$, and $\{\bar{0}, \bar{2}, \bar{4}\} \nleq g(T)$. Hence, $g(N) \nleq_{Sg(T).e} \mathbb{Z}_6$.

The small singular (s-singular) submodule of M is denoted by $Z^s(M)$.
$Z^s(M) = \{m \in M | ml = 0$ for some s-essential right ideal l of R$\}$, if $Z^s(M) = 0$, then M is called a s-non singular module, and if $Z^s(M) = M$, then M is called s-singular [6].

Proposition (2.13): Let T and N be submodules of a module D, such that N⊈T, and let D be finitely generated. Then $\frac{D}{N+T}$ is s-singular.

Proof: Let $N+T \leq_{ST.e} D$. Then by (2.6), $\frac{N+T}{T} \leq_{s.e} \frac{D}{T}$. Hence by[6](2.2.6). Then $\frac{D/T}{N+T/T}$ is s-singular. Then, by the third isomorphic theorem, $\frac{D/T}{N+T/T} \cong \frac{D}{N+T}$. Therefore, $\frac{D}{N+T}$ is s-singular.

Proposition (2.14): Let T, N be submodules of a module D, such that N⊈T. If $\frac{D}{T}$ is s-nonsingular, then $\frac{D}{N+T}$ is s-singular if and only if $N + T \leq_{ST.e} D$.

Proof: ⇒) Let $\frac{D}{T}$ be S- non singular module and $\frac{D}{N+T}$ is S-singular, by the third isomorphic theorem $\frac{D}{N+T} \cong \frac{\frac{D}{T}}{\frac{N+T}{T}}$. Therefore, $\frac{D/T}{N+T/T}$ is s-singular, by the previously mentioned study [6](2.2.6). Then $\frac{N+T}{T} \leq_{S.e} \frac{D}{T}$. By(2.6), $N + T \leq_{ST.e} D$.

(⇐ Clear. By(2.15).

Proposition (2.15): Let $\{D_\alpha : \alpha \in \Lambda\}$ be a family of modules and T_α, H_α be submodules of D_α $, \forall \alpha \in \Lambda$, such that $H_\alpha \nsubseteq T$. If $H_\alpha + T_\alpha \leq_{ST_\alpha.e} D_\alpha, \forall \alpha \in \Lambda$, then $\oplus_{\alpha \in \Lambda} \frac{H_\alpha + T_\alpha}{T_\alpha} \leq_{S(\oplus_{\alpha \in \Lambda} T_\alpha).e} \oplus_{\alpha \in \Lambda} D_\alpha$.

Proof: Assume that $H_\alpha + T_\alpha \leq_{S(T_\alpha)-e} D_\alpha, \forall \alpha \in \Lambda$. By (2.6), $\frac{H_\alpha + T_\alpha}{T_\alpha} \leq_{S.e} \frac{D_\alpha}{T_\alpha}, \forall \alpha \in \Lambda$. By the above study [6] (1,1,17-3), $\oplus_{\alpha \in \Lambda} (\frac{H_\alpha + T_\alpha}{T_\alpha}) \leq_{S.e} \oplus_{\alpha \in \Lambda} (\frac{D_\alpha}{T_\alpha})$.

Then $\frac{\oplus_{\alpha \in \Lambda}(H_\alpha + T_\alpha)}{\oplus_{\alpha \in \Lambda} T_\alpha} = \frac{\oplus_{\alpha \in \Lambda} H_\alpha + \oplus_{\alpha \in \Lambda} T_\alpha}{\oplus_{\alpha \in \Lambda} T_\alpha} \leq_{S.e} \frac{\oplus_{\alpha \in \Lambda} D_\alpha}{\oplus_{\alpha \in \Lambda} T_\alpha}$. Then $\oplus_{\alpha \in \Lambda} H_\alpha + T_\alpha \leq_{S(\oplus_{\alpha \in \Lambda} T_\alpha).e} \oplus_{\alpha \in \Lambda} D_\alpha$, by (2.6).

Let D be a module. D is called a faithful module if Ann(D)=0. Let D be a module. D is called a multiplication module if for each submodule N of D, there exists an ideal I of R such that N=ID. Let M be an R-module and N≤M. The residual of M in N (denoted by N:M)={r∈R|rM⊆N} [7].

Proposition (2.16): Let T and N be submodules of a finitely generated and multiplication module D, such that N⊈T. If $N \leq_{ST.e} D$, then $[N:D] \leq_{S[T:D].e} R$.

Proof: Let V be small ideal in R, such that $[N:D] \cap V \leq [T:D]$. Then $[N:D]D \cap VD \leq [T:D]D$. Since D is multiplication , then $N \cap VD \leq T$, $N \leq_{ST.e} D$. Then $VD \leq T, V \leq [T:D]$. As $N \leq_{ST.e} D$, so $V \leq T$. Hence $[N:D] \leq_{S[T:D].e} R$.

Theorem (2.17): Let T,N be submodules of finitely generated, faithful and multiplication module D, such that N⊈T. Then $N \leq_{ST.e} D$ if and only if $[N:D] \leq_{S[T:D].e} R$.

Proof: ⇒) It is clear by (2.16).
(⇍Suppose that $[N:D] \leq_{S[T:D].e} R$. Let L be small a submodule of D, such that $N \cap L \leq T$. And let D be multiplication module, then $[N:D]D \cap [L:D]D \leq [T:D]D$. Since D is finitely generated, faithful and multiplication module, then D is cancellation module. Since L is small, then it is easy to show that [L:D] is small. So $[N:D] \cap [L:D] \leq [T:D]$. But $[N:D] \leq_{S[T:D].e} R$, therefore $[L:D] \leq [T:D]$ and hence $[L:D]D \leq [T:D]D$. Thus $L \leq T$.

3- ST. complement submodules

In this section, we introduce ST. complement submodules and study some of their properties and examples.

Definition (3.1): Let N,T be a submodule of a module D. A small submodule L is called small T-complement for N of D, if L is maximal with respect to the property that $N \cap L \leq T$.

Remarks and Examples (3.2):

1- The module $\mathbb{Z}_{12}$ as $\mathbb{Z}$- module. Let $T = \{\bar{0}, \bar{6}\}, N = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\}$. The small submodules of $\mathbb{Z}_{12}$ are $\{\bar{0}\}, \{\bar{0}, \bar{6}\}$. If $L = \{\bar{0}, \bar{6}\}$, then L is maximal with respect to the property $N \cap L \leq T$. Then L is ST-complement of A of $\mathbb{Z}_{12}$.

2- It is clear that ST-complement is not unique.

3- Let T and N be submodules of a module D. And let L be small submodule of a module D. If L is ST- complement of N of D, then it is not necessary that N is ST- complement L of D, as in the following example: The module $\mathbb{Z}_8$ as $\mathbb{Z}$-module. Let $T = \{\bar{0}, \bar{4}\}, N = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$ and let $L = \{\bar{0}, \bar{4}\}$ be small submodule of $\mathbb{Z}_8$, $N \cap L \leq T$. But N is not ST- complement for L of $\mathbb{Z}_8$ (since N is not small submodule of $\mathbb{Z}_8$).

Proposition (3.3): Let T and N be submodules of a module D. Then N has a ST- complement of D.

Proof: Let T and N be submodules of a module D. Let $G = \{L \text{ small submodule of } D \backslash N \cap L \leq T\}$, G≠ $\emptyset$, where $0 \in$G. Let $\{C_\propto\}_{\propto \in \wedge}$ be a chain in G. Clearly, $\cup_{\propto \in \wedge} C_\propto$ is small submodule of D, since $N \cap (\cup_{\propto \in \wedge} C_\propto) = \cup_{\propto \in \wedge} (N \cap C_\propto) \leq T$. Then $\cup_{\propto \in \wedge} C_\propto \in G$. By Zoren's lemma , G has a maximal element say H. Claim that, H is ST- complement for N in D. To show that, let V be small submodule of D with $H \nleq V$ and $N \cap V \leq T$. Therefore V∈G, which is a contradiction. Thus, H=V.

Proposition (3.4): Let T and N be submodules of the module D and let C be small submodule of D, with N≤C. If L is ST- complement for N of D and C is a ST-complement for L of D, then L is a ST- complement for C of D.

Proof: Let L be ST- complement for N of D and C be a ST- complement for L of D, such that N≤C. Then $L \cap C \leq T$. We want to show that L is ST- complement for C of D. Let V be small submodule of D, such that L≤V and $C \cap V \leq T$. As N≤C, then $N \cap V \leq C \cap V \leq T$. But L is maximal with respect to the property that $N \cap L \leq T$, therefore L=V. Hence, L is ST- complement for C of D.

Proposition (3.5): Let T,N,C be submodules of a module D, such that T≤N≤C. If L is a ST-complement for N of D and C is a ST- complement for L of D. Then C is maximal ST- essential of N of D.

Proof: Assume that L is an ST- complement for N of D and C is a ST- complement for L of D and T≤N≤C. First, we prove that $N \leq_{ST.e} C$. Let K be small submodule of C, such that N∩K≤T. Claim that N∩ (L+K)≤T. To show that, let $n = b + k$, where $n \in N$, $b \in L, k \in K$. Then $b = n - k \in L \cap C \leq T \leq N$. So $n - b = k \in N \cap K \leq T$ and hence $n \in T$. But L is maximal with respect to the property that $N \cap L \leq T$, therefore $L + K = L$. Then $K \leq B$, which implies that $K = K \cap C \leq B \cap C \leq T$. Thus $N \leq_{ST.e} C$. Now, to show that C is maximal ST-essential for N of D, let V be small submodule of D, such that $C \leq V$ with $N \leq_{ST.e} V$. Since $N \cap L \leq T$, then $(N \cap L) \cap V \leq T \cap V \leq T$, implies that $N \cap (L \cap V) \leq T$. Since $N \leq_{ST-e} V$, then $L \cap V \leq T$. But C is maximal with respect to the property that $L \cap C \leq T$, therefore V=C. Thus, C is maximal ST-essential for N of D.

Proposition (3.6): Let T, N, L be submodules of a module D, such that $T = N \cap L$. Then N is a ST-complement for L of D if and only if $\frac{N+L}{N} \leq_{s.e} \frac{D}{L}$.

Proof: ⇒) Suppose that N is a ST- complement for L of D. Let $\frac{B}{N}$ be small submodule of $\frac{D}{N}$, such that $\frac{N+L}{N} \cap \frac{B}{N} = 0$, then $\frac{(N+L)\cap B}{N} = 0$. Hence, by the modular law, $\frac{(L\cap B)+N}{N} = 0$, implies that (L∩B)+N=N. Then L∩B≤N, and then L∩B≤N∩L=T. But L is maximal with respect to the property that N∩L≤T, therefore B=N. Thus, $\frac{N+L}{N} \leq_{s.e} \frac{D}{N}$.

(⇐ Let B be small submodule of D, such that L∩B≤T. Then $\frac{B}{N}$ be small submodule of $\frac{D}{N}$, and $\frac{N+L}{N} \cap \frac{B}{N} = \frac{(N+L)\cap B}{N} = \frac{(L\cap B)+N}{N}$, by the modular law. Since L∩B≤T and N∩L=T, then L∩B≤N∩L. But L≤B, therefore N∩L≤L∩B, so N∩L=L∩B. Thus, $\frac{(L\cap B)+N}{N} = \frac{(N\cap L)+N}{N} = \frac{N}{N} = 0$. Since $\frac{N+L}{N} \leq_{s.e} \frac{D}{N}$, then $\frac{B}{N} = 0$, hence B=N. Thus, N is ST-complement for L of D.

Theorem (3.7): Let T, N be submodules of a module D. If L is a ST- complement to N+T of D, then (N+T)+L$\leq_{ST.e}$ M and $\frac{(N+T)+L}{T} = \frac{N+T}{T} \oplus \frac{L+T}{T}$.

Proof: Assume that L is ST- complement for N+T of D. Then L is maximal small submodule of D, (N+T)∩L≤T. Let C be small submodule of D, such that ((N+T)+L)∩C≤T. Claim that (N+T)∩(L+C)≤T. To show that , let $n + t = b + c$, where $n \in N, t \in T, b \in L, c \in C$. Since $(n + t) - b = c \in ((N + T) + L)$, then $(n + t) - c = b \in (N + T) \cap L \leq T$. Thus, $n + t \in T$ and hence (N+T)∩(L+C)≤T. As L is maximal with respect to the property that (N+T)∩L≤T, therefore L=L+C and hence C≤L , implies that C≤L∩C≤((N+T)+L)∩C≤T. Hence, (N+T)+L$\leq_{ST.e}$D. For the second part, it is enough to show that $\frac{N+T}{T} \cap \frac{L+T}{T} = 0$. Let $n + t = b + t_1$, where $n \in N, b \in B, t, t_1 \in T$. Then $n + t - t_1 = b$, and hence b∈(N+T)∩L≤T, then $n + t$ ∈T, implies that $\frac{N+T}{T} \cap \frac{L+T}{T} = \frac{(N+T)\cap(L+T)}{T} \leq \frac{T}{T} = 0$.

Proposition (3.8): Let T,N,L and C be submodules of a module D, such that $T \leq N$. If A$\leq_{S(T+C).e}$ D and C is ST-complement of L of D, then $\frac{N+C}{C} \leq_{S\left(\frac{T+C}{C}\right).e} \frac{D}{C}$.

Proof: Let A$\leq_{S(T+C).e}$ D and C be ST-complement to L of D. Let $\frac{L}{C}$ be a small submodule of $\frac{D}{C}$, such that $\frac{N+C}{C} \cap \frac{L}{C} \leq \frac{(T+C)}{C}$. Since $\frac{(N+C)}{C} \cap \frac{L}{C} = \frac{(A+C)\cap L}{C} = \frac{(A\cap L)+C}{C} \leq \frac{T+C}{C}$, then $(N \cap L) + C \leq T + C$, and hence $N \cap L \leq T + C$. But $N \leq_{S(T+C).e} D$, therefore $L \leq T + C$. So $\frac{L}{C} \leq \frac{(T+C)}{C}$. Thus, $\frac{N+C}{C} \leq_{S\left(\frac{T+C}{C}\right).e} \frac{D}{C}$.

1. **Proposition (3.9):** Let T,N be submodules of a module D. If $N \cap (T + L) \leq T$ and $\frac{N+L}{L} \leq_{S\left(\frac{T+L}{L}\right).e} \frac{D}{L}$, then (T+L) is a S(T+L)- complement for N of D.

Proof: Let C be small submodule of a module D, such that T+L≤C and N∩C≤T+L. Then $\frac{N+L}{L} \cap \frac{C}{L} = \frac{(N+L)\cap C}{L} = \frac{(N\cap C)+L}{L}$, by modular law. Then $\frac{(N\cap C)+L}{L} \leq \frac{(T+L)+L}{L} = \frac{T+L}{L}$, since $\frac{N+L}{L} \leq_{S\left(\frac{T+L}{L}\right).es} \frac{D}{L}$. Then $\frac{C}{L} \leq \frac{T+L}{L}$ and hence $C \leq T + L$. Thus, T+L=C.

Proposition (3.10): Let T,N be a submodule of a finitely generated, faithful and multiplication module D. Then L is a ST- complement for N in D if and only if [L:D] is a S[T:D]- complement for [N:D] in R.

Proof: ⇒) Let L is ST- complement for N of D, then L is maximal small submodule of D, such that $N \cap L \leq T$. Since D is multiplication module , then[N:D]$D \cap [L: D]D \leq [T: D]D$. Since D is finitely generated, faithful and multiplication module, then it is a cancellation module, as previously. So

$[N:D] \cap [L:D] \leq [T:D]$. Let L is small submodule of D, $[L:D] \leq [C:D]$, such that $[N:D] \cap [C:D] \leq [T:D]$,then $[N:D]D \cap [C:D]D \leq [T:D]D$. Since D is finitely generated, then $N \cap C \leq T$. But L is maximal with respect to the property that $N \cap L \leq T$, then L=C, and hence [N:D]=[C:D]. Thus [L:D] is S[T:D]-complement for [N:D] in R.($\Leftarrow$ Suppose that [L:D] is S[T:D]- complement for [N:D] in R. Then $[N:D] \cap [L:D] \leq [T:D]$, implies that $[N:D]D \cap [L:D]D \leq [T:D]D$.. Since D is finitely generated, then $N \cap L \leq T$. Let $L \leq C$, where C is small submodule of D, such that $N \cap C \leq T$. Now, as D is a multiplication module, then $[N:D]D \cap [C:D]D \leq [T:D]D$. Since D is finitely generated, faithful and multiplication module, then D is a cancellation module, as shown previously[8]. So $[N:D] \cap [C:D] \leq [T:D]$. But [L:D] is a maximal with respect to the property that $[N:D] \cap [L:D] \leq [T:D]$, therefore [L:D]=[C:D], implies that [L:D]D=[C:D]D, thus L=C.

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